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BERNSTEIN OPERATIONAL MATRIX OF DIFFERENTIATION AND COLLOCATION APPROACH FOR A CLASS OF THREE-POINT SINGULAR BVPS: ERROR ESTIMATE AND CONVERGENCE ANALYSIS

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Abstract. Singular boundary value problems (BVPs) have widespread applications in the field of engineering, chemical science, astrophysics and mathematical biology. Finding an approximate solution to a problem with both singularity and non-linearity is highly challenging. The goal of the current study is to establish a numerical approach for dealing with problems involving three-point boundary conditions. The Bernstein polynomials and collocation nodes of a domain are used for developing the proposed numerical approach. The straightforward mathematical formulation and easy to code, makes the proposed numerical method accessible and adaptable for the researchers working in the field of engineering and sciences. The priori error estimate and convergence analysis are carried out to affirm the viability of the proposed method. Various examples are considered and worked out in order to illustrate its applicability and effectiveness. The results demonstrate excellent accuracy and efficiency compared to the other existing methods.

Keywords: Bernstein polynomials, collocation method, three-point singular BVPs, convergence analysis, error estimate.

Mathematics Subject Classification: 34B05, 34B15, 34B16, 34B18, 34B27, 34B60.

1. INTRODUCTION

A massive community of mathematicians and physicists have been involved in the research on singular boundary value problems (SBVPs), regarding its solution with analytical and numerical approaches. These communities are showing their interest in the SBVPs due to its frequent occurrence in mathematical modelling of various physical phenomenon *viz.* electro hydrodynamics, chemical kinetics, shallow membrane cap theory and astrophysics (see [5, 31] and the references therein). In the last few decades, multipoint BVPs have been intensively used for modelling real-life applications including the vibration of a guy wire of uniform cross section and composed of N parts, theory of elastic stability, large bridges with multi-point support, the elasticity of an equally loaded three layered sandwich beam (for details see [11,53] and references therein).

In the present work, a class of three-point SBVPs [2,12] is considered

$$-(p(x)y'(x))' = p(x)f(x, y(x)), \quad 0 < x \leq 1, \tag{1.1}$$

subject to the boundary conditions

$$y(0) = 0, \quad y(1) = \alpha y(\eta), \tag{1.2}$$

or else to

$$y^{'}(0) = 0, \quad y(1) = \alpha y(\eta). \tag{1.3}$$

Here, we assume that

$$\begin{cases} 0 < \alpha < \dfrac{h(1)}{h(\eta)}, \quad h(x) = \displaystyle\int_0^x \frac{1}{p(t)}dt, & \text{in the case of boundary conditions (1.2)}, \\ \alpha > 0, & \text{in the case of boundary conditions (1.3)}, \end{cases}$$

$0 < \eta < 1$.

The following conditions are met throughout the entire article by p and f.

(P1) $p \in \mathcal{C}[0,1] \cap \mathcal{C}^1(0,1]$ with $p(x) > 0$ on $(0,1]$ and $\frac{1}{p}$ is locally integrable on $(0,1]$,
(P2) $f \in \mathcal{C}([0,1] \times [0,\infty), [0,\infty))$ and $f(x,0) \not\equiv 0$.

When $p(0) = 0$ the problems (1.1), (1.2) or (1.1), (1.3) are singular, which constitutes a challenge to find analytical or numerical solutions.

One particular form of singular differential equation (1.1) is the Lane–Emden equation, for $p(x) = x^2$ and $f(x,y) = y^n$. The Lane–Emden equation is a particular form of Poisson's equations for gravitational potential of a Newtonian self-gravitating, spherically symmetric, polytropic fluid, and some process in a chemical reactor (see [7,25–27,49,50] and the references therein). Under uniform normal pressure, an equation of circular membrane cap $y'' + \frac{3}{x}y' + \frac{k}{y^2} = 0$ is considered by Agarwal and O'Regan [1].

The singular behaviour at $x = 0$, of differential equation (1.1) impose great difficulty to the researchers in order to get the solution. In literature, there are numerous research works available regarding analytical and numerical approach for an approximate solution of SBVPs. In [43], the authors have proposed sufficient theorems for the existence of a positive solution of the SBVPs $y^{''} + \mu a(t)f(t, y(t)) = 0$ with boundary conditions (BC) $y(0) - \beta y^{'}(0) = 0, y(1) = \alpha y(\eta)$. Kiguradze [18] has established some existence and uniqueness result for the multi-point boundary value problems subject to the singular differential equation (DE). He has used fixed point theory on an equivalent integral operator for establishing these results. A two-point boundary value problem subject to the DE (1.1) with $p(x) = x^\alpha$ has been studied by Kiguradze and Shekhter [19] for the analytical solutions. Cecchi *et al.* [9] have provided some sufficient condition for a unique non-negative solution of Emden–Fowler equation (1.1) for $p(x) = x^{\alpha}$. Jin and Yan [14] have studied the existence of a positive solution of SBVPs

$$-u^{''}(t) + |u^{'}(t)|^{\mu} + K(t)u^{-q} = \lambda u^{p}, \quad t \in (0,1) \setminus \left\{\frac{1}{2}\right\},$$

with BC $u(0) = u(1) = 0$, using upper and lower solution method. Agarwal and O'Regan [1] have also utilized the upper and lower solution method for the existence result on

$$\frac{1}{q(x)}(p(x)y'(x))' = q(x)f(x, y(x))$$

subject to Sturm–Liouville boundary conditions

$$-\alpha_0 y'(0) + \beta_0 \lim_{x\to 0^+} p(x)y'(x) = c_0, \quad \alpha_1 y'(1) + \beta_1 \lim_{x\to 1^-} p(x)y'(x) = c_1.$$

Further an existence result on the DE (1.1) without any restriction of growth on f is also established. Azbelev *et al.* [5] has established some solvability theorem for unique positive solution of Emden–Fowler equation (1.1) for $p(x) = x^k$ using Alves scheme [4] by transforming it into an equivalent integral equation (see pp. 142–147). One can find some more analytical results on the existence of solutions of SBVPs in the literature (see [24, 32] and references therein). These analytical works motivate us for study of SBVPs with a numerical aspect, since the analytical results are not good enough for the researchers from application point of view of the mathematical models. In fact, the solution of the model is needed in terms of some specific function, thus numerical approximation of the solution becomes very essential for the use in real world problems.

