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GENERAL SOLUTIONS OF SECOND-ORDER LINEAR DIFFERENCE EQUATIONS OF EULER TYPE

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Abstract. The purpose of this paper is to give general solutions of linear difference equations which are related to the Euler-Cauchy differential equation $y'' + (\lambda/t^2)y = 0$ or more general linear differential equations. We also show that the asymptotic behavior of solutions of the linear difference equations are similar to solutions of the linear differential equations.

Keywords: Euler-Cauchy equations, oscillation, conditionally oscillatory.

Mathematics Subject Classification: 39A06, 39A12, 39A21.

1. INTRODUCTION

In this paper, we consider linear difference equations which are related to the Euler-Cauchy differential equation

$$y'' + \frac{\lambda}{t^2} y = 0, \quad t > 0, \tag{1.1}$$

or the more general linear differential equation

$$y'' + \left\{ \frac{1}{4} \sum_{k=1}^{m-1} \left(\prod_{i=0}^{k-1} \log_i(t) \right)^{-2} + \lambda \left(\prod_{i=0}^{m-1} \log_i(t) \right)^{-2} \right\} y = 0, \quad t > t_{m-1}, \tag{1.2}$$

where $\lambda > 0$, $\log_0(t) = t$ and $\log_m(t) = \log(\log_{m-1}(t))$, $t_0 = 0$ and $t_m = \exp(t_{m-1})$ for $m \in \mathbb{N}$. Note that the function $\log_m(t)$ is positive for $t > t_m$. Moreover, notice that we have adopted the notation $\sum_{i=j}^{k} a_i = 0$ and $\prod_{i=j}^{k} a_i = 1$ if $j > k$. Then it is easy to check that equation (1.2) with $m = 1$ becomes equation (1.1). Furthermore, it is known that equation (1.2) with $m = 2$ is called the Riemann-Weber version of the Euler type differential equation (see [8]).

By using the Liouville transformation $s = \log t$, $u(s) = t^{-1/2}y(t)$ successively, equation (1.2) is transformed into equation (1.1). Thus, we can get general solutions of equation (1.2) (for example, see [6, 8, 13, 14, 16]).

Theorem 1.1. *Let $m \in \mathbb{N}$. Then equation* (1.2) *has the general solution*

$$y(t) = \begin{cases} \left(\prod_{k=0}^{m-2} \log_k(t)\right)^{1/2} \{K_1(\log_{m-1}(t))^z + K_2(\log_{m-1}(t))^{1-z}\} & \text{if } \lambda \neq \dfrac{1}{4}, \\ \left(\prod_{k=0}^{m-1} \log_k(t)\right)^{1/2} \{K_3 + K_4 \log_m(t)\} & \text{if } \lambda = \dfrac{1}{4}, \end{cases}$$

where K_i $(i = 1, 2, 3, 4)$ are arbitrary constants and z is the root of the characteristic equation

$$z^2 - z + \lambda = 0. \tag{1.3}$$

As for linear difference equations which are related to equation (1.1), we can consider various types. But, from a viewpoint of general solutions, we choose the difference equation

$$\Delta^2 x(n) + \frac{\lambda}{n(n+1)} x(n) = 0, \quad n \in \mathbb{N}, \tag{1.4}$$

where $\Delta x(n) = x(n+1) - x(n)$, $\Delta^2 x(n) = \Delta(\Delta x(n))$. Note that equation (1.4) has the general solution

$$x(n) = \begin{cases} K_1 \prod_{j=n_0}^{n-1} \left(1 + \dfrac{z}{j}\right) + K_2 \prod_{j=n_0}^{n-1} \left(1 + \dfrac{1-z}{j}\right) & \text{if } \lambda \neq \dfrac{1}{4}, \\ \prod_{j=n_0}^{n-1} \left(1 + \dfrac{1}{2j}\right) \left\{K_3 + K_4 \sum_{k=n_0}^{n-1} \dfrac{2}{2k+1}\right\} & \text{if } \lambda = \dfrac{1}{4}, \end{cases}$$

where z satisfies (1.3) (for example, see [1, 4] and [15, Appendix]). However, since we have not found a transformation such as the Liouville transformation, it is not easy to get linear difference equations which correspond to equation (1.2) and which have general solutions. Here a natural question now arises. What is the best difference equation which corresponds to equation (1.2)? The purpose of this paper is to answer the question.

