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GENERAL DECAY RATE OF A WEAKLY DISSIPATIVE VISCOELASTIC EQUATION WITH A GENERAL DAMPING

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Abstract. In this paper, we consider a weakly dissipative viscoelastic equation with a nonlinear damping. A general decay rate is proved for a wide class of relaxation functions. To support our theoretical findings, some numerical results are provided.

Keywords: general decay, relaxation function, viscoelastic, weakly dissipative equation.

Mathematics Subject Classification: 34G10, 35B40, 35L90, 45K05.

1. INTRODUCTION

The modeling of a generalized Kirchhoff viscoelastic plate, where a bending moment relation with memory is considered, can be described by the following nonlinear weakly dissipative viscoelastic equation; for a given non-negative relaxation function g,

$$\begin{cases} u'' + \Delta^2 u + \int\limits_0^t g(t-s)\Delta u(s)ds + h(u') = 0 & \text{in } \Omega \times (0, +\infty), \\ u = \Delta u = 0, & \text{on } \partial\Omega, \\ u(x,0) = u_0(x), \quad u'(x,0) = u_1(x), & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with a smooth (or piecewise smooth) boundary $\partial\Omega$, h is a function satisfying some conditions, see (2.4) below, and the initial data u_0, u_1 are given.

The main focus of this work is on investigating the decay of the energy of the above viscoelastic problem. The presence of the weakly viscoelastic dissapative term enforces us to deal with what so-called a modified (or second) energy to achieve our goals. Indeed this will make the analysis more technical and also delicate. To confirm our theoretical finding numerically, we provide graphical illustrations of the decay of the energy, where the solution of problem (1.1) is approximated via finite differences in time and finite elements in space.

Over the last five decades, the asymptotic or decaying behaviour of various types of viscoelastic equations were the subject of study of many researchers since the pioneer work of Dafermos [5,6]. We focus on those related to our problem (1.1). In the absence of damping term but infinite memory, Revira *et al.* [19] studied the energy decay of the following problem:

$$u''(t) + Au(t) - \int_0^\infty g(\tau)A^\alpha u(t-\tau)d\tau = 0, \quad \text{for} \quad t > 0,$$

for $\alpha \in (0,1)$, where A is a positive definite self-adjoint operator on a Hilbert space H. For $t \geq 0$, it is assumed that $g'(t) \leq -\delta\, g(t)$ for some positive constant δ.

For the case of finite memory and a more general relaxation function g (see hypothesis (A1) for more details), Hassan and Messaoudi [9] investigated recently the energy decay rate. In the absent of memory term, but with a nonlinear source term, Messaoudi [14] discussed the energy decay of the model problem:

$$u''(t) + \Delta^2 u(t) + |u'(t)|^{m-2}u'(t) = |u(t)|^{p-2}u(t), \quad \text{for} \quad t \in (0,T), \quad \text{with} \quad m,p > 1.$$

Another recent related work by Al-Gharabli, Guesmia and Messaoudi [1] where the energy decay rate of a viscoelastic plate equation with a logarithmic nonlinearity of the form

$$u''(t) + \Delta^2 u(t) + u(t) - \int_0^t g(t-\tau)\Delta^2 u(\tau)d\tau = ku(t)\ln|u(t)|, \quad \text{for} \quad t > 0,$$

was investigated, assuming that $g'(t) \leq -\xi(t)g^p(t)$ for $1 \leq p < \frac{3}{2}$. For more studies on the energy decay analysis of various types of viscoelastic equations, we refer the readers to [3, 4, 7, 8, 11–13, 20] and references therein.

The outline of the paper is the following. In the next section, we introduce some necessary notations and assumptions, and state and prove a few technical lemmas that will be used in the forthcoming decaying analysis. Section 3 is dedicated to show the decay rates of the energy functional E (see (3.1)). Having a weakly dissepative in problem (1.1) leads us to introduce a second energy functional $\mathcal{E}$ (see (3.3) below) to overcome the difficulties in proving the decay of E. For sake of illustrating the theoretical decaying rate of E numerically, we develop a fully-discrete numerical method in Section 4. Owing to the presence of the biharmonic operator in problem (1.1), we use the $\mathcal{C}^2$ Galerkin finite element method for the spatial discretization. In the time variable and to avoid solving any nonlinear systems, our scheme is based on an appropriate combination between backward-forward Euler and second central differences. We show the decay of both, the numerical solution of problem (1.1) and the approximation of E.

2. PRELIMINARIES

Let $H^\ell(\Omega)$ $(\ell \geq 0)$ be the standard Sobolev space. For $\ell = 0$, $H^\ell(\Omega)$ reduces to the standard Lesbesgue space $L^2(\Omega)$. On this space, $\langle \cdot, \cdot \rangle$ denotes the usual inner product and $\| \cdot \|$ is the associated $L^2(\Omega)$-norm. For $p \neq 2$, the norm on $L^p(\Omega)$ is denoted by $\| \cdot \|_p$. We introduce the Sobolev space

$$\mathcal{H}(\Omega) = \{u \in H^3(\Omega) : u = \Delta u = 0 \text{ on } \partial\Omega\}.$$

An application of the Poincaré inequality and by using the elliptic regularity property, there exist two positive constants ω_0 and ω_1 such that

$$\|\nabla w\|^2 \leq \omega_0 \, \|\Delta w\|^2 \quad \text{and} \quad \|\Delta w\|^2 \leq \omega_1 \, \|\nabla(\Delta w)\|^2, \ \forall w \in \mathcal{H}(\Omega). \tag{2.1}$$

Throughout this paper, c is a generic positive constant. In the decay energy analysis, the following hypothesis is imposed.

(A1) The relaxation function $g \in \mathcal{C}^1(\mathbb{R}^+)$ is assumed to be non-increasing,

$$g(0) > 0, \quad 1 - \max\{\omega_0, \omega_1\} \int_0^\infty g(s)ds =: l > 0, \tag{2.2}$$

and there exists a $\mathcal{C}^1$ function $G : (0, \infty) \to (0, \infty)$ which is strictly increasing, and strictly convex $\mathcal{C}^2$ function on $(0, g(0)]$, with $G(0) = G'(0) = 0$, such that

$$g'(t) \leq -\xi(t)G(g(t)), \quad \forall\, t \geq 0, \tag{2.3}$$

where ξ is a positive non-increasing $\mathcal{C}^1$ function. Moreover, the function $h \in \mathcal{C}^1(\mathbb{R})$ is assumed to be non-decreasing,

$$h(0) = 0, \quad sh(s) \geq 0 \quad \text{and} \quad \alpha_1|s| \leq |h(s)| \leq \alpha_2|s|, \ \forall s \in \mathbb{R}, \tag{2.4}$$

for some positive contants α_1 and α_2.