There are several numerical techniques available for the numerical solution of singular boundary value problems [17,28,29,40,47]. Other methods to find approximate solutions of SBVPs are cubic spline method [16, 30], B-spline method [10], Green's function solution [22], modified homotopy analysis method [38, 39], Haar wavelet resolution technique [34], Chebyshev collocation method [42], Bernstein collocation method [41], Hermite polynomial and collocation method [23], Adomian decomposition method [45] and the combination of iterative method and homotopy perturbation method [13, 35], variational iteration method [44, 51]. Ahmad *et al.* [3] have presented a bio-inspired numerical technique for solving a boundary value problem arising in the modeling of corneal shape. Although these methods have various advantages, the implementation is not easy and time-consuming. The Bernstein polynomial and collocation approach is an excellent numerical technique that has been used to obtain the numerical solution of BVPs. The Bernstein collocation method has been used intensively to solve nonlinear Fredholm–Volterra integro-differential equations, strongly nonlinear damped system and integro-differential-difference equations [8, 52]. Shahni and Singh [33] have solved the system of Emden–Fowler type equations arising in dusty fluid model using Bernstein collocation method. Maleknejad *et al.* [21] has applied the collocation method together with Bernstein operational matrix method for a numerical solution of the Volterra–Fredholm integro-differential equation. In the literature, there exist some prime results on the existence of SBVPs (1.1)–(1.2) and (1.1)–(1.3). [6] has established some result for the existence of a positive solution of a system of singular differential equations of the class (1.1) subject to the boundary condition (1.2). Additionally, [36] has shown some existence results for the solution of (1.1) subject to the boundary condition (1.3). According to a thorough review of the available literature, the three-point singular boundary value problems (1.1)–(1.2) and (1.1)–(1.3) have not been solved numerically using the Bernstein collocation method.

In this article, the collocation method in the presence of Bernstein polynomials is employed to obtain numerical solutions of the three-point SBVPs (1.1)–(1.2) and (1.1)–(1.3). The method is based on the representation of the unknown solution as a linear combination of Bernstein polynomials with unknown coefficients. This leads to the transformation of the SBVP into a matrix form. The next step uses the collocation points to convert the matrix form of the SBVP into a system of nonlinear algebraic equations. Consequently, the solution of the system of algebraic equations yields the approximate solution of the SBVP.

The article is structured as follows. Section 2 introduces the Bernstein polynomials, operational matrix of differentiation and the approximation of the unknown solution using Bernstein polynomials. In Section 3, the methodology to deal with the SBVPs (1.1)–(1.3) is developed. The error estimate and the convergence analysis is given in Section 4. Section 5 is devoted to compare the performance of the proposed method with other existing numerical methods, considering several examples of three-point SBVP. Finally, some remarks and conclusions are made in Section 6.

2. BERNSTEIN POLYNOMIALS

This section briefly introduces Bernstein polynomials and its properties.

2.1. BASICS

The Bernstein polynomials of degree n are given by

$$\begin{cases} B_n^i(x) = \binom{n}{i} x^i (1-x)^{n-i}, & 0 \le i \le n, \\ 0, & i < 0, \quad i > n, \end{cases} \tag{2.1}$$

where $\binom{n}{i} = \frac{n!}{i!(n-i)!}$, $n \in \mathbb{N}$, $i = 0, 1, \ldots, n$ and $x \in [0,1]$. The set of polynomials $\{B_n^i(x), i = 0, 1, \ldots, n\}$ for any n forms a complete basis. Some basic notions of the Bernstein polynomials are the following:

(i) $B_n^i(x) \ge 0$ for all $x \in [0,1]$,
(ii) $B_n^0(0) = B_n^n(1) = 1$,
(iii) $B_n^i(0) = B_n^i(1) = 0$ for $1 \le i \le n-1$,
(iv) $\sum_{i=0}^{n} B_n^i(x) = 1$.

2.2. FUNCTIONAL APPROXIMATION

A square integrable function $y(x)$ on $(0,1)$ can be approximated in a linear combination of Bernstein basis as

$$y(x) \approx y_n(x) = \sum_{i=0}^{n} a_i B_n^i(x) = A^T B(x), \tag{2.2}$$

where

$$A^T = [a_0, a_1, \ldots, a_n], \tag{2.3}$$

and

$$B(x) = [B_n^0(x), B_n^1(x), \ldots, B_n^n(x)]^T. \tag{2.4}$$

The Bernstein polynomial $B_n^i(x)$ can be expressed in the series of integer power of x as

$$B_n^i(x) = \binom{n}{i} x^i (1-x)^{n-i} = \sum_{j=0}^{n-i} (-1)^j \binom{n}{i} \binom{n-i}{j} x^{i+j}. \tag{2.5}$$

The equation (2.2) can be expressed in the matrix form, by add of equation (2.5) as

$$y_n(x) = A^T D X(x), \tag{2.6}$$

where

$$D = \begin{bmatrix} (-1)^0\binom{n}{0} & (-1)^1\binom{n}{0}\binom{n-0}{1} & \ldots & (-1)^{n-0}\binom{n}{0}\binom{n-0}{n-0} \\ 0 & (-1)^0\binom{n-1}{1} & \ldots & (-1)^{n-1}\binom{n}{1}\binom{n-1}{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & (-1)^0\binom{n}{n} \end{bmatrix}, \tag{2.7}$$

and

$$X(x) = [1, x, \ldots, x^n]^T. \tag{2.8}$$

2.3. THE OPERATIONAL MATRIX OF DIFFERENTIATION

The derivative of the array $X(x)$ is given by the following relation

$$X^{'}(x) = [0, 1, 2x, \ldots, nx^{n-1}]^T = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 \\ 1 & 0 & 0 & \ldots & 0 \\ 0 & 2 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & n & 0 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \\ \vdots \\ x^n \end{bmatrix}.$$

Thus, the derivatives of the function $y_n(x)$ in terms of Bernstein basis is given by

$$y_n^k(x) = A^T D C^k X(x), \quad k = 1, 2, \ldots, \tag{2.9}$$

where

$$C = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & n & 0 \end{bmatrix}. \tag{2.10}$$

The matrix C is of order $(n+1)$, called an operational matrix of differentiation.

3. METHODOLOGY

In this section, a numerical technique for the solution of the SBVP (1.1)–(1.2) or (1.1)–(1.3) is developed. This method is based on the Bernstein polynomials and the collocation approach. Using the approximations of $y(x)$ and its derivatives given in (2.9) the differential equation in (1.1) can be expressed in a matrix form given by

$$\begin{aligned} &p(x)\,(X(x))^T \left(C^T\right)^2 D^T A + p^{'}(x)\,(X(x))^T\, C^T D^T A \\ &+ p(x) f(x, (X(x))^T\, D^T A) = 0. \end{aligned} \tag{3.1}$$

Now, we introduce here the collocation points x_i, given by

$$x_{i-1} = \frac{i}{n+1}, \quad i = 1, 2, \dots, n+1, \tag{3.2}$$

corresponding to an integer n, to evaluate the equation (3.1). This gives rise to a system of algebraic equations

$$P\bar{X}\left(C^T\right)^2 D^T A + P_1 \bar{X} C^T D^T A + F = 0, \tag{3.3}$$

where F is given by

$$F = [p(x_0) f(x_0, (X(x_0))^T\, D^T A), \dots, p(x_n) f(x_n, (X(x_n))^T\, D^T A)$$

and

$$P = \begin{bmatrix} p(x_0) & 0 & 0 & \cdots & 0 \\ 0 & p(x_1) & 0 & \cdots & 0 \\ 0 & 0 & p(x_2) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & p(x_n) \end{bmatrix}$$

$$P_1 = \begin{bmatrix} p^{'}(x_0) & 0 & 0 & \cdots & 0 \\ 0 & p^{'}(x_1) & 0 & \cdots & 0 \\ 0 & 0 & p^{'}(x_2) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & p^{'}(x_n) \end{bmatrix}, \quad \bar{X} = \begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ 1 & x_2 & x_2^2 & \cdots & x_2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{bmatrix}.$$