This paper is organized as follows. In Section 2, we will construct discrete functions such as the logarithm function. In Section 3, we give linear difference equations which correspond to equation (1.2) and solve them explicitly. Finally, in Section 4, we discuss the oscillatory behavior of their solutions as $n \to \infty$.

Remark 1.2. From the general solution of equation (1.4), we see that $\lambda > 1/4$ is necessary and sufficient for all nontrivial solutions of equation (1.4) to be oscillatory (see [15]). Here a function $x(n)$ is said to be *oscillatory* if it is neither eventually positive nor eventually negative, otherwise it is said to be *nonoscillatory*. As another type difference equation which corresponds to equation (1.1), we can consider the linear difference equation

$$\Delta^2 x(n) + \frac{\lambda}{(n+1)^2} x(n+1) = 0. \tag{1.5}$$

According to Zhang and Cheng [17], without using general solutions, they showed that $\lambda > 1/4$ is necessary and sufficient for all nontrivial solutions of equation (1.5) to be oscillatory. Note that we can use Sturm's comparison theorem for equation (1.5) because it is self-adjoint (see [9]). Moreover, we see that advanced results in this direction can be found in [2, 5, 7, 10–12] and the references contained therein.

2. LOGARITHM LIKE FUNCTIONS

To begin with, we introduce some functions such as the function $\log_k(t)$. Define

$$L_m(n) = \prod_{k=0}^{m-1} l_k(n), \quad m \in \mathbb{N},$$

where $l_m(n)$ is positive and satisfies

$$\Delta l_0(n) = 1 \quad \text{and} \quad \Delta l_m(n) = \left(\frac{l_{m-1}(n)}{\Delta l_{m-1}(n)} + \frac{1}{2} \right)^{-1}.$$

Remark 2.1. For any $m \in \mathbb{N}$, the function $l_m(n)$ is increasing because $\Delta l_m(n)$ represents as

$$\Delta l_m(n) = \frac{2\Delta l_{m-1}(n)}{2l_{m-1}(n) + \Delta l_{m-1}(n)} = \prod_{k=0}^{m-1} \frac{2}{l_k(n) + l_k(n+1)} > 0. \tag{2.1}$$

In this section, we prepare some lemmas which are useful in proving our results.

Lemma 2.2. *For any* $m \in \mathbb{N}$,

$$\Delta l_m(n) = \left(L_m(n) + \frac{1}{2} \sum_{k=1}^{m} \frac{L_m(n)}{L_k(n)} \right)^{-1}, \tag{2.2}$$

and

$$\Delta L_m(n) = \frac{1}{2} \sum_{k=1}^{m} \left(\frac{L_m(n+1)}{L_k(n+1)} + \frac{L_m(n)}{L_k(n)} \right). \tag{2.3}$$

Proof. We first show (2.2). Since

$$\Delta l_1(n) = \left(l_0(n) + \frac{1}{2}\right)^{-1} = \left(L_1(n) + \frac{1}{2}\sum_{k=1}^{1} \frac{L_1(n)}{L_k(n)}\right)^{-1},$$

(2.2) with $m = 1$ holds. Suppose that (2.2) with $m = p$ holds. Then we have

$$\begin{aligned}
\Delta l_{p+1}(n) &= \left(\frac{l_p(n)}{\Delta l_p(n)} + \frac{1}{2}\right)^{-1} = \left\{l_p(n)\left(L_p(n) + \frac{1}{2}\sum_{k=1}^{p}\frac{L_p(n)}{L_k(n)}\right) + \frac{1}{2}\right\}^{-1} \\
&= \left(L_{p+1}(n) + \frac{1}{2}\sum_{k=1}^{p}\frac{L_{p+1}(n)}{L_k(n)} + \frac{1}{2}\frac{L_{p+1}(n)}{L_{p+1}(n)}\right)^{-1} \\
&= \left(L_{p+1}(n) + \frac{1}{2}\sum_{k=1}^{p+1}\frac{L_{p+1}(n)}{L_k(n)}\right)^{-1},
\end{aligned}$$

and therefore, (2.2) with $m = p + 1$ holds.