Remark 2.1. As in [16], we present the following:

(1) From assumption (A1), we deduce that $\lim_{t\to\infty} g(t) = 0$, and there exists $t_0 > 0$ such that $g(t_0) = r$, while $g(t) \leq r$ for $t \geq t_0$. The non-increasing property of g implies that

$$0 < g(t_0) \leq g(t) \leq g(0), \quad \forall\, t \in [0, t_0].$$

Continuity of G on $[0, r]$ yields $a \leq G(g(t)) \leq b$ on $[0, t_0]$, for some constants $a, b > 0$. Consequently, for any $t \in [0, t_0]$, we have

$$g'(t) \leq -\xi(t)G(g(t)) \leq -a\xi(t) = -\frac{a}{g(0)}\xi(t)g(0) \leq -\frac{a}{g(0)}\xi(t)g(t),$$

and hence

$$\xi(t)g(t) \leq -\frac{g(0)}{a}g'(t), \quad \forall\, t \in [0, t_0]. \tag{2.5}$$

(2) If G is a strictly increasing and strictly convex $\mathcal{C}^2$ function on $(0, r]$, with $G(0) = G'(0) = 0$, then there is a function $\bar{G} : [0, +\infty) \longrightarrow [0, +\infty)$ that extends G and its properties. For instance, we can define $\bar{G}$, for any $t > r$, by

$$\bar{G}(t) := \frac{G''(r)}{2}t^2 + \big(G'(r) - G''(r)r\big)t + \left(G(r) + \frac{G''(r)}{2}r^2 - G'(r)r\right).$$

For later use, by (2.1) and the second inequality in (2.2), we have

$$\|\Delta u(t)\|^2 - \int_0^t g(s)ds\|\nabla u(t)\|^2 \geq 0 \;\text{ and }\; \|\nabla(\Delta u(t))\|^2 - \int_0^t g(s)ds\|\Delta u(t)\|^2 \geq 0. \quad (2.6)$$

For convenience, we introduce the following notations: for $t > 0$,

$$(g \circ w)(t) := \int_0^t g(t-s)\|w(t) - w(s)\|^2 ds,$$

and for $0 < \varepsilon < 1$, we put

$$C_\varepsilon := \int_0^\infty \frac{g^2(s)}{h_\varepsilon(s)} ds \quad \text{with} \quad h_\varepsilon(t) := \varepsilon g(t) - g'(t).$$

The next three lemmas will be used in the forthcoming decay analysis section.

Lemma 2.2 ([10])**.** *Assume that* (A1) *holds true. Then for any* $v \in L^2_{loc}([0, +\infty); L^2(\Omega))$,

$$\int_\Omega \left(\int_0^t g(t-s)(v(t) - v(s))ds\right)^2 dx \leq C_\varepsilon(h_\varepsilon \circ v)(t), \quad \text{for} \quad t \geq 0. \quad (2.7)$$

Lemma 2.3 (Jensen's inequality)**.** *Let* $F : [a, b] \longrightarrow \mathbb{R}$ *be a convex function. Assume that the functions* $f : \Omega \longrightarrow [a, b]$ *and* $h : \Omega \longrightarrow \mathbb{R}$ *are integrable such that* $h(x) \geq 0$, *for any* $x \in \Omega$ *and* $\int_\Omega h(x)dx = k > 0$. *Then,*

$$F\left(\frac{1}{k}\int_\Omega f(x)h(x)dx\right) \leq \frac{1}{k}\int_\Omega F(f(x))h(x)dx.$$

Lemma 2.4 ([21])**.** *Assume that* (A1) *holds true. Then for any* $w \in H^1\big([0, \infty); L^2(\Omega)\big)$,

$$\begin{aligned} &\int_0^t g(t-s)\langle w(s), w'(t)\rangle ds \\ &= \frac{1}{2}\frac{d}{dt}\left[\int_0^t g(s)ds\|w(t)\|^2 - (g \circ w)(t)\right] - \frac{1}{2}g(t)\|w(t)\|^2 + \frac{1}{2}(g' \circ w)(t). \end{aligned}$$

3. DECAY

In this section, we aim to find the best possible estimate of the energy functional of problem (1.1). First, taking the inner product of (1.1) with u' gives

$$\langle u'', u'\rangle + \langle \Delta^2 u, u'\rangle + \int_0^t g(t-s)\langle \Delta u(s), u'(t)\rangle ds + \langle h(u'), u'\rangle = 0.$$

Applying Green's formula (twice for the second term and once for the third term) and using the fact that $u' = \Delta u' = 0$ on $\partial\Omega$, yield the following weak formulation of (1.1):

$$\langle u'', u'\rangle + \langle \Delta u, \Delta u'\rangle - \int_0^t g(t-s)\langle \nabla u(s), \nabla u'(t)\rangle ds + \langle h(u'), u'\rangle = 0.$$

Using Lemma 2.4 with $w = \nabla u$, this equation can be rewritten as

$$\frac{1}{2}\frac{d}{dt}\left[\|u'(t)\|^2 + \|\Delta u(t)\|^2 - \int_0^t g(s)ds\|\nabla u(t)\|^2 + (g\circ\nabla u)(t)\right]$$
$$= \frac{1}{2}(g'\circ\nabla u)(t) - \frac{1}{2}g(t)\|\nabla u(t)\|^2 - \langle h(u'), u'\rangle.$$

Introducing the first energy functional

$$E(t) := \frac{1}{2}\left[\|u'(t)\|^2 + \|\Delta u(t)\|^2 - \int_0^t g(s)ds\|\nabla u(t)\|^2 + (g\circ\nabla u)(t)\right] \geq 0, \tag{3.1}$$

where, we used (2.6) in the last inequality. Hence,

$$E'(t) = \frac{1}{2}(g'\circ\nabla u)(t) - \frac{1}{2}g(t)\|\nabla u(t)\|^2 - \langle h(u'(t)), u'(t)\rangle \leq 0. \tag{3.2}$$

Turning now to the second energy functional of (1.1). Taking the inner product of problem (1.1) with $-\Delta u'$ and then, applying Green's formula to the first, second and fourth terms, we get

$$\langle \nabla u'', \nabla u'\rangle + \langle \nabla(\Delta u), \nabla(\Delta u')\rangle - \int_0^t g(t-s)\langle \Delta u(s), \Delta u'(t)\rangle ds + \langle \nabla h(u'), \nabla u'\rangle = 0.$$

Hence, by Lemma 2.4, the above equations can be rewritten as

$$\frac{1}{2}\frac{d}{dt}\left[\|\nabla u'(t)\|^2 + \|\nabla(\Delta u)(t)\|^2 - \int_0^t g(s)ds\|\Delta u(t)\|^2 + (g\circ\Delta u)(t)\right]$$
$$= -\frac{1}{2}g(t)\|\Delta u(t)\|^2 + \frac{1}{2}(g'\circ\Delta u)(t) - \langle \nabla h(u'), \nabla u'\rangle.$$

Therefore, using (2.6), the second energy functional of (1.1) is

$$\mathcal{E}(t) := \frac{1}{2}\left[\|\nabla u'(t)\|^2 + \|\nabla(\Delta u(t))\|^2 - \int_0^t g(s)ds\|\Delta u(t)\|^2 + (g\circ\Delta u)(t)\right] \geq 0. \tag{3.3}$$

Moreover, since $\langle \nabla h(u'), \nabla u'\rangle = \langle h'(u')\nabla u', \nabla u'\rangle \geq 0$,

$$\mathcal{E}'(t) = -\frac{1}{2}g(t)\|\Delta u(t)\|^2 + \frac{1}{2}(g'\circ\Delta u)(t) - \langle h'(u'(t))\nabla u'(t), \nabla u'(t)\rangle \leq 0. \tag{3.4}$$