Furthermore, the matrix representation of the boundary conditions in (1.2) is given by

$$(X(0))^T D^T A = 0, \quad (X(1))^T D^T A - \alpha (X(\eta))^T D^T A = 0, \tag{3.4}$$

and the boundary conditions in (1.3) are given by

$$(X(0))^T C^T D^T A = 0, \quad (X(1))^T D^T A - \alpha (X(\eta))^T D^T A = 0. \tag{3.5}$$

The aforementioned procedure leads to a system of $(n+1)$ equations in (3.3), which is a discretization of the considered BVP. Moreover, two pair of algebraic equations (3.4) and (3.5) corresponding to the conditions (1.2) and (1.3) are obtained. In order to get the solution of a SBVP, replace the two equations given in (3.3) by (3.4) or (3.5) according to the boundary conditions. The solution of the unknown coefficients corresponding to the $(n+1)$ system of algebraic equations are obtained using the "Maple 18" software. Further, the desired numerical solution can be found by replacing the values of the coefficients in equation (2.2).

It is worth mentioning that most of the numerical examples involve an integer power of y. Therefore, it is necessary to show how an integer power of y can be approximated using Bernstein polynomials and the collocation method. The approximation of y^m in terms of Bernstein polynomials using collocation points is given by

$$[(y(x_0))^m, (y(x_1))^m, \ldots, (y(x_n))^m]^T = (B_1)^{m-1} \bar{X} D^T A, \tag{3.6}$$

where

$$B_1 = \begin{bmatrix} A^T B(x_0) & 0 & 0 & \cdots & 0 \\ 0 & A^T B(x_1) & 0 & \cdots & 0 \\ 0 & 0 & A^T B(x_2) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A^T B(x_n) \end{bmatrix}. \tag{3.7}$$

The flowchart of the methodology is depicted in Figure 1.

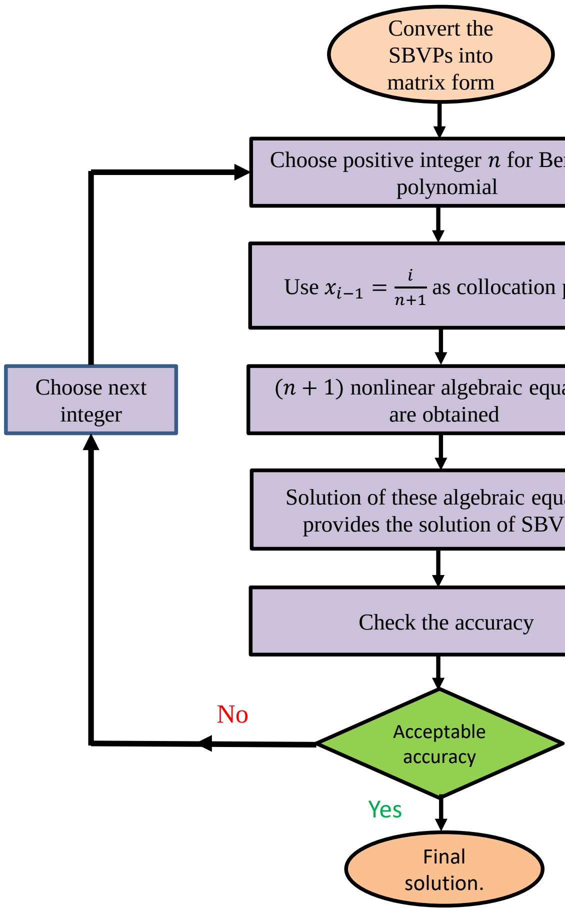


Fig. 1. Schematic representation of the new approach

4. ERROR ESTIMATION AND CONVERGENCE ANALYSIS

4.1. ERROR ESTIMATION

This section provides upper bounds for error norms. To analyze the upper bounds, we have used the Taylor series expansion. The residual correction term is also given, using the error estimation process.

The differential equation (1.1) can also be expressed as

$$p(x)y^{''}(x) + p^{'}y^{'}(x) + p(x)f(x,y) = 0. \tag{4.1}$$

Let the function $f(x, y)$ can be expressed as $f(x, y) = r(x) + y^m$. Thus, the differential equation (4.1) is given by

$$p(x)y^{''}(x) + p^{'}(x)y^{'}(x) + p(x)y^m = r_1(x), \tag{4.2}$$

where $r_1(x) = -p(x)r(x)$. We can have the matrix form of the differential equation (4.2), using the methodology developed in Section 3. We have

$$\left[P\bar{X}\left(C^T\right)^2 D^T + P_1\bar{X}C^T D^T + PB_1^{m-1}\bar{X}D^T\right] A = R, \tag{4.3}$$

$$ZA = R. \tag{4.4}$$

The choice of the collocation points are such that the matrix Z is non-singular. Thus, equation (4.4) can be expressed as

$$A = Z^{-1}R, \tag{4.5}$$

where

$$Z = \left[P\bar{X}\left(C^T\right)^2 D^T + P_1\bar{X}C^T D^T + B_1^{m-1}\bar{X}D^T\right], \quad s(y_n(x_j)) = \sum_{j=0}^{n} S(i,j)a_j$$

and $r_1(x_i) = R[i]$.

Theorem 4.1. *Let x and $\hat{x} = x + \delta x$ be solutions of the system of equations $Ax = b$ and $A\hat{x} = b + \delta b$, where A is a non-singular matrix and $b \neq 0$ is a vector* [48]*, then*

$$\|\delta x\| \le \|A^{-1}\|\|\delta b\|. \tag{4.6}$$

Let $u(x) \in C^\infty[0,1]$ be the exact solution of the differential equation (4.2), the Taylor series expansion of $u(x)$ about $\xi \in [0,1]$ is given by

$$u(x) = 1+\frac{u^{'}(\xi)}{1!}(x-\xi)+\frac{u^{''}(\xi)}{2!}(x-\xi)^2+\ldots+\frac{u^{(n)}(\xi)}{n!}(x-\xi)^n+\frac{u^{(n+1)}(\xi)}{n+1!}(x-\xi)^{n+1}+\ldots \tag{4.7}$$

The following series can also be expressed as

$$u(x) = P_n(x) + Rw(x), \tag{4.8}$$

where

$$P_n(x) = \sum_{i=0}^{n} \frac{u^{(i)}(\xi)}{i!}(x-\xi)^i \quad \text{and} \quad Rw(x) = \sum_{i=n+1}^{\infty} \frac{u^{(i)}(\xi)}{i!}(x-\xi)^i.$$