We next show (2.3). Since $\Delta L_1(n) = \Delta l_0(n) = 1$, (2.3) with $m = 1$ holds. Suppose that (2.3) with $m = p$ holds. Then, using (2.2), we have

$$\begin{aligned}
\Delta L_{p+1}(n) =& \Delta(l_p(n)\, L_p(n)) = \Delta l_p(n)\, L_p(n) + l_p(n+1)\, \Delta L_p(n) \\
=& \Delta l_p(n) L_p(n) + \frac{l_p(n+1)}{2}\sum_{k=1}^{p}\left(\frac{L_p(n+1)}{L_k(n+1)} + \frac{L_p(n)}{L_k(n)}\right) \\
=& \Delta l_p(n) L_p(n) + \frac{1}{2}\sum_{k=1}^{p}\left(\frac{L_{p+1}(n+1)}{L_k(n+1)} + \frac{L_p(n)(l_p(n) + \Delta l_p(n))}{L_k(n)}\right) \\
=& \Delta l_p(n) L_p(n) + \frac{1}{2}\sum_{k=1}^{p}\left(\frac{L_{p+1}(n+1)}{L_k(n+1)} + \frac{L_{p+1}(n)}{L_k(n)}\right) + \frac{\Delta l_p(n)}{2}\sum_{k=1}^{p}\frac{L_p(n)}{L_k(n)} \\
=& \Delta l_p(n)\left(L_p(n) + \frac{1}{2}\sum_{k=1}^{p}\frac{L_p(n)}{L_k(n)}\right) + \frac{1}{2}\sum_{k=1}^{p}\left(\frac{L_{p+1}(n+1)}{L_k(n+1)} + \frac{L_{p+1}(n)}{L_k(n)}\right) \\
=& 1 + \frac{1}{2}\sum_{i=1}^{p}\left(\frac{L_{p+1}(n+1)}{L_i(n+1)} + \frac{L_{p+1}(n)}{L_i(n)}\right) \\
=& \frac{1}{2}\sum_{k=1}^{p+1}\left(\frac{L_{p+1}(n+1)}{L_k(n+1)} + \frac{L_{p+1}(n)}{L_k(n)}\right),
\end{aligned}$$

and therefore, (2.3) with $m = p + 1$ holds. $\square$

Lemma 2.3. *Let $k \in \mathbb{N} \cup \{0\}$. Then there exists $C > 1$ such that*

$$|l_k(n) - \log_k(n)| < C \tag{2.4}$$

for n sufficiently large.

Proof. We use the mathematical induction on k. Since $l_0(n) = n + c$, (2.4) with $k = 0$ holds. Assume that (2.4) holds for $0 \le i \le k$. Then there exists $n_1 \in \mathbb{N}$ such that

$$0 < 2(\log_i(n) - C) \le 2l_i(n) \le l_i(n) + l_i(n+1) \le 2l_i(n+1) \le 2(\log_i(n+1) + C)$$

for $n \ge n_1$. Hence we have

$$\sum_{j=n_1}^{n-1} \prod_{i=0}^{k} \frac{1}{\log_i(j+1) + C} \le \sum_{j=n_1}^{n-1} \prod_{i=0}^{k} \frac{2}{l_i(j) + l_i(j+1)} \le \sum_{j=n_1}^{n-1} \prod_{i=0}^{k} \frac{1}{\log_i(j) - C}$$

for $n \ge n_1$. Let $\log_0(t;\alpha) = t + \alpha$ and $\log_m(t;\alpha) = \log(\log_{m-1}(t;\alpha) + \alpha)$, where $\alpha \in \mathbb{R}$. Then, using equality (2.1) and the inequalities

$$\begin{aligned}
\sum_{j=n_1}^{n-1} \prod_{i=0}^{k} \frac{1}{\log_i(j) - C} &\le \int_{n_1}^{n} \prod_{i=0}^{k} \frac{1}{\log_i(t-1) - C} dt \le \int_{n_1}^{n} \prod_{i=0}^{k} \frac{1}{\log_i(t;-C) - C} dt \\
&= \log_{k+1}(n;-C) - \log_{k+1}(n_1;-C) \le \log_{k+1}(n), \\
\sum_{j=n_1}^{n-1} \prod_{i=0}^{k} \frac{1}{\log_i(j+1) + C} &\ge \int_{n_1}^{n} \prod_{i=0}^{k} \frac{1}{\log_i(t+1) + C} dt \ge \int_{n_1}^{n} \prod_{i=0}^{k} \frac{1}{\log_i(t;C) + C} dt \\
&= \log_{k+1}(n;C) - \log_{k+1}(n_1;C) \\
&\ge \log_{k+1}(n) - \log_{k+1}(n_1;C),
\end{aligned}$$

we obtain

$$\log_{k+1}(n) - \log_{k+1}(n_1;C) \le l_{k+1}(n) - l_{k+1}(n_1) \le \log_{k+1}(n).$$

Thus we conclude that

$$|l_{k+1}(n) - \log_{k+1}(n)| \le \max\{l_{k+1}(n_1), |l_{k+1}(n_1) - \log_{k+1}(n_1;C)|\},$$

that is, (2.4) with $k+1$ holds. This completes the proof. □

Remark 2.4. From Lemma 2.3, for any $k \in \mathbb{N} \cup \{0\}$, $l_k(n) \to \infty$ as $n \to \infty$.