In the next two lemmas, assuming that (A1) holds, we estimate the time derivative of the functionals

$$I_1(t) = \langle u(t), u'(t)\rangle \quad \text{and} \quad I_2(t) = -\int_0^t g(t-s)\langle u(t)-u(s), u'(t)\rangle ds.$$

Lemma 3.1. *Along the solution of* (1.1)*, and for* $\delta > 0$*, we have*

$$I_1'(t) \leq \|u'(t)\|^2 + \Big(\delta - \frac{l}{2}\Big)\|\Delta u(t)\|^2 + c\,C_\varepsilon(h_\varepsilon\circ\nabla u)(t) - cE'(t).$$

Proof. Differentiating I_1 and exploiting the differential equation in (1.1), we get

$$\begin{aligned} I_1'(t) &= \|u'(t)\|^2 - \langle u(t), \Delta^2 u(t)\rangle - \int_0^t g(t-s)\langle \Delta u(s), u(t)\rangle ds - \langle h(u'(t)), u(t)\rangle \\ &= \|u'(t)\|^2 - \|\Delta u(t)\|^2 + \int_0^t g(t-s)\langle \nabla u(s), \nabla u(t)\rangle ds - \langle h(u'(t)), u(t)\rangle. \end{aligned}$$

Young's inequality, Lemma 2.2, and the inequalities in (2.1) imply that the third term in the right-hand side equals

$$\begin{aligned} &\left\langle \nabla u(t), \int_0^t g(t-s)\nabla\big(u(s)-u(t)\big)ds\right\rangle + \int_0^t g(s)ds\|\nabla u(t)\|^2 \\ &\leq \frac{l}{2\omega_0}\|\nabla u(t)\|^2 + \frac{\omega_0}{2l}\int_\Omega\left(\int_0^t g(t-s)\nabla\big(u(s)-u(t)\big)ds\right)^2 dx + \int_0^t g(s)ds\|\nabla u(t)\|^2 \\ &\leq \frac{l}{2\omega_0}\|\nabla u(t)\|^2 + c\,C_\varepsilon(h_\varepsilon\circ\nabla u)(t) + \omega_0\int_0^t g(s)ds\|\Delta u(t)\|^2 \\ &\leq \frac{l}{2}\|\Delta u(t)\|^2 + c\,C_\varepsilon(h_\varepsilon\circ\nabla u)(t) + (1-l)\|\Delta u(t)\|^2 \\ &\leq \Big(1-\frac{l}{2}\Big)\|\Delta u(t)\|^2 + c\,C_\varepsilon(h_\varepsilon\circ\nabla u)(t). \end{aligned}$$

Also, the forth term is simplified using Cauchy–Schwarz, Young's inequalities and (2.4) as follows:

$$\begin{aligned}\langle h(u'(t)), u(t)\rangle &\le \|u(t)\|\|h(u'(t))\|\\ &\le c\|\Delta u(t)\|\|h(u'(t))\|\\ &\le \delta\|\Delta u(t)\|^2 + \frac{c}{\delta}\langle |h(u'(t))|^2, 1\rangle\\ &\le \delta\|\Delta u(t)\|^2 + \frac{c\alpha_2}{\delta}\langle h(u'(t)), u'(t)\rangle\\ &\le \delta\|\Delta u(t)\|^2 - cE'(t).\end{aligned}$$

Combining the above results completes the proof of the lemma. □

Lemma 3.2. *Along the solution of* (1.1)*, and for* $\delta > 0$*, we have*

$$I_2'(t) \le \delta\|\Delta u\|^2 - \left(\int_0^t g(s)ds - \delta\right)\|u'\|^2 + \frac{c}{\delta}(C_\varepsilon + 1)(h_\varepsilon \circ \Delta u)(t) - cE'(t).$$

Proof. Differentiating I_2 and exploiting the differential equation in (1.1), we get

$$I_2'(t) = I_{2,1}(t) + I_{2,2}(t) + I_{2,3}(t) + I_{2,4}(t), \tag{3.5}$$

where

$$\begin{aligned}I_{2,1}(t) &= \int_0^t g(t-s)\langle u(t)-u(s), \Delta^2 u(t)\rangle ds\\ I_{2,2}(t) &= \left\langle \int_0^t g(t-s)\Delta u(s)ds, \int_0^t g(t-s)\big(u(t)-u(s)\big)ds\right\rangle\\ I_{2,3}(t) &= \left\langle \int_0^t g(t-s)(u(t)-u(s))\,ds, h(u'(t))\right\rangle\\ I_{2,4}(t) &= -\left\langle u'(t), \int_0^t \Big[g'(t-s)\big(u(t)-u(s)\big) + g(t-s)u'(t)\Big]ds\right\rangle.\end{aligned}$$

The current task is to estimate these four terms. By Green's formula, Young's inequality and Lemma 2.2, we have

$$I_{2,1}(t) \le \|\Delta u(t)\|\int_0^t g(t-s)\|\Delta\big(u(t)-u(s)\big)\|ds \le \frac{\delta}{2}\|\Delta u(t)\|^2 + \frac{c}{\delta}C_\varepsilon(h_\varepsilon\circ\Delta u)(t),$$

and

$$\begin{aligned}I_{2,2}(t) &= \left\| \int_0^t g(t-s)\nabla\big(u(t)-u(s)\big)ds \right\|^2 \\ &\quad + \left\langle \int_0^t g(t-s)\nabla u(t)ds, \int_0^t g(t-s)\nabla\big(u(t)-u(s)\big)ds \right\rangle \\ &\le \left(\int_0^t g(t-s)\|\nabla u(t)-\nabla u(s)\|ds \right)^2 + c\|\nabla u(t)\| \int_0^t g(t-s)\|\nabla u(t)-\nabla u(s)\|ds \\ &\le C_\varepsilon(h_\varepsilon \circ \nabla u)(t) + \frac{\delta}{2\omega_0}\|\nabla u(t)\|^2 + \frac{c}{\delta}C_\varepsilon(h_\varepsilon \circ \nabla u)(t).\end{aligned}$$

For $I_{2,3}(t)$, Young's inequality gives

$$I_{2,3}(t) \le \delta \int_\Omega \left| \int_0^t g(t-s)\big(u(t)-u(s)\big)ds \right|^2 dx + \frac{c}{\delta}\langle |h(u'(t))|^2, 1\rangle.$$

Since

$$\begin{aligned}\left| \int_0^t g(t-s)\big(u(t)-u(s)\big)ds \right| &\le \int_0^t g^{\frac{1}{2}}(t-s)g^{\frac{1}{2}}(t-s)\big|u(t)-u(s)\big|ds \\ &\le \left(\int_0^t g(s)ds \right)^{\frac{1}{2}} \left(\int_0^t g(t-s)|(u(t)-u(s)|^2 ds \right)^{\frac{1}{2}},\end{aligned}$$

application of Hölder's inequality and Lemma 2.2 yield

$$\begin{aligned}\int_\Omega \left| \int_0^t g(t-s)\left(u(t)-u(s)\right)ds \right|^2 dx &\le \left(\int_0^t g(s)ds \right)^2 \int_0^t g(t-s)\|u(t)-u(s)\|^2\, ds \\ &\le c(g \circ \nabla u)(t).\end{aligned}$$