Let $Q_n(x)$ be the solution of the equation (4.2) using Bernstein polynomials. So, $Q_n(x)$ satisfies the equation (4.2) on the collocation points. Thus, the polynomials $Q_n(x)$ and $P_n(x)$ are solution of the system $\hat{Z}A = R$ and $\hat{Z}\hat{A} = R + \Delta R$, where

$$\Delta R[i] = -p(x_i)Rw^{''}(x_i) - p^{'}(x_i)Rw^{'}(x_i) - \sum_{j=1}^{n}\binom{n}{j}\left(P_n(x_i)\right)^{n-j}\left(Rw(x_i)\right)^j.$$

Theorem 4.2. *Let $u(x)$ be the exact solution of the differential equation* (4.2)*, and $Q_n(x)$ is the solution using Bernstein polynomials. If $P_n(x)$ be the truncation of the of $u(x)$ and $Rw(x)$, its residual, then the error function $e(x) = u(x) - Q_n(x)$ holds the following inequality*

$$|e(x)| \leq |Rw(x)| + \|B^T(x)\| \|\Delta R\| \|\hat{Z}^{-1}\|.$$

Proof. The absolute norm of an error function $e(x)$ is given by

$$|e(x)| = |u(x) - Q_n(x)|. \tag{4.9}$$

Now, we establish a triangular inequality on (4.9), by the use of the function $P_n(x)$, as follows:

$$|e(x)| \leq |u(x) - P_n(x)| + |Q_n(x) - P_n(x)| = |Rw(x)| + |Q_n(x) - P_n(x)|. \tag{4.10}$$

Since $Q_n(x)$ and $P_n(x)$ are polynomials of degree n, they can be approximated using Bernstein basis. Let $Q_n(x) = B^T(x)A$ and $P_n(x) = B^T(x)\hat{A}$. Thus we have

$$|e(x)| \leq |Rw(x)| + |B^T(x)(A - \hat{A})|,$$

which implies

$$|e(x)| \leq |Rw(x)| + \|B^T(x)\| \|(A - \hat{A})\|.$$

Using Theorem 4.1 to the above inequality, we have

$$|e(x)| \leq |Rw(x)| + \|B^T(x)\| \|\Delta R\| \|\hat{Z}^{-1}\|. \tag{4.11}$$

This completes the proof. □

Corollary 4.3. *If the SBVPs corresponding to the differential equation* (4.2) *constitute an exact solution, which is a polynomial, then the proposed method provides exact solution for $n \geq deg(u(x))$.*

Proof. For $n \geq deg(u(x))$, $Rw(x) = 0$. Thus, the inequality (4.11) provides $e(x) = 0$, which completes the proof. □

4.2. CONVERGENCE ANALYSIS

To show the convergence of the methodology, the Bernstein polynomials [20] have been used for the Weierstrass approximation theorem.

Theorem 4.4. *If $y(x)$ is a continuous function on* $[0, 1]$ *and*

$$B_n(y, x) = \sum_{i=0}^{n} B_n^i(x) y\left(\frac{i}{n}\right)$$

is the Bernstein polynomial of degree n in terms of Bernstein basis, then $B_n(y, x)$ converges uniformly to $y(x)$.

Proof. Some results on the Bernstein polynomials are as follows:

$$\sum_{i=0}^{n} \begin{pmatrix} n \\ i \end{pmatrix} B_n^i(x) = 1, \tag{4.12}$$

$$\sum_{i=0}^{n} \begin{pmatrix} n \\ i \end{pmatrix} \left(\frac{i}{n}\right) B_n^i(x) = x, \tag{4.13}$$

$$\sum_{i=0}^{n} \begin{pmatrix} n \\ i \end{pmatrix} \left(\frac{i}{n}\right)^2 B_n^i(x) = \frac{n-1}{n}x^2 + \frac{x}{n}. \tag{4.14}$$

The difference of $B_n(y,x)$ and $y(x)$ is given by

$$B_n(y,x) - y(x) = \sum_{i=0}^{n} f\left(\frac{i}{n}\right) B_n^i(x) - y(x).1.$$

Using the relation (4.12), in the following equation, we have

$$B_n(y,x) - y(x) = \sum_{i=0}^{n} \left\{ y\left(\frac{i}{n}\right) - y(x) \right\} B_n^i(x), \tag{4.15}$$

so

$$|B_n(y,x) - y(x)| \leq \sum_{i=0}^{n} \left| y\left(\frac{i}{n}\right) - y(x) \right| B_n^i(x). \tag{4.16}$$

Since the function $y(x)$ is uniformly continuous on $[0,1]$, thus there exists a positive real number δ for a given real number $\epsilon > 0$, so that

$$|x_1 - x_2| < \delta \quad \Longrightarrow \quad |y(x_1) - y(x_2)| < \epsilon. \tag{4.17}$$

Corresponding to the real number $\delta > 0$ and $x \in [0,1]$, we can divide the set of nodes $\frac{i}{n}$ into two sets

$$A = \left\{ \frac{i}{n} : |\frac{i}{n} - x| < \delta \right\} \quad \text{and} \quad B = \left\{ \frac{i}{n} : |\frac{i}{n} - x| \geq \delta \right\}.$$

Thus the series on the right hand side of the inequality (4.16) can be divided into two series $\sum'$ and $\sum''$ as follows:

$$\begin{aligned} |B_n(y,x) - y(x)| \leq & \sum_{i=0}^{n}{}' \left| y\left(\frac{i}{n}\right) - y(x) \right|_{\left(\frac{i}{n} \in A\right)} B_n^i(x) \\ & + \sum_{i=0}^{n}{}'' \left| y\left(\frac{i}{n}\right) - y(x) \right|_{\left(\frac{i}{n} \in B\right)} B_n^i(x). \end{aligned} \tag{4.18}$$

Let ϵ be given corresponding to the real number δ such that

$$\left| y\left(\frac{i}{n}\right) - y(x)\right| < \frac{\epsilon}{2} \quad \text{for} \quad \left|\frac{i}{n} - x\right| < \delta. \tag{4.19}$$

Now, for $\left|\frac{i}{n} - x\right| \geq \delta$, we have

$$1 \leq \frac{\left(\frac{i}{n} - x\right)^2}{\delta^2}. \tag{4.20}$$

Let $|f(x)| \leq M$, then by using relation (4.20), and assuming that $\frac{i}{n} \in B$, we have

$$\begin{aligned}\sum_{i=0}^{n}{}'' \left| y\left(\frac{i}{n}\right) - y(x)\right| B_n^i(x) &\leq \frac{1}{\delta^2}\sum_{i=0}^{n}{}'' \left(\frac{i}{n} - x\right)^2 \left| y\left(\frac{i}{n}\right) - y(x)\right| B_n^i(x),\\ &< \frac{2M}{\delta^2}\sum_{i=0}^{n}{}'' \left(\frac{i}{n} - x\right)^2 B_n^i(x).\end{aligned}$$

Using the results (4.12), (4.13) and (4.14) in the above inequality, we have

$$\sum_{i=0}^{n}{}'' \left| y\left(\frac{i}{n}\right) - y(x)\right|_{\left(\frac{i}{n} \in B\right)} B_n^i(x) < \frac{2M}{\delta^2}\left(\frac{x(1-x)}{n}\right) < \frac{M}{2\delta^2 n}.$$

For a positive real number $\epsilon > 0$, there exists a natural number N such that for all $n \geq N$, $\frac{M}{2\delta^2 n} < \frac{\epsilon}{2}$. Therefore, for all $x \in [0, 1]$, we have

$$|B_n(y, x) - y(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \tag{4.21}$$

Thus the Bernstein polynomial $B_n(y, x)$ converges uniformly to $y(x)$. □

5. NUMERICAL TESTING AND DISCUSSION

In this section, the accuracy and efficiency of the proposed method is tested against exact and existing numerical methods for six numerical examples.