3. THE MAIN RESULT

Let us consider the linear difference equation

$$\Delta^2 x(n) + \left\{ \frac{1}{4} \sum_{k=1}^{m-1} \frac{1}{L_k(n) L_k(n+1)} + \frac{\lambda}{L_m(n) L_m(n+1)} \right\} x(n) = 0. \tag{3.1}$$

This equation can regard as a difference equation which correspond to linear differential equation (1.2). In fact, if $m = 1$, then equation (3.1) becomes the difference equation

$$\Delta^2 x(n) + \frac{\lambda}{l_0(n) l_0(n+1)} x(n) = 0,$$

and therefore, this equation includes equation (1.4). Moreover, in case $m = 2$, equation (3.1) reduces the equation

$$\Delta^2 x(n) + \left\{ \frac{1}{4\, l_0(n)\, l_0(n+1)} + \frac{\lambda}{l_0(n)\, l_0(n+1)\, l_1(n)\, l_1(n+1)} \right\} x(n) = 0,$$

which corresponds to Riemann-Weber version of the Euler type differential equation. Let us consider the function $\xi_m(n, \mu)$ defined by

$$\xi_m(n, \mu) = \prod_{j=n_0}^{n-1} \left(1 + \frac{1}{2} \sum_{k=1}^{m-1} \frac{1}{L_k(j)} + \frac{\mu}{L_m(j)} \right).$$

Then we get general solutions of equation (3.1). Our main result is as follows.

Theorem 3.1. *Equation* (3.1) *has the general solution*

$$x(n) = \begin{cases} K_1 \xi_m(n, z) + K_2 \xi_m(n, 1 - z) & \textit{if } \lambda \neq 1/4, \\ \xi_m(n, 1/2) \left\{ K_3 + K_4\, l_m(n) \right\} & \textit{if } \lambda = 1/4, \end{cases}$$

where K_i $(i = 1, 2, 3, 4)$ *are arbitrary constants and* z *satisfies* (1.3).

Proof. Let $x(n) = \xi_m(n, z)$. Then $x(n)$ satisfies

$$\Delta x(n) = \left(\frac{1}{2} \sum_{k=1}^{m-1} \frac{1}{L_k(n)} + \frac{z}{L_m(n)} \right) x(n),$$

and therefore, by using (2.3), we have

$$\begin{aligned} \Delta^2 x(n) &= \left\{ \frac{1}{2} \sum_{k=1}^{m-1} \Delta \left(\frac{1}{L_k(n)} \right) + z\, \Delta \left(\frac{1}{L_m(n)} \right) \right\} x(n) \\ &\quad + \left(\frac{1}{2} \sum_{k=1}^{m-1} \frac{1}{L_k(n+1)} + \frac{z}{L_m(n+1)} \right) \Delta x(n) \\ &= \left\{ \frac{1}{2} \sum_{k=1}^{m-1} \Delta \left(\frac{1}{L_k(n)} \right) + z\, \Delta \left(\frac{1}{L_m(n)} \right) \right\} x(n) \\ &\quad + \left(\frac{1}{2} \sum_{k=1}^{m-1} \frac{1}{L_k(n+1)} + \frac{z}{L_m(n+1)} \right) \left(\frac{1}{2} \sum_{k=1}^{m-1} \frac{1}{L_k(n)} + \frac{z}{L_m(n)} \right) x(n) \\ &= \left[-\frac{1}{2} \sum_{k=1}^{m-1} \frac{\Delta L_k(n)}{L_k(n) L_k(n+1)} - z \frac{\Delta L_m(n)}{L_m(n) L_m(n+1)} \right. \\ &\quad + \frac{1}{4} \left(\sum_{k=1}^{m-1} \frac{1}{L_k(n+1)} \right) \left(\sum_{k=1}^{m-1} \frac{1}{L_k(n)} \right) + \frac{z}{2 L_m(n)} \sum_{k=1}^{m-1} \frac{1}{L_k(n+1)} \end{aligned}$$