By merging the above three inequalities and with the help of (2.4), we obtain

$$\begin{aligned}I_{2,3}(t) &\le c\,\delta(g \circ \nabla u)(t) + \frac{c}{\delta}\langle |h(u'(t))|^2, 1\rangle \\ &\le c\,\delta(g \circ \nabla u)(t) + \frac{c\alpha_2}{\delta}\langle h(u'(t)), u'(t)\rangle \\ &\le c\,\delta(g \circ \nabla u)(t) - cE'(t).\end{aligned}$$

To estimate the last term $I_{2,4}(t)$, we use again Young's inequality, the fact that $|g'| = |\varepsilon g - h_\varepsilon| \le \varepsilon g + h_\varepsilon$, and Lemma 2.2. So, we have

$$\begin{aligned}
I_{2,4}(t) \le \delta\|u'(t)\|^2 + \frac{c}{\delta}\left(\int_0^t (\varepsilon g(t-s) + h_\varepsilon(t-s))\|u(t)-u(s)\|ds\right)^2 \\
- \|u'(t)\|^2 \int_0^t g(s)\,ds \\
\le \left(\delta - \int_0^t g(s)ds\right)\|u'(t)\|^2 + \frac{c}{\delta}\left(\varepsilon^2(g\circ u)(t) + (h_\varepsilon \circ u)(t)\right) \\
\le \left(\delta - \int_0^t g(s)ds\right)\|u'(t)\|^2 + \frac{c}{\delta}C_\varepsilon(h_\varepsilon \circ u)(t).
\end{aligned}$$

Inserting the obtained estimates of $I_{2,1}$, $I_{2,2}$, $I_{2,3}$, and $I_{2,4}$ in (3.5) will complete the proof. □

The achieved convolution estimates in the next lemma will also be needed in our forthcoming analysis. For convenience, we introduce the following notations. With $f(t) := \int_t^\infty g(s)ds$, let

$$J_1(t) := \int_0^t f(t-s)\|\nabla u(s)\|^2 ds \quad \text{and} \quad J_2(t) := \int_0^t f(t-s)\|\Delta u(s)\|^2 ds.$$

Lemma 3.3 ([9])**.** *Assume that* (A1) *holds, then for* $t \ge 0$,

$$J_1'(t) \le \frac{3}{\omega_0}(1-l)\|\nabla u\|^2 - \frac{1}{2}(g\circ \nabla u)(t) \quad \text{and} \quad J_2'(t) \le \frac{3}{\omega_0}(1-l)\|\Delta u\|^2 - \frac{1}{2}(g\circ \Delta u)(t).$$

Lemma 3.4. *For* ε_1, $\varepsilon_2 > 0$, *the functional* $\mathcal{L}(t) := N(E(t) + \mathcal{E}(t)) + \varepsilon_1 I_1(t) + \varepsilon_2 I_2(t)$ *satisfies*

$$\mathcal{L} \sim E + \mathcal{E} \quad \text{for a sufficiently large } N. \tag{3.6}$$

Moreover, for any $t \ge t_0$, *with* t_0 *being introduced in Remark 2.1, we have*

$$\begin{aligned}
\mathcal{L}'(t) \le -(1-l)\Big(4 + \frac{3}{2\omega_0}\Big)\Big(\frac{2}{l}\|u'(t)\|^2 + \|\Delta u(t)\|^2\Big) \\
+ \frac{1}{4}\Big((g\circ\nabla u)(t) + (g\circ \Delta u)(t)\Big) - cE'(t).
\end{aligned} \tag{3.7}$$

Proof. The proof of (3.6) is done in [15]. To show (3.7), differentiating and get

$$\mathcal{L}'(t) = N(E'(t) + \mathcal{E}'(t)) + \varepsilon_1 I_1'(t) + \varepsilon_2 I_2'(t).$$

Using (3.2), (3.4), and Lemmas 3.1 and 3.2, yield

$$\begin{aligned}\mathcal{L}'(t) \le N\Bigg[&\frac{1}{2}(g' \circ \nabla u)(t) - \frac{1}{2}g(t)\|\nabla u\|^2 - \langle h(u'(t)), u'(t)\rangle - \frac{1}{2}g(t)\|\Delta u\|^2 \\ &+ \frac{1}{2}(g' \circ \Delta u)(t) - \langle h'(u'(t))\nabla u'(t), \nabla u'(t)\rangle\Bigg] \\ &+ \varepsilon_1\Big[\|u'\|^2 + \Big(\delta - \frac{l}{2}\Big)\|\Delta u\|^2 + c\, C_\varepsilon(h_\varepsilon \circ \nabla u)(t) - cE'(t)\Big] \\ &+ \varepsilon_2\Bigg[\delta\|\Delta u\|^2 - \Bigg(\int_0^t g(s)ds - \delta\Bigg)\|u'\|^2 + \frac{c}{\delta}(C_\varepsilon + 1)((h_\varepsilon \circ \Delta u)(t)) - cE'(t)\Bigg].\end{aligned}$$

Since the relaxation function $g > 0$,

$$\begin{aligned}\mathcal{L}'(t) \le &-\Bigg[\Bigg(\int_0^t g(s)ds - \delta\Bigg)\varepsilon_2 - \varepsilon_1\Bigg]\|u'\|^2 - \Bigg(\Big(\frac{l}{2} - \delta\Big)\varepsilon_1 - \varepsilon_2\delta\Bigg)\|\Delta u\|^2 \\ &+ \frac{N}{2}(g' \circ \nabla u)(t) + \frac{N}{2}(g' \circ \Delta u)(t) + \varepsilon_1 c\, C_\varepsilon(h_\varepsilon \circ \nabla u)(t) \\ &+ \frac{\varepsilon_2 c}{\delta}(C_\varepsilon + 1)(h_\varepsilon \circ \Delta u)(t) - c(\varepsilon_1 + \varepsilon_2)E'(t).\end{aligned} \tag{3.8}$$

Using $g'(t) := \varepsilon g(t) - h_\varepsilon(t)$, and noting that $h_\varepsilon > 0$, we observe

$$\begin{aligned}\mathcal{L}'(t) \le &-\Bigg[\Bigg(\int_0^t g(s)ds - \delta\Bigg)\varepsilon_2 - \varepsilon_1\Bigg]\|u'\|^2 - \Bigg(\Big(\frac{l}{2} - \delta\Big)\varepsilon_1 - \varepsilon_2\delta\Bigg)\|\Delta u\|^2 \\ &+ \frac{N\varepsilon}{2}\Big[(g \circ \nabla u)(t) + (g \circ \Delta u)(t)\Big] \\ &- \Big[\frac{N}{2} - \frac{c}{\delta}(\varepsilon_1 + \varepsilon_2) - \frac{c}{\delta}C_\varepsilon(\varepsilon_1 + \varepsilon_2)\Big](h_\varepsilon \circ \Delta u)(t) - c(\varepsilon_1 + \varepsilon_2)E'(t).\end{aligned} \tag{3.9}$$