Example 5.1.

$$-y'' - \frac{2}{x}y' = 1 - 2y^3, \quad 0 < x < 1, \tag{5.1}$$

subject to

$$y'(0) = 0, \quad y(1) = \frac{1}{3}y\left(\frac{1}{4}\right). \tag{5.2}$$

Applying the methodology developed in Section 3, we have the following matrix form of the differential equation (5.1)

$$\begin{aligned}&-(X(x))^T \left(C^T\right)^2 D^T A - \left(\frac{2}{x}\right)(X(x))^T C^T D^T A\\ &+ 2(A^T B(x))^2 (X(x))^T D^T A = 1.\end{aligned} \tag{5.3}$$

Also, the matrix representation of boundary conditions (5.2) is given by

$$(X(0))^T C^T D^T A = 0, \tag{5.4}$$

$$(X(1))^T \left(D^T\right)^{-1} C - \frac{1}{3}\left(X\left(\frac{1}{4}\right)\right)^T D^T A = 0. \tag{5.5}$$

Now, using collocation points (3.2), the matrix form (5.3) of the differential equation (5.1) has a new matrix representation which is a system of $(n+1)$ algebraic equations given by

$$-\bar{X}\left(C^T\right)^2 D^T A - H\bar{X}C^T D^T A + 2{B_1}^2\bar{X}D^T A - g = 0. \tag{5.6}$$

Here, the matrices C, D and A are defined in Section 2 and $\bar{X}$ and B_1 are defined in Section 3. The matrices H and g are given by

$$H = \begin{bmatrix} \frac{2}{x_0} & 0 & 0 & \cdots & 0 \\ 0 & \frac{2}{x_1} & 0 & \cdots & 0 \\ 0 & 0 & \frac{2}{x_2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & \cdots & \frac{2}{x_n} \end{bmatrix}, \quad g = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}.$$

In order to get the solution, we replace two equations of (5.6) by (5.4) and (5.5) among $(n+1)$ algebraic equations. Now, on solving these equations for unknown coefficients and substituting them to the equation (2.2), the required numerical solution is obtained. The approximated results obtained using the proposed Bernstein Collocation Method (BCM(n)) are compared against the modified variational iteration (MVIM) [37] and He's variational iteration method (He's VIM) [15] due to the nonavailability of the exact solution. Numerical results are also listed and presented in comparison with MVIM and He's VIM in Table 1 and Figure 2 for $n = 5$. The approximate result using MVIM is presented for the parameter $\omega = 0$ at which better result have shown in [37]. Also, the values of approximate solution using He's VIM is given at first iteration. The methods are also compared in terms of the residual error $Rw = |-y^{''} - \frac{2}{x}y^{'} - 1 + 2y^3|$, which are provided in Table 1. The values of Bernstein coefficients have presented in Table 2 for $n = 5$.

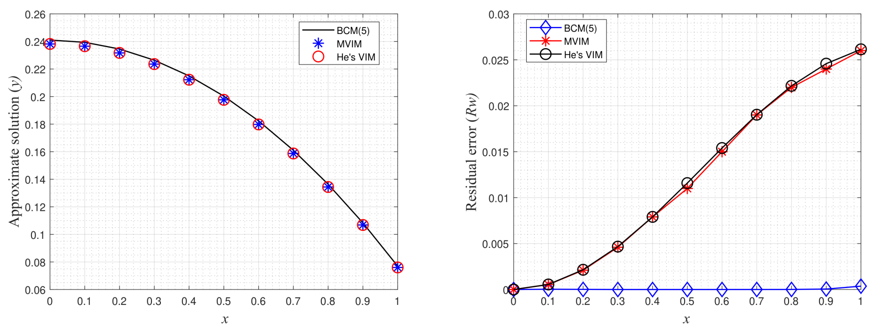
From Figure 2, one can easily conclude that the proposed approach is very promising (in terms of accuracy and efficiency) for solving three point SBVPs as it computes the results almost equally good as the MVIM and He's VIM (see Figures 2(a) and 2(b)). Moreover, in terms of residual errors, the proposed method shows better precision than the existing method and the similar trends to the previous case are obtained for errors. From the above results and discussion, it can easily be concluded that the new approach not only approximate the three point SBVPs with higher precision, but also consumes lesser computations to obtained these results.

Table 1
Comparison of BCM(n) with VIM at $n = 5$ of Example 5.1

x	BCM(5)	Rw	MVIM	Rw	He's VIM	Rw
0.0	0.241005	0.0	0.238177	4.5×10^{-17}	0.238177	2.8×10^{-08}
0.1	0.239384	3.0×10^{-05}	0.236555	5.4×10^{-04}	0.236555	5.4×10^{-04}
0.2	0.234519	1.3×10^{-05}	0.231690	2.1×10^{-03}	0.231690	2.1×10^{-03}
0.3	0.226400	1.2×10^{-06}	0.223582	4.6×10^{-03}	0.223582	4.6×10^{-03}
0.4	0.215013	1.9×10^{-07}	0.212231	7.9×10^{-03}	0.212231	7.9×10^{-03}
0.5	0.200338	1.3×10^{-14}	0.197636	1.1×10^{-02}	0.197636	1.1×10^{-02}
0.6	0.182354	1.3×10^{-06}	0.179798	1.5×10^{-02}	0.179798	1.5×10^{-02}
0.7	0.161041	1.9×10^{-06}	0.158717	1.9×10^{-02}	0.158717	1.9×10^{-02}
0.8	0.136378	5.9×10^{-06}	0.134392	2.2×10^{-02}	0.134392	2.2×10^{-02}
0.9	0.108352	4.9×10^{-05}	0.106825	2.4×10^{-02}	0.106825	2.4×10^{-02}
1.0	0.076956	3.6×10^{-04}	0.076013	2.6×10^{-02}	0.076014	2.6×10^{-02}

Table 2
Bernstein coefficients at $n = 5$ for Example 5.1

	a_0	a_1	a_2	a_3	a_4	a_5
$n = 5$	0.241005	0.241005	0.224790	0.192433	0.143121	0.0769556



(a) Numerical vs VIM (b) Residual error (Rw)

Fig. 2. Comparison of results for Example 5.1

Example 5.2.