$$
\begin{aligned}
&\quad + \frac{z}{2L_m(n+1)} \sum_{k=1}^{m-1} \frac{1}{L_k(n)} + \frac{z^2}{L_m(n)L_m(n+1)} \Bigg] x(n) \\
&= \Bigg[\left\{ -\frac{1}{2} \sum_{k=1}^{m-1} \frac{\Delta L_k(n)}{L_k(n)L_k(n+1)} + \frac{1}{4} \left(\sum_{k=1}^{m-1} \frac{1}{L_k(n+1)} \right) \left(\sum_{k=1}^{m-1} \frac{1}{L_k(n)} \right) \right\} \\
&\quad + \left\{ -\Delta L_m(n) + \frac{1}{2} \sum_{k=1}^{m-1} \left(\frac{L_m(n+1)}{L_k(n+1)} + \frac{L_m(n)}{L_k(n)} \right) \right\} \frac{z}{L_m(n)L_m(n+1)} \\
&\quad + \frac{z^2}{L_m(n)L_m(n+1)} \Bigg] x(n) \\
&= - \Bigg[\left\{ \frac{1}{2} \sum_{k=1}^{m-1} \frac{\Delta L_k(n)}{L_k(n)L_k(n+1)} - \frac{1}{4} \left(\sum_{k=1}^{m-1} \frac{1}{L_k(n+1)} \right) \left(\sum_{k=1}^{m-1} \frac{1}{L_k(n)} \right) \right\} \\
&\quad + \frac{z - z^2}{L_m(n)L_m(n+1)} \Bigg] x(n) \\
&= - \left\{ \frac{1}{4} \sum_{k=1}^{m-1} \frac{1}{L_k(n)L_k(n+1)} + \frac{z - z^2}{L_m(n)L_m(n+1)} \right\} x(n).
\end{aligned}
$$

Here the last equality is calculated as follows:

$$
\begin{aligned}
&\frac{1}{2} \sum_{k=1}^{m-1} \frac{\Delta L_k(n)}{L_k(n)L_k(n+1)} - \frac{1}{4} \left(\sum_{k=1}^{m-1} \frac{1}{L_k(n+1)} \right) \left(\sum_{k=1}^{m-1} \frac{1}{L_k(n)} \right) \\
&= \frac{1}{2} \sum_{k=1}^{m-1} \frac{\Delta L_k(n)}{L_k(n)L_k(n+1)} \\
&\quad - \frac{1}{4} \sum_{k=1}^{m-1} \left\{ \frac{1}{L_k(n)L_k(n+1)} + \sum_{j=1}^{k-1} \left(\frac{1}{L_k(n)L_j(n+1)} + \frac{1}{L_j(n)L_k(n+1)} \right) \right\} \\
&= \frac{1}{4} \sum_{k=1}^{m-1} \left\{ \frac{2\Delta L_k(n) - 1}{L_k(n)L_k(n+1)} - \frac{1}{L_k(n)L_k(n+1)} \sum_{j=1}^{k-1} \left(\frac{L_k(n+1)}{L_j(n+1)} + \frac{L_k(n)}{L_j(n)} \right) \right\} \\
&= \frac{1}{4} \sum_{k=1}^{m-1} \left\{ \frac{2\Delta L_k(n) - 1}{L_k(n)L_k(n+1)} - \frac{2\Delta L_k(n) - 2}{L_k(n)L_k(n+1)} \right\} = \frac{1}{4} \sum_{k=1}^{m-1} \frac{1}{L_k(n)L_k(n+1)}.
\end{aligned}
$$

Hence we get the characteristic equation (1.3). Note that if z is a root of (1.3), $1-z$ is also a root of (1.3).

In case $\lambda \neq 1/4$, $\xi_m(n, z)$ and $\xi_m(n, 1-z)$ are linearly independent solutions of equation (3.1). In fact, we have

$$
\det \begin{pmatrix} \xi_m(n, z) & \xi_m(n, 1-z) \\ \Delta\xi_m(n, z) & \Delta\xi_m(n, 1-z) \end{pmatrix} = \frac{1-2z}{L_m(n)} \xi_m(n, z)\xi_m(n, 1-z) \neq 0.
$$

Hence, the linear combination of the functions $\xi_m(n, z)$ and $\xi_m(n, 1-z)$ is the general solution of equation (3.1) (see [3, Theorem 2.15]).