Now choose $\delta < \frac{l}{8}g_0$ with $g_0 = \int_0^{t_0} g(s)ds$. So, for

$$\varepsilon_1 = \frac{3}{8}g_0\varepsilon_2 \quad \text{with} \quad \varepsilon_2 = \frac{16(1-l)}{lg_0}\Big(4 + \frac{3}{2\omega_0}\Big),$$

a simple calculation shows that

$$(g_0 - \delta)\varepsilon_2 - \varepsilon_1 > \frac{2}{l}(1-l)\Big(4 + \frac{3}{2\omega_0}\Big) \text{ and } \Big(\frac{l}{2} - \delta\Big)\varepsilon_1 - \delta\varepsilon_2 > (1-l)\Big(4 + \frac{3}{2\omega_0}\Big). \tag{3.10}$$

From $\dfrac{\varepsilon g^2(s)}{\varepsilon g(s) - g'(s)} < g(s)$, and by the Lebesgue dominated convergence theorem,

$$\lim_{\varepsilon \to 0^+} \varepsilon C_\varepsilon = \lim_{\varepsilon \to 0^+} \int_0^\infty \frac{\varepsilon g^2(s)}{\varepsilon g(s) - g'(s)}ds = 0.$$

So, there exists $0 < \varepsilon_0 < 1$ such that if $\varepsilon < \varepsilon_0$, then

$$\varepsilon C_\varepsilon < \frac{1}{\frac{8c}{\delta}(\varepsilon_1 + \varepsilon_2)}.$$

Putting $\varepsilon = \frac{1}{2N}$ and choosing $N > \max\left\{\frac{4c}{\delta}(\varepsilon_1 + \varepsilon_2),\ \frac{1}{2\varepsilon_0}\right\}$, then

$$\frac{N}{4} - \frac{c}{\delta}(\varepsilon_1 + \varepsilon_2) > 0 \quad \text{and} \quad \varepsilon < \varepsilon_0.$$

From the above two equations,

$$\frac{N}{2} - \frac{c}{\delta}(\varepsilon_1 + \varepsilon_2) - \frac{c}{\delta}C_\varepsilon(\varepsilon_1 + \varepsilon_2) > \frac{N}{2} - \frac{c}{\delta}(\varepsilon_1 + \varepsilon_2) - \frac{1}{8\varepsilon} = \frac{N}{4} - \frac{c}{\delta}(\varepsilon_1 + \varepsilon_2) > 0. \quad (3.11)$$

Finally, substituting (3.10) and (3.11) in (3.9) gives the desired result. □

We are ready now to show our energy decay result. Our subsequent analysis makes a frequent use of the quadratic functional defined, for a purely time dependent function ϕ, and for $0 \le t_1 \le t_2 \le t$,

$$\mathcal{I}(\phi, t_1, t_2, t) := \int_{t_1}^{t_2} \phi(s)\left(\|\nabla(u(t) - u(t-s))\|^2 + \|\Delta(u(t) - u(t-s))\|^2\right) ds.$$

For convenience, if $t_2 = t$, we let $\mathcal{I}(\phi, t_1, t) := \mathcal{I}(\phi, t_1, t, t)$.

Theorem 3.5. *Putting $G_0(t) = tG'(t)$. Assume that hypothesis* (A1) *holds and the initial data $u_0 \in \mathcal{H}(\Omega)$ and $u_1 \in L^2(\Omega)$. Then, there exist positive constants λ_1, λ_2 such that the energy functional associated to problem* (1.1) *satisfies the estimate*

$$E(t) \le \lambda_2 G_0^{-1}\left(\frac{\lambda_1}{\int_{t_0}^{t}\xi(s)ds}\right), \quad \forall\, t > t_0. \quad (3.12)$$

Proof. From the non-increasing property of ξ, and the inequalities (3.2), (3.4) and (2.5),

$$\begin{aligned} \mathcal{I}(g, 0, t_0, t) &\le \frac{1}{\xi(t_0)}\mathcal{I}(\xi\, g, 0, t_0, t) \le -\frac{g(0)}{a\,\xi(t_0)}\mathcal{I}(g', 0, t_0, t) \\ &\le -\frac{g(0)}{a\,\xi(t_0)}\mathcal{I}(g', 0, t \le -c\big(E'(t) + \mathcal{E}'(t)\big), \end{aligned}$$

for any $t \ge t_0$. Inserting this estimate in (3.7), we obtain

$$\mathcal{L}'(t) \le -mE(t) - c\big(E'(t) + \mathcal{E}'(t)\big) + c\,\mathcal{I}(g, t_0, t), \quad \forall t \ge t_0. \quad (3.13)$$

By Lemmas 3.3 and 3.4, the functional $\mathcal{L}_1(t) := \mathcal{L}(t) + J_1(t) + \frac{1}{2}J_2(t)$ is nonnegative and satisfies

$$\mathcal{L}_1'(t) \le -\frac{2}{l}(1-l)\Big(4 + \frac{3}{2\omega_0}\Big)\|u'\|^2 - (1-l)\,\|\Delta u\|^2 - \frac{1}{4}\,(g \circ \nabla u)\,(t) \le -c_0 E(t),$$

for some $c_0 > 0$ and for any $t \geq t_0$. This estimate leads to the following bound

$$\int_0^\infty E(s)ds < +\infty. \tag{3.14}$$

From the first inequality in (2.1), we easily see that

$$E(t) \geq \frac{l}{2}\|\Delta u(t)\|^2 \quad \text{and} \quad E(t) \geq \frac{l}{2\omega_0}\|\nabla u(t)\|^2, \quad \forall\, t \geq 0. \tag{3.15}$$

For a positive constant γ, these last estimates together with (3.14) yield

$$\begin{aligned}\mathcal{I}(\gamma, t_0, t) &\leq 2\gamma \int_{t_0}^{t} \Big(\|\nabla u(t)\|^2 + \|\nabla u(t-s)\|^2 + \|\Delta u(t)\|^2 + \|\Delta u(t-s)\|^2\Big)\, ds \\ &\leq \frac{4\gamma}{l}(1+\omega_0)\int_{t_0}^{t}\big(E(t) + E(t-s)\big)ds.\end{aligned} \tag{3.16}$$

However, the last integral is finite (due to (3.14) and the inequality $E(t) \geq 0$), and hence, choose $0 < \gamma < 1$ such that

$$\mathcal{I}(\gamma, t_0, t) < 1, \quad \forall\, t \geq t_0. \tag{3.17}$$

The strict convexity of G and the fact that $G(0) = 0$ gives that

$$G(s\tau) \leq sG(\tau), \quad \text{for} \quad 0 \leq s \leq 1 \quad \text{and} \quad \tau \in (0, r]. \tag{3.18}$$