$$-y'' - \frac{2}{x}y' = \frac{3}{4}e^{y}, \quad 0 < x < 1, \tag{5.7}$$

subject to

$$y'(0) = 0, \quad y(1) = \frac{2}{5}y\left(\frac{1}{2}\right) \tag{5.8}$$

In this problem $f(x,y) = \dfrac{3}{4}e^{y}$. Expanding the function $f(x,y)$ in Taylor series, we have

$$\frac{3}{4}e^{y} = \frac{3}{4}\left(1 + y + \frac{y^2}{2} + \frac{y^3}{6} + \frac{y^4}{24} + \frac{y^5}{120} + \dots\right).$$

Using the methodology developed in Section 3, we have the following matrix form of the differential equation (5.7):

$$\begin{aligned}
&-(X(x))^T \left(C^T\right)^2 D^T A - \left(\frac{2}{x}\right)(X(x))^T C^T D^T A - \frac{3}{4}(X(x))^T D^T A \\
&-\frac{3}{8}(A^T B(x))(X(x))^T D^T A - \frac{1}{8}(A^T B(x))^2 (X(x))^T D^T A \\
&-\frac{1}{32}(A^T B(x))^3 (X(x))^T D^T A - \frac{1}{160}(A^T B(x))^4 (X(x))^T D^T A = \frac{3}{4}.
\end{aligned} \tag{5.9}$$

Also, the matrix representation of boundary conditions (5.8) is given by

$$(X(0))^T C^T D^T A = 0, \tag{5.10}$$

$$(X(1))^T D^T A - \frac{2}{5}\left(X\left(\frac{1}{2}\right)\right)^T D^T A = 0. \tag{5.11}$$

Now, using collocation points (3.2), the matrix form (5.9) of the differential equation (5.7) has a new matrix representation which is a system of $(n+1)$ algebraic equations given by

$$\begin{aligned}
&-\bar{X}\left(C^T\right)^2 D^T A - H\bar{X}C^T D^T A - \frac{3}{4}\bar{X}D^T A - \frac{3}{8}B_1\bar{X}D^T A - \frac{1}{8}{B_1}^2\bar{X}D^T A \\
&-\frac{1}{32}{B_1}^3\bar{X}D^T A - \frac{1}{160}{B_1}^4\bar{X}D^T A - g = 0.
\end{aligned}$$

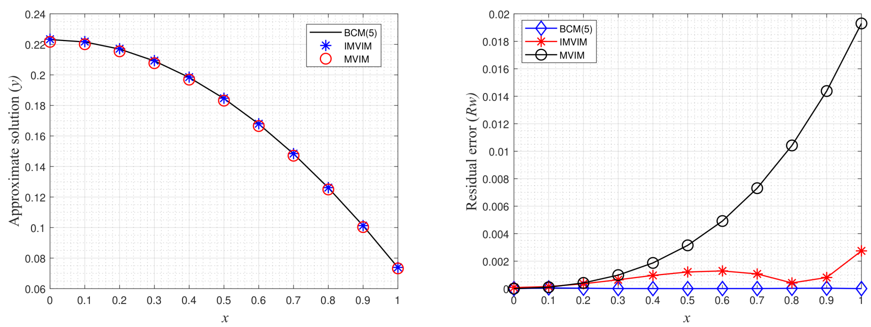
Here, the matrices C, D and A are defined in Section 2 and $\bar{X}$ and B_1 are defined in Section 3. The matrices H and g are given by

$$H = \begin{bmatrix} \frac{2}{x_0} & 0 & 0 & \cdots & 0 \\ 0 & \frac{2}{x_1} & 0 & \cdots & 0 \\ 0 & 0 & \frac{2}{x_2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & \cdots & \frac{2}{x_n} \end{bmatrix}, \quad g = \begin{bmatrix} \frac{3}{4} \\ \frac{3}{4} \\ \frac{3}{4} \\ \vdots \\ \frac{3}{4} \end{bmatrix}.$$

In order to get the solution, replacing two equations of (5) by (5.10) and (5.11) among $(n+1)$ algebraic equations. Now, on solving these equations for unknown coefficients $a_i, i = 0, 1, 2, \ldots, n$ and substituting them to the equation (2.2), the required numerical solution is obtained. The approximated results are presented against MVIM [37] and the improved modified variational iteration method (IMVIM) [46] due to non-availability of the exact solution. Numerical results are listed and presented in comparison with MVIM and IMVIM in Table 3 and Figure 3 for $n = 5$. To verify the accuracy and efficiency the residual error

$$Rw = | - y^{''} - \frac{2}{x} y^{'} - \frac{3}{4} e^{y} |$$

is also presented in Table 3. The values of Bernstein coefficient have presented in Table 4 for $n = 5$. From Figure 3 and Tables 3 and 4, one can easily concludes that the proposed approach is very promising for solving a three point SBVPs as it compute the results almost equally good as the IMVIM.



(a) Numerical vs VIM (b) Residual error (Rw)

Fig. 3. Comparison of results for Example 5.2

Table 3
Comparison of BCM(n) with IMVIM and MVIM at $n = 5$ of Example 5.2

x	BCM(5)	Rw	IMVIM	Rw	MVIM	Rw
0.0	0.223144	0.0	0.223377	7.8×10^{-05}	0.221653	0.000000
0.1	0.221584	5.1×10^{-05}	0.221814	1.5×10^{-04}	0.220093	1.0×10^{-04}
0.2	0.216910	3.0×10^{-05}	0.217135	3.5×10^{-04}	0.215425	4.1×10^{-04}
0.3	0.209146	5.0×10^{-06}	0.209366	6.4×10^{-04}	0.207674	9.8×10^{-04}
0.4	0.198336	3.9×10^{-06}	0.198547	9.6×10^{-04}	0.196889	1.8×10^{-03}
0.5	0.184540	3.8×10^{-14}	0.184736	1.2×10^{-03}	0.183133	3.1×10^{-03}
0.6	0.167831	3.5×10^{-06}	0.168007	1.2×10^{-03}	0.166489	4.9×10^{-03}
0.7	0.148298	4.0×10^{-06}	0.148447	1.0×10^{-03}	0.147056	7.3×10^{-03}
0.8	0.126042	2.1×10^{-05}	0.126161	4.1×10^{-04}	0.124949	1.0×10^{-02}
0.9	0.101174	3.2×10^{-05}	0.101267	8.1×10^{-04}	0.100299	1.4×10^{-02}
1.0	0.073816	1.0×10^{-14}	0.073894	2.7×10^{-03}	0.073253	1.9×10^{-02}

Table 4
Bernstein coefficients at $n = 5$ for Example 5.2

	a_0	a_1	a_2	a_3	a_4	a_5
$n = 5$	0.223143	0.223143	0.207537	0.176249	0.130935	0.0738151