On the other hand, in case $\lambda = 1/4$, the characteristic equation (1.3) has double root $1/2$. As in the proof of the case $\lambda \neq 1/4$, $\xi_m(n, 1/2)$ is one of the solutions of equation (3.1). To get another solution of equation (3.1) with $\lambda = 1/4$, we put

$$u(n) = \Delta y(n) - \frac{1}{2}\left(\sum_{k=1}^{m} \frac{1}{L_k(n)}\right) y(n), \tag{3.2}$$

where $y(n)$ is a solution of (3.1) with $\lambda = 1/4$ satisfying $y(n_0) = 0$, $\Delta y(n_0) = 1/L_m(n_0)$. Then we see that $u(n)$ satisfies $u(n_0) = 1/L_m(n_0)$ and

$$\begin{aligned}\Delta u(n) =& \Delta^2 y(n) - \frac{1}{2}\left\{\sum_{k=1}^{m} \Delta\left(\frac{1}{L_k(n)}\right)\right\} y(n) - \frac{1}{2}\left(\sum_{k=1}^{m} \frac{1}{L_k(n+1)}\right)\Delta y(n)\\ =& -\frac{1}{2}\left(\sum_{k=1}^{m} \frac{1}{L_k(n+1)}\right) u(n).\end{aligned}$$

Since (2.2) and (2.3) can be rewritten as

$$\frac{1}{L_m(n)} = \Delta l_m(n)\left(1 + \frac{1}{2}\sum_{k=1}^{m} \frac{1}{L_k(n)}\right)$$

and

$$L_m(n+1) - \frac{1}{2}\sum_{k=1}^{m} \frac{L_m(n+1)}{L_k(n+1)} = L_{m+1}(n) + \frac{1}{2}\sum_{k=1}^{m} \frac{L_m(n)}{L_k(n)},$$

we have

$$\begin{aligned}u(n) &= u(n_0)\prod_{j=n_0}^{n-1}\left(1 - \frac{1}{2}\sum_{k=1}^{m} \frac{1}{L_k(j+1)}\right)\\ &= \frac{1}{L_m(n_0)}\prod_{j=n_0}^{n-1} \frac{1}{L_m(j+1)}\left(L_m(j+1) - \frac{1}{2}\sum_{k=1}^{m} \frac{L_m(j+1)}{L_k(j+1)}\right)\\ &= \frac{1}{L_m(n_0)}\prod_{j=n_0}^{n-1} \frac{1}{L_m(j+1)}\left(L_m(j) + \frac{1}{2}\sum_{k=1}^{m} \frac{L_m(j)}{L_k(j)}\right)\\ &= \frac{1}{L_m(n)}\prod_{j=n_0}^{n-1} \frac{1}{L_m(j)}\left(L_m(j) + \frac{1}{2}\sum_{k=1}^{m} \frac{L_m(j)}{L_k(j)}\right)\\ &= \frac{1}{L_m(n)}\prod_{j=n_0}^{n-1}\left(1 + \frac{1}{2}\sum_{k=1}^{m} \frac{1}{L_k(j)}\right)\\ &= \Delta l_m(n)\prod_{j=n_0}^{n}\left(1 + \frac{1}{2}\sum_{k=1}^{m} \frac{1}{L_k(j)}\right) = \Delta l_m(n)\,\xi_m(n+1, 1/2).\end{aligned}$$

Hence, together with (3.2), we have

$$\Delta y(n) = \frac{1}{2}\left(\sum_{k=1}^{m}\frac{1}{L_k(n)}\right) y(n) + \Delta l_m(n)\, \xi_m(n+1, 1/2).$$

Solving this first order nonhomogeneous equation with $y(n_0) = 0$, we obtain

$$\begin{aligned} y(n) &= \sum_{k=n_0}^{n-1}\left\{\Delta l_m(k)\, \xi_m(k+1, 1/2) \prod_{j=k+1}^{n-1}\left(1 + \frac{1}{2}\sum_{k=1}^{m}\frac{1}{L_k(j)}\right)\right\} \\ &= \xi_m(n, 1/2)\sum_{k=n_0}^{n-1}\Delta l_m(k) = \xi_m(n, 1/2)\,(l_m(n) - l_m(n_0)). \end{aligned}$$