Combining this with the hypothesis (A1), Jensen's inequality and (3.17), we obtain

$$\begin{aligned}-\mathcal{I}(g', t_0, t) &= -\frac{1}{\mathcal{I}(\gamma, t_0, t)}\mathcal{I}(\mathcal{I}(\gamma, t_0, t)\, g', t_0, t) \\ &\geq \frac{1}{\mathcal{I}(\gamma, t_0, t)}\mathcal{I}(\mathcal{I}(\gamma, t_0, t)\, \xi\, G(g), t_0, t) \\ &\geq \frac{\xi(t)}{\mathcal{I}(\gamma, t_0, t)}\mathcal{I}(G(\mathcal{I}(\gamma, t_0, t)\, g), t_0, t) \\ &\geq \frac{\xi(t)}{\gamma} G\left(\gamma\, \mathcal{I}(g, t_0, t)\right) \\ &= \frac{\xi(t)}{\gamma}\bar{G}\left(\gamma\, \mathcal{I}(g, t_0, t)\right),\end{aligned}$$

for any $t > t_0$, where $\bar{G}$ is introduced in Remark 2.1. Thus,

$$\mathcal{I}(g, t_0, t) \leq \frac{1}{\gamma}\bar{G}^{-1}\left(-\frac{\gamma\mathcal{I}(g', t_0, t)}{\xi(t)}\right), \quad \text{for any} \quad t \geq t_0,$$

and with $\mathcal{F} := \mathcal{L} + cE + c\mathcal{E}$ (the constant c here is the one occurred in (3.13)), (3.13) becomes

$$\mathcal{F}'(t) \leq -mE(t) + \frac{c}{\gamma}\bar{G}^{-1}\left(-\frac{\gamma\mathcal{I}(g', t_0, t)}{\xi(t)}\right), \quad \forall\, t \geq t_0. \tag{3.19}$$

Let $0 < r_1 < r$, then define a functional $\mathcal{F}_1$ by

$$\mathcal{F}_1(t) := \bar{G}'\left(r_1 E_0(t)\right)\mathcal{F}(t), \quad \forall\, t \geq t_0, \quad \text{with } \; E_0(t) = \frac{E(t)}{E(0)}.$$

and so,

$$\mathcal{F}_1'(t) = r_1 E_0'(t)\bar{G}''\left(r_1 E_0(t)\right)\mathcal{F}(t) + \bar{G}'\left(r_1 E_0(t)\right)\mathcal{F}'(t).$$

Then, estimate (3.19) together with the facts that $E_0' \leq 0$, $G' > 0$ and $G'' > 0$ lead to

$$\mathcal{F}_1'(t) \leq -mE(t)\bar{G}'\left(r_1 E_0(t)\right) + \frac{c}{\gamma}\bar{G}'\left(r_1 E_0(t)\right)\bar{G}^{-1}\left(-\frac{\gamma\mathcal{I}(g', t_0, t)}{\xi(t)}\right), \quad \forall\, t \geq t_0. \tag{3.20}$$

Let $\bar{G}^*$ be the convex conjugate of $\bar{G}$ in the sense of Young (see [2, pp. 61–64]), given by

$$\bar{G}^*(s) = s\big(\bar{G}'\big)^{-1}(s) - \bar{G}\left[\big(\bar{G}'\big)^{-1}(s)\right] \tag{3.21}$$

and satisfies the following generalized Young inequality

$$AB \leq \bar{G}^*(A) + \bar{G}(B). \tag{3.22}$$

Set $A = \bar{G}'\left(r_1 E_0(t)\right)$ and $B = \bar{G}^{-1}\left(-\gamma\mathcal{I}(g', t_0, t)/\xi(t)\right)$, then it follows from a combination of (3.20) and (3.22) that

$$\begin{aligned}\mathcal{F}_1'(t) &\leq -mE(t)\bar{G}'\left(r_1 E_0(t)\right) + c\gamma^{-1}\bar{G}^*\left[\bar{G}'\left(r_1 E_0(t)\right)\right] - c\mathcal{I}(g', t_0, t)/\xi(t)\\ &\leq -m(E(0) - cr_1)E_0(t)\bar{G}'\left(r_1 E_0(t)\right) - c\mathcal{I}(g', t_0, t)/\xi(t), \quad \forall\, t \geq t_0.\end{aligned}$$

After fixing r_1, we arrive at

$$\mathcal{F}_1'(t) \leq -m_1 E_0(t)\bar{G}'\left(r_1 E_0(t)\right) - c\mathcal{I}(g', t_0, t)/\xi(t), \quad \forall\, t \geq t_0,$$

where $m_1 > 0$. Hence, multiplying both sides by $\xi(t)$, and using $r_1 E_0(t) < r$ and the inequality

$$-\mathcal{I}(g', t_0, t) \leq -c\big(E'(t) + \mathcal{E}'(t)\big), \quad \forall\, t \geq t_0, \tag{3.23}$$

it follows from the definition of $\mathcal{I}$ and the estimates $E'(t) \leq \frac{1}{2}(g' \circ \nabla u)(t)$ (by (3.2)) and $\mathcal{E}'(t) \leq \frac{1}{2}(g' \circ \Delta u)(t)$ (by (3.4)) that

$$\begin{aligned}\xi(t)\mathcal{F}_1'(t) &\leq -m_1 E_0(t)G'\left(r_1 E_0(t)\right)\xi(t) - c\mathcal{I}(g', t_0, t)\\ &\leq -m_1 E_0(t)G'\left(r_1 E_0(t)\right)\xi(t) - c\big(E'(t) + \mathcal{E}'(t)\big), \quad \forall\, t \geq t_0.\end{aligned}$$

Let $\mathcal{F}_2 = \xi\mathcal{F}_1 + c(E + \mathcal{E})$, then we obtain, from the non-increasing property of ξ, that

$$m_1 E_0(t)G'\left(r_1 E_0(t)\right)\xi(t) \leq -\mathcal{F}_2'(t), \quad \forall\, t \geq t_0. \tag{3.24}$$

Since $G'' > 0$ and by the non-increasing property of E, the map $t \longmapsto E(t)G'(\varepsilon_1 E_0(t))$ is non-increasing. Consequently, an integration of (3.24) over (t_0, t) yields

$$m_1 E_0(t) G'(r_1 E_0(t)) \int_{t_0}^{t} \xi(s)ds \leq \int_{t_0}^{t} m_1 E_0(s) G'(r_1 E_0(s)) \xi(s) ds$$
$$\leq \mathcal{F}_2(t_0) - \mathcal{F}_2(t) \leq \mathcal{F}_2(t_0).$$

Since $G_0(\tau) = \tau G'(\tau)$ is strictly increasing, then the desired bound follows immediately. □

Example 3.6.

(1) Choose $g(t) = ae^{-bt^{\nu}}$ with $0 < \nu < 1$, where a and b are positive constants such that $1 - \max\{\omega_0, \omega_1\}\frac{a}{b} > 0$ and $h(s) = s$ so, hypothesis (A1) is satisfied. Then, $g'(t) = -\xi(t)G(g(t))$ with $G(t) = t$, but $\xi(t) = \nu\, bt^{\nu-1}$. By Theorem 3.5, we conclude that $E(t) \leq c(t - t_0)^{-\nu}$ for a sufficiently large t.

(2) Choose $g(t) = a(1 + t)^{-\nu}$ where $\nu > 1$ and a is chosen so that hypothesis (A1) remains valid and $h(s) = s$. Here, $g'(t) = -\xi(t)G(g(t))$ with $G(t) = t^{1+1/\nu}$ and $\xi(t) = b$, where b is a fixed constant. By Theorem 3.5, $E(t) \leq c(1 + t)^{-\nu/(\nu+1)}$ for a sufficiently large t.