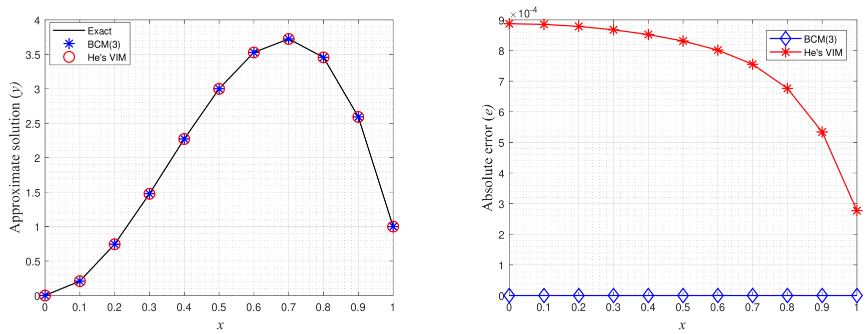
Example 5.3.

$$-(xy')' = x(-92 + 198x - 23x^2 + 22x^3 + y) \tag{5.12}$$

subject to

$$y^{'}(0) = 0, \quad y(1) = \frac{1}{3}y\left(\frac{1}{2}\right) \tag{5.13}$$

The exact solution of SBVPs (5.12)–(5.13) is $-22x^3 + 23x^2$. The qualitative and quantitative comparison of the approximate solution of the proposed method BCM(n) at $n = 3$ against the exact solution (Exact) and He's VIM [15] is provided in Table 5 and Figure 4(a), respectively. Table 5 and Figure 4(b) provides the absolute error $e = |\text{exact solution} - \text{approximate solution}|$. The values of unknown Bernstein coefficient have presented in Table 6. We have provided the numerical result of He's VIM at third iteration. From Figures and Table, it can be observed that the proposed method is computing highly accurate results and matching well with the exact results.



(a) Numerical vs Exact (b) Absolute error (e)

Fig. 4. Comparison of results for Example 5.3

Table 5

Comparison of BCM(n) with exact solution and He's VIM at $n = 3$ of Example 5.3

x	Exact	BCM(3)	e	He's VIM	e
0.0	0.000000	-1.5×10^{-16}	6.8×10^{-09}	-0.000887	0.000887
0.1	0.208000	0.208000	1.3×10^{-16}	0.207114	0.000885
0.2	0.744000	0.744000	1.1×10^{-16}	0.743121	0.000878
0.3	1.476000	1.476000	2.2×10^{-16}	1.475132	0.000867
0.4	2.272000	2.272000	0.0	2.271147	0.000852
0.5	3.000000	3.000000	0.0	2.999169	0.000830
0.6	3.528000	3.528000	0.0	3.527198	0.000801
0.7	3.724000	3.724000	0.0	3.723244	0.000755
0.8	3.456000	3.456000	0.0	3.455323	0.000676
0.9	2.592000	2.592000	0.0	2.591465	0.000534
1.0	1.000000	1.000000	1.1×10^{-16}	0.999722	0.000277

Table 6

Bernstein coefficients at $n = 3$ for Example 5.3

	a_0	a_1	a_2	a_3
$n = 3$	-1.5×10^{-16}	-1.5×10^{-16}	7.666667	1.0

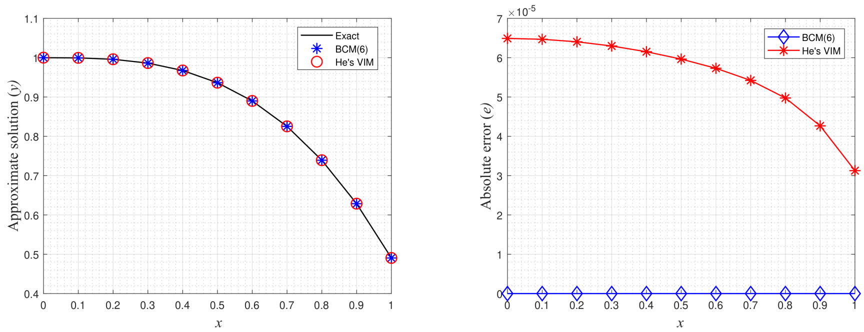
Example 5.4.

$$-\left(x^2y^{'}\right)^{'} = x^2\left(-1+\frac{324}{53}x+\frac{54}{53}x^3-\frac{729}{2809}x^6+y^2\right), \quad 0<x<1, \tag{5.14}$$

subject to

$$y'(0)=0, \quad y(1)=\frac{1}{2}y\left(\frac{1}{3}\right) \tag{5.15}$$

The exact solution of SBVPs (5.14)–(5.15) is $-\frac{27}{53}x^3+1$. For the problem in Example 5.4, the approximate solution using present method BCM(n) is compared with exact solution and He's VIM [15] graphically in Figure 5 along with the quantitative values of the solutions at different values of x in Table 7. The absolute error e are also calculated and provided in Table 7 and Figure 5. The values of unknown Chebyshev coefficients are listed in Table 8 for $n=6$. The numerical results of He's VIM at third iteration is presented in Table 6 and Figure 5. Once again the plots show that the proposed approach estimate the results well and overlap with the exact solutions (refer to Figures 5(a) and 5(b)).



(a) Numerical vs Exact (b) Absolute error (e)

Fig. 5. Comparison of results for Example 5.4

Example 5.5.

$$-(x^{0.7}y')' = x^{0.7}\left(\frac{567}{85}x-\frac{17}{5}\right), \tag{5.16}$$

subject to

$$y(0)=0, \quad y(1)=1.2y\left(\frac{1}{2}\right). \tag{5.17}$$

The exact solution of the boundary value problem (5.16)–(5.17) is

$$y(x) = -\frac{14}{17}x^3+x^2. \tag{5.18}$$

Table 7

Comparison of BCM(n) with exact solution and He's VIM at $n = 6$ of Example 5.4

x	Exact	Cheb(6)	e	He's VIM	e
0.0	1.000000	1.000000	0.0	1.000065	6.4×10^{-05}
0.1	0.999491	0.999491	0.0	0.999555	6.4×10^{-05}
0.2	0.995925	0.995925	0.0	0.995989	6.4×10^{-05}
0.3	0.986245	0.986245	0.0	0.986308	6.2×10^{-05}
0.4	0.967396	0.967396	0.0	0.967458	6.1×10^{-05}
0.5	0.936321	0.936321	1.1×10^{-16}	0.936380	5.9×10^{-05}
0.6	0.889962	0.889962	2.2×10^{-16}	0.890020	5.7×10^{-05}
0.7	0.825264	0.825264	2.2×10^{-16}	0.825318	5.4×10^{-05}
0.8	0.739170	0.739170	4.4×10^{-16}	0.739220	4.9×10^{-05}
0.9	0.628623	0.628623	6.6×10^{-16}	0.628665	4.2×10^{-05}
1.0	0.490566	0.490566	8.8×10^{-16}	0.490597	3.1×10^{-05}

Table 8

Bernstein coefficients at $n = 6$ for Example 5.4

	a_0	a_1	a_2	a_3	a_4	a_5	a_6
$n = 6$	1.0	1.0	1.0	0.974528	0.898113	0.745283	0.490566