Hence, the linear combination of $\xi_m(n, 1/2)$ and $\xi_m(n, 1/2)\, l_m(n)$ is the general solution of equation (3.1). As a matter of fact, the Casoratian of the functions $\xi_m(n, 1/2)$ and $\xi_m(n, 1/2)\, l_m(n)$ is

$$\det\begin{pmatrix} \xi_m(n, 1/2) & \xi_m(n, 1/2)\, l_m(n) \\ \Delta\xi_m(n, 1/2) & \Delta(\xi_m(n, 1/2)\, l_m(n)) \end{pmatrix} = \frac{(\xi_m(n, 1/2))^2}{L_m(n)} \neq 0.$$

The proof is now complete. □

In case $\lambda > 1/4$, the characteristic equation (1.3) has the conjugate roots $z = 1/2 \pm \mathrm{i}\alpha/2$, where $\alpha = \sqrt{4\lambda - 1}$. Hence, by Euler's formula, we have

$$\begin{aligned} \xi_m(n, z) &= \prod_{j=n_0}^{n-1}\left(1 + \frac{1}{2}\sum_{k=1}^{m-1}\frac{1}{L_k(j)} + \frac{z}{L_m(j)}\right) = \prod_{j=n_0}^{n-1}\left(1 + \frac{1}{2}\sum_{k=1}^{m}\frac{1}{L_k(j)} \pm \frac{\mathrm{i}\alpha}{2L_m(j)}\right) \\ &= \prod_{j=n_0}^{n-1} r(j)e^{\pm\mathrm{i}\theta(j)} = \left(\prod_{j=n_0}^{n-1} r(j)\right)\left\{\cos\left(\sum_{j=n_0}^{n-1}\theta(j)\right) \pm \mathrm{i}\sin\left(\sum_{j=n_0}^{n-1}\theta(j)\right)\right\}, \end{aligned}$$

where $r(n)$ and $\theta(n)$ satisfy

$$r(n)\cos\theta(n) = 1 + \frac{1}{2}\sum_{k=1}^{m}\frac{1}{L_k(n)} \quad \text{and} \quad r(n)\sin\theta(n) = \frac{\alpha}{2L_m(n)}. \tag{3.3}$$

Let

$$\varphi(n) = \left(\prod_{j=n_0}^{n-1} r(j)\right)\cos\left(\sum_{j=n_0}^{n-1}\theta(j)\right), \quad \psi(n) = \left(\prod_{j=n_0}^{n-1} r(j)\right)\sin\left(\sum_{j=n_0}^{n-1}\theta(j)\right).$$

Then $\varphi(n)$ and $\psi(n)$ are linearly independent solutions of equation (3.1). In fact, since

$$\Delta\varphi(n) = \varphi(n+1) - \varphi(n) = r(n)\left(\prod_{j=n_0}^{n-1} r(j)\right)\cos\left(\sum_{j=n_0}^{n-1}\theta(j) + \theta(n)\right) - \varphi(n)$$

$$
\begin{aligned}
&= r(n)\left\{\varphi(n)\cos\theta(n) - \psi(n)\sin\theta(n)\right\} - \varphi(n)\\
&= \{r(n)\cos\theta(n) - 1\}\varphi(n) - \{r(n)\sin\theta(n)\}\psi(n),
\end{aligned}
$$

$$
\begin{aligned}
\Delta\psi(n) = \psi(n+1) - \psi(n) &= r(n)\left(\prod_{j=n_0}^{n-1} r(j)\right)\sin\left(\sum_{j=n_0}^{n-1}\theta(j) + \theta(n)\right) - \psi(n)\\
&= r(n)\{\psi(n)\cos\theta(n) + \varphi(n)\sin\theta(n)\} - \psi(n)\\
&= \{r(n)\cos\theta(n) - 1\}\psi(n) + \{r(n)\sin\theta(n)\}\varphi(n),
\end{aligned}
$$

we have

$$
\det\begin{pmatrix} \varphi(n) & \psi(n) \\ \Delta\varphi(n) & \Delta\psi(n) \end{pmatrix} = r(n)\sin\theta(n)\left\{\varphi(n)^2 + \psi(n)^2\right\} = \frac{\alpha}{2L_m(n)}\Big(\prod_{j=n_0}^{n-1} r(j)\Big)^2 \neq 0.
$$

Thus the real solutions of equation (3.1) can be written as follows.

Corollary 3.2. *Let $\lambda > 1/4$. Then*

$$
x(n) = \left(\prod_{j=n_0}^{n-1} r(j)\right)\left\{K_5\sin\left(\sum_{j=n_0}^{n-1}\theta(j)\right) + K_6\cos\left(\sum_{j=n_0}^{n-1}\theta(j)\right)\right\},
$$

is a general solution of equation (3.1), *where $r(n)$ and $\theta(n)$ satisfy* (3.3).