4. NUMERICAL STUDY

This section is devoted to illustrate numerically the achieved theoretical decaying results in Theorem 3.5 on a sample test problem of the form (1.1). To do so, we develop a numerical scheme for the nonlinear model problem (1.1) using finite differences for the time discretization combined with the $\mathcal{C}^2$ continuous bicubic Galerkin method in space [18]. To avoid solving any nonlinear algebraic systems of equations, the approximation of the damping term is based on an extrapolation technique.

To discretize in time, we truncate the interval $(0, \infty)$ and work instead on the finite interval $(0, T]$ where T is large enough. Divide $[0, T]$ uniformly into N subintervals with size τ each and nodes $\{t_n\}_{n=0}^{N}$, that is, $t_n = n\tau$ for $0 \leq n \leq N$, where $\tau = T/N$. For the grid function w^n, let

$$\delta_t w^n = \frac{w^n - w^{n-1}}{\tau}, \quad \delta_{tt} w^n = \frac{w^{n+1} - 2w^n + w^{n-1}}{\tau^2},$$
$$w^{n+\frac{1}{2}} = \frac{w^n + w^{n-1}}{2}, \quad w^{n+\frac{1}{4}} = \frac{w^{n+1} + 2w^n + w^{n-1}}{4}.$$

Turning into the spatial discretization, choose $\Omega = (a, b) \times (c, d)$ and then divide both (a, b) (in the x-direction) and (c, d) (in the y-direction) into a family of uniform (quasi-uniform) cells. To elaborate, let $x_i = i\, h_x$ for $0 \leq i \leq M_x$ with $h_x = (b - a)/M_x$

and let $y_j = j\,h_y$ for $0 \le j \le M_y$ with $h_y = (d-c)/M_y$. Then, the $\mathcal{C}^2$ Galerkin finite dimensional space $S_h := S_{h_x} \otimes S_{h_y}$, where

$$S_{h_x} = \{v \in H^3(a,b) :\ v|_{[x_{i-1},x_i]} \in P_3 \text{ for } 1 \le i \le N_x,$$

$$\text{with } v(x)|_{x=a,b} = v''(x)|_{x=a,b} = 0\},$$

$$S_{h_y} = \{v \in H^3(c,d) :\ v|_{[y_{i-1},x_i]} \in P_3 \text{ for } 1 \le i \le N_y,$$

$$\text{with } v(y)|_{y=c,d} = v''(y)|_{y=c,d} = 0\},$$

where P_3 is the space of polynomials of degree at most 3 in each direction x or y.

Usually, continuous Galerkin finite element schemes are motivated by the weak formulation of the model problem. So, we take the inner product of (1.1) with $\phi \in \mathcal{H}(\Omega)$ then using Green's formula. This leads to

$$\langle u'', \phi\rangle + \langle \Delta u, \Delta\phi\rangle - \int_0^t g(t-s)\langle \nabla u(s), \nabla\phi\rangle\, ds + \langle h(u'), \phi\rangle = 0. \tag{4.1}$$

Consequently, for each $t > 0$, the semi-discrete finite element solution $u_h(t) \in S_h$ is defined by

$$\langle u_h'', \phi\rangle + \langle \Delta u_h, \Delta\phi\rangle - \int_0^t g(t-s)\,\langle \nabla u_h(s), \nabla\phi\rangle\ ds + \langle h(u_h'), \phi\rangle = 0, \quad \forall\ \phi \in S_h.$$

Our fully-discrete numerical solution U_h^n which approximates $u(t_n)$ is defined by

$$\langle \delta_{tt} U_h^n, \phi\rangle + \langle \Delta U_h^{n+\frac14}, \Delta\phi\rangle - \int_0^{t_{n+1}} g(t_{n+1}-s)\langle \nabla \bar U_h(s), \nabla\phi\rangle ds + \langle h(\delta_t U_h^n), \phi\rangle = 0, \tag{4.2}$$

for all $\phi \in S_h$, and for $1 \le n \le N-1$, where the piecewise constant function $\bar U_h(s) = U_h^{j+\frac12}$ for $t_j < s < t_{j+1}$ with $1 \le j \le N-1$.

At each time level, the above scheme amounts to a square linear system (see the matrix form below). So the existence of the approximate solution U_h^{n+1} follows from its uniqueness. For uniqueness, we need to show that if

$$\frac{1}{\tau^2}\langle U_h^{n+1}, \phi\rangle + \frac14\langle \Delta U_h^{n+1}, \Delta\phi\rangle - \frac12 \int_{t_n}^{t_{n+1}} g(t_{n+1}-s)\, ds \langle \nabla U_h^{n+1}, \nabla\phi\rangle ds = 0, \quad \phi \in S_h,$$

then $U_h^{n+1} \equiv 0$. Choose $\phi = U_h^{n+1}$ and then, use the first inequality in (2.1) in addition to the non-increasing and positivity properties of g, we observe

$$\frac{1}{\tau^2}\|U_h^{n+1}\|^2 + \frac{1}{4}\|\Delta U_h^{n+1}\|^2 < \frac{\omega_0}{2} g(0)\tau\|\Delta U_h^{n+1}\|^2.$$

Hence, for a sufficiently small τ ($\frac{\omega_0}{2} g(0)\tau \leq \frac{1}{4}$), we get $\frac{1}{\tau^2}\|U_h^{n+1}\|^2 \leq 0$ and thus $\frac{1}{\tau^2}\|U_h^{n+1}\| \equiv 0$. This completes the proof of uniqueness, and consequently the existence, of the numerical solution.

To write this numerical scheme in a matrix form, let

$$d_{hx} := \dim S_{hx} = N_x - 1 \text{ and let } \{\phi_p\}_{p=1}^{d_{hx}}$$

denote the basis function of S_{h_x}. We define $d_{hx} \times d_{hx}$ matrices:

$$\boldsymbol{M}_x = \left[\int_a^b \phi_q \phi_p dx \right], \quad \boldsymbol{G}_x = \left[\int_a^b \phi'_q \phi'_p dx \right], \quad \text{and} \quad \boldsymbol{S}_x = \left[\int_a^b \phi''_q \phi''_p dx \right].$$

In a similar fashion, let $d_{hy} := \dim S_{hy} = N_y - 1$ and let $\{\psi_p\}_{p=1}^{d_{hy}}$ denote the basis function of S_{h_y}, and consequently, the $d_{hy} \times d_{hy}$ matrices in the y-direction are:

$$\boldsymbol{M}_y = \left[\int_c^d \psi_q \psi_p dy \right], \quad \boldsymbol{G}_y = \left[\int_c^d \psi'_q \psi'_p dy \right], \quad \text{and} \quad \boldsymbol{S}_y = \left[\int_c^d \psi''_q \psi''_p dy \right].$$

The $(d_{hx} \times d_{hy})$-dimensional column vectors $\mathbf{b}^n$ and $\mathbf{F}^n$ are the transpose of the vectors

$$[b_{1,1}^n, b_{1,2}^n, \ldots, b_{1,d_{hy}}^n, \ldots, b_{d_{hx},1}^n, \ldots, b_{d_{hx},d_{hy}}^n],$$

and

$$[f_{1,1}^n, f_{1,2}^n, \ldots, f_{1,d_{hy}}^n, \ldots, f_{d_{hx},1}^n, \ldots, f_{d_{hx},d_{hy}}^n],$$

with $f_{i,j}^n := \langle h(\delta_t U_h^n), \phi_i \psi_j \rangle$, respectively.