Applying the methodology developed in Section 3, the following matrix form of the differential equation (5.12) is obtained

$$-\left(X(x)\right)^T \left(C^T\right)^2 D^T A - \left(\frac{7}{10x}\right)\left(X(x)\right)^T C^T D^T A = \left(\frac{567}{85}x - \frac{17}{5}\right), \tag{5.19}$$

and the matrix representation of boundary conditions (5.17) is given by

$$\left(X(0)\right)^T D^T A = 0, \tag{5.20}$$

$$\left(X(1)\right)^T D^T A - 1.2\left(X\left(\frac{1}{2}\right)\right)^T D^T A = 0. \tag{5.21}$$

Now, using collocation points (3.2), the matrix form (5.19) of the differential equation (5.16) has a new matrix representation which is a system of $(n+1)$ algebraic equation given by

$$-\bar{X}\left(C^T\right)^2 D^T A - H\bar{X}C^T D^T A - g = 0. \tag{5.22}$$

Here, the matrices H and g are given by

$$H = \begin{bmatrix} \frac{7}{10x_0} & 0 & 0 & \cdots & 0 \\ 0 & \frac{7}{10x_1} & 0 & \cdots & 0 \\ 0 & 0 & \frac{7}{10x_2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & \cdots & \frac{7}{10x_n} \end{bmatrix}, \quad g = \begin{bmatrix} \left(\frac{567}{85}x_0 - \frac{17}{5}\right) \\ \left(\frac{567}{85}x_1 - \frac{17}{5}\right) \\ \left(\frac{567}{85}x_2 - \frac{17}{5}\right) \\ \vdots \\ \left(\frac{567}{85}x_n - \frac{17}{5}\right) \end{bmatrix}.$$

In order to get the solution of problem 5.5, following from the proposed methodology replace the boundary conditions given in equations (5.20) and (5.21) into two out of $(n+1)$ algebraic equations (5.22). Now, on solving these equations for unknown coefficient a_i, $i = 0, 1, 2, \ldots, n$ at $n = 3$, the approximate solution given as follows is obtained:

$$\begin{aligned} y(x) = A^T B(x) &= -0.8235294115x^3 + 0.9999999997x^2 - 6.8 \times 10^{-11}x \\ &\approx -\frac{14}{17}x^3 + x^2. \end{aligned} \tag{5.23}$$

Here,

$$A = [0.0 \quad 2.3 \times 10^{-09} \quad 0.333333 \quad 0.176471]^T.$$

From equation (5.23), it can be seen that the numerical solution is approximately equal to the exact solution (5.18). Hence, the proposed method has the ability to find the solution with higher precision at a less computational cost.

Example 5.6.

$$-(x^{0.5}y')' = x^{0.5}\left(-3 + \frac{45}{7}x - x^4 + \frac{12}{7}x^5 - \frac{36}{49}x^6 + y^2\right) \tag{5.24}$$

subject to

$$y(0) = 0, \quad y(1) = y\left(\frac{1}{2}\right). \tag{5.25}$$

The exact solution of boundary value problem (5.24)–(5.25) is $y(x) = -\frac{6}{7}x^3 + x^2$.

We have presented the approximate solution BCM(n) in comparison with the exact solution (Exact) for $n = 6$ quantitatively and qualitatively in Table 9 and Figure 6, respectively. To ensure the accuracy and reliability of the method, the absolute error $e = |\text{Exact} - \text{BCM}(n)|$ have also presented in Table 9 and Figure 6. The values of Bernstein coefficients have presented in Table 10 for $n = 6$.

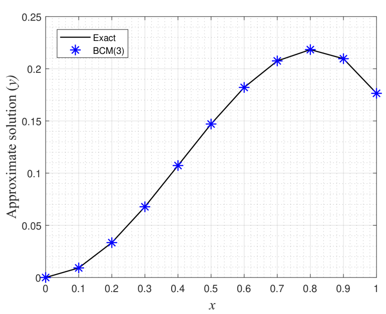
One can observe that the numerical solution matches well with the exact solution for $n = 6$ (refer to Figure 6(a)). This shows the tendency of the new approach to approximate all numerical results very efficiently. Moreover, Table 9 demonstrates that as the value of N increased from 5 to 6, the value of the errors (e) decreased significantly, leads to converge the numerical solution to the exact solution with less computations.

Table 9
Comparison of BCM(n) with exact solution at $N = 5$ and $N = 6$ of Example 5.6

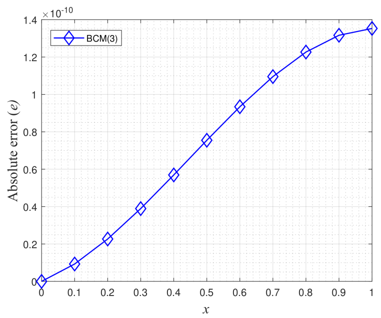
x	Exact	BCM(5)	e	BCM(6)	e
0.0	0.000000	0.000000	0.0	0.0	0.0
0.1	0.009143	0.009142	2.3×10^{-09}	0.009143	3.6×10^{-11}
0.2	0.033143	0.033142	3.9×10^{-09}	0.033143	3.8×10^{-11}
0.3	0.066857	0.066857	5.5×10^{-09}	0.066857	2.8×10^{-11}
0.4	0.105143	0.105142	6.7×10^{-09}	0.105143	1.8×10^{-11}
0.5	0.142857	0.142857	7.7×10^{-09}	0.142857	9.6×10^{-12}
0.6	0.174857	0.174857	8.3×10^{-09}	0.174857	4.1×10^{-12}
0.7	0.196000	0.195999	8.6×10^{-09}	0.196000	3.1×10^{-12}
0.8	0.201143	0.201142	8.6×10^{-09}	0.201142	9.2×10^{-12}
0.9	0.185143	0.185142	8.3×10^{-09}	0.185143	2.6×10^{-11}
1.0	0.142857	0.142857	7.7×10^{-09}	0.142857	5.7×10^{-11}

Table 10
Bernstein coefficients at different values of N for Example 5.6

	a_0	a_1	a_2	a_3	a_4	a_5	a_6
$N = 5$	0.0	-4.9×10^{-09}	0.1	0.214286	0.257143	0.142857	–
$N = 6$	0.0	-1.1×10^{-09}	0.066667	0.157143	0.228571	0.238095	0.142857



(a) Numerical vs Exact



(b) Absolute error (e)

Fig. 6. Comparison of results for Example 5.6

6. CONCLUSIONS AND REMARKS

This work introduced a novel numerical technique using the notion of the Bernstein polynomials and a collocation method for finding the approximate solution of a class of three-point SBVPs. The operational matrix of differentiation and collocation approach discretize the differential equation on $[0, 1]$. The errors (absolute errors and residual errors) listed in the tables for numerical results exhibit excellent accuracy for the proposed method over He's VIM, MVIM and IMVIM [15, 37, 46]. The computer program of the algorithm is simple and modification is easy in terms of implementation over various numerical examples, which makes this scheme cost effective. Thus one can adopt the BCM(n) over variational iteration methods (VIM), because VIM involves computation of unnecessary terms, which consumes time and effort. The convergence analysis of the Bernstein polynomials and its error estimate over the problem of consideration has also been discussed.

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