Therefore, through tensor products of one-dimensional $\mathcal{C}^2$ splines, the fully-discrete scheme (4.2) has the following matrix representation:

$$\begin{aligned}
&\Big(\boldsymbol{M}_x \otimes \boldsymbol{M}_y\Big)\delta_{tt}\mathbf{b}^n + \Big(\boldsymbol{S}_x \otimes \boldsymbol{M}_y + 2\boldsymbol{G}_x \otimes \boldsymbol{G}_y + \boldsymbol{M}_x \otimes \boldsymbol{S}_y\Big)\mathbf{b}^{n+\frac{1}{4}} \\
&+ \sum_{j=0}^{n} g_{n+1}^j \Big(\boldsymbol{G}_x \otimes \boldsymbol{M}_y + \boldsymbol{M}_x \otimes \boldsymbol{G}_y\Big)\mathbf{b}^{j+\frac{1}{2}} = -\mathbf{F}^n,
\end{aligned}$$

with

$$g_{n+1}^{j} := \int_{t_j}^{t_{j+1}} g(t_{n+1} - s)\,ds.$$

Alternatively, this can be rewritten as: for $1 \le n \le N-1$,

$$\begin{aligned}
&\Big(4\boldsymbol{M}_x \otimes \boldsymbol{M}_y + \tau^2\big(\boldsymbol{S}_x \otimes \boldsymbol{M}_y + 2\boldsymbol{G}_x \otimes \boldsymbol{G}_y + \boldsymbol{M}_x \otimes \boldsymbol{S}_y\big) \\
&\quad + 2\tau^2 g_{n+1}^{n}\big(\boldsymbol{G}_x \otimes \boldsymbol{M}_y + \boldsymbol{M}_x \otimes \boldsymbol{G}_y\big)\Big)\mathbf{b}^{n+1} \\
&\quad = 4\boldsymbol{M}_x \otimes \boldsymbol{M}_y(2\mathbf{b}^n - \mathbf{b}^{n-1}) \\
&\qquad - \tau^2\Big(\boldsymbol{S}_x \otimes \boldsymbol{M}_y + 2\boldsymbol{G}_x \otimes \boldsymbol{G}_y + \boldsymbol{M}_x \otimes \boldsymbol{S}_y\Big)(2\mathbf{b}^n + \mathbf{b}^{n-1}) \\
&\qquad - 2\tau^2 g_{n+1}^{n}\big(\boldsymbol{G}_x \otimes \boldsymbol{M}_y + \boldsymbol{M}_x \otimes \boldsymbol{G}_y\big)\mathbf{b}^n \\
&\qquad - 2\tau^2\big(\boldsymbol{G}_x \otimes \boldsymbol{M}_y + \boldsymbol{M}_x \otimes \boldsymbol{G}_y\big)\Big(\sum_{j=0}^{n-1} g_{n+1}^{j}(\mathbf{b}^{j+1} + \mathbf{b}^{j})\Big) - \mathbf{F}^n.
\end{aligned}$$

Therefore, at each time level t_{n+1}, we solve a finite square linear system, where the unknown is the column vector $\mathbf{b}^{n+1}$.

Furthermore, from the above matrix form, it is clear that our scheme (4.2) is a three-time level scheme. That is, the approximate solutions U_h^0 and U_h^1 need to be determined first, and then U_h^j for $2 \le j \le N$ can be computed by solving the above linear system recursively. We choose $U_h^0 \in S_h$ to be the bicubic spline polynomial that interpolates u_0 at the interior nodal nodes. However, motivated by the Taylor series expansion of u about $t = 0$, we choose $U_h^1 \in S_h$ to be the bicubic spline polynomial that interpolates $u_0 + t_1 u_1$ at the interior nodal nodes.

For the computer implementation of the linear system, it is important to consider the discretization of spatial Galerkin-type integrals in the scheme. To this end, on each cell of our two-dimensional partition, the integrals are approximated using 2-point Gauss quadrature rule in each direction (x and y).

In our test problem, we choose $\Omega = (0,1) \times (0,1)$, the time interval is $(0, 80)$, the initial data

$$u_0(x,y) = \frac{1}{64^2}[xy(1-x)(1-y)]^3, \quad u_1(x,y) = 0,$$

the relaxation function $g(t) = \mathrm{e}^{-t}$, and the damping function $h(s) = s$. The spatial mesh consists of 400 (square) cells of equal areas, while the time domain consists of 80000 subintervals.

Figure 1 shows that the numerical solution U_h converges to zero as the time t gets far away from 0. The graphical plots of the numerical approximations of the weighted energy in Figure 2 confirms that $tE(t) \le 1$ for a sufficiently large t. This is compatible with the achieved theoretical results in Theorem 3.5 (see Example 3.6).

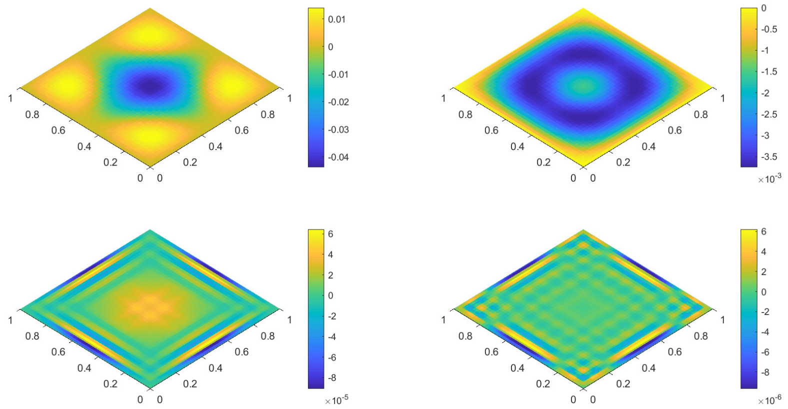

Fig. 1. The numerical solutions for $t = 5$ (top-left), $t = 10$ (top-right), $t = 20$ (bottom-left), and $t = 30$ (bottom-right)

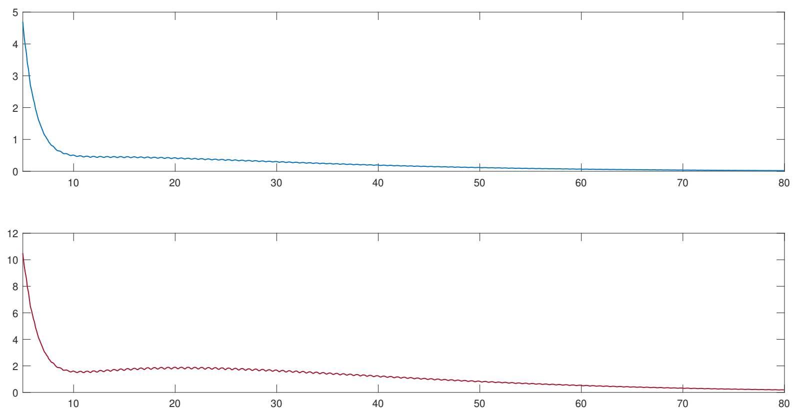

Fig. 2. The numerical weighted energy plots against $t \in [5, 80]$. The top and the bottom are the approximations of $tE(t)$ and $t^{1.5}E(t)$, respectively

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