4. OSCILLATORY BEHAVIOR

In this section, we examine the oscillatory behavior of solutions of equation (3.1). To get oscillation criteria, we need the following lemma.

Lemma 4.1. *Let $m \in \mathbb{N}$ and $0 < \mu \le 1/4$. Then there exist positive constants C_1 and C_2 such that*

$$
C_1\left(\prod_{i=0}^{m-2}\log_i(n)\right)^{1/2}(\log_{m-1}(n))^{\mu} \le \xi_m(n,\mu) \le C_2\left(\prod_{i=0}^{m-2}\log_i(n)\right)^{1/2}(\log_{m-1}(n))^{\mu} \tag{4.1}
$$

for n sufficiently large.

Proof. To begin with, we show the equality

$$
\xi_m(n,\mu) = \left[\prod_{k=0}^{m-2}\left\{\prod_{j=n_0}^{n-1}\left(1 + \frac{\Delta l_k(j)}{2l_k(j)}\right)\right\}\right]\prod_{j=n_0}^{n-1}\left(1 + \frac{\mu\Delta l_{m-1}(j)}{l_{m-1}(j)}\right). \tag{4.2}
$$

Since

$$
\xi_1(n,\mu) = \prod_{j=n_0}^{n-1}\left(1 + \frac{\mu}{L_1(j)}\right) = \prod_{j=n_0}^{n-1}\left(1 + \frac{\mu\Delta l_0(j)}{l_0(j)}\right),
$$

(4.2) with $m = 1$ is true. Assume that (4.2) with $m = p$ holds. Then, using (2.2), we have

$$\begin{aligned}
\xi_{p+1}(n,\mu) &= \prod_{j=n_0}^{n-1}\left(1+\frac{1}{2}\sum_{k=1}^{p}\frac{1}{L_k(j)}+\frac{\mu}{L_{p+1}(j)}\right)\\
&= \xi_p(n,1/2)\prod_{j=n_0}^{n-1}\left\{1+\left(1+\frac{1}{2}\sum_{k=1}^{p}\frac{1}{L_k(j)}\right)^{-1}\frac{\mu}{L_{p+1}(j)}\right\}\\
&= \prod_{k=0}^{p-1}\left\{\prod_{j=n_0}^{n-1}\left(1+\frac{\Delta l_k(j)}{2l_k(j)}\right)\right\}\prod_{j=n_0}^{n-1}\left(1+\frac{\mu\Delta l_p(j)}{l_p(j)}\right)
\end{aligned}$$

Thus, (4.2) with $m = p+1$ is also true.

As in the proof of Lemma 2.3, we can show that there exist $n_1 \in \mathbb{N}$ and $C > 0$ such that

$$\sum_{j=n_1}^{n-1}\prod_{i=0}^{k}\frac{1}{l_i(j)} \le \log_{k+1}(n) \quad \text{and} \quad \sum_{j=n_1}^{n-1}\prod_{i=0}^{k}\frac{1}{l_i(j+1)} \ge \log_{k+1}(n) - \log_{k+1}(n_1;C)$$

for $n \ge n_1$. Since $z - z^2/2 < \log(1+z) < z$ for $0 < z < 1$, we have

$$\begin{aligned}
\log\left\{\prod_{j=n_1}^{n-1}\left(1+\frac{\mu\Delta l_k(j)}{l_k(j)}\right)\right\} &= \sum_{j=n_1}^{n-1}\log\left(1+\frac{\mu\Delta l_k(j)}{l_k(j)}\right) \le \sum_{j=n_1}^{n-1}\frac{\mu\Delta l_k(j)}{l_k(j)}\\
&= \sum_{j=n_1}^{n-1}\frac{\mu}{l_k(j)}\prod_{i=0}^{k-1}\frac{2}{l_i(j)+l_i(j+1)} \le \mu\sum_{j=n_1}^{n-1}\prod_{i=0}^{k}\frac{1}{l_i(j)}\\
&\le \mu\log_{k+1}(n),\\
\log\left\{\prod_{j=n_1}^{n-1}\left(1+\frac{\mu\Delta l_k(j)}{l_k(j)}\right)\right\} &\ge \sum_{j=n_1}^{n-1}\left\{\left(\frac{\mu\Delta l_k(j)}{l_k(j)}\right)-\frac{1}{2}\left(\frac{\mu\Delta l_k(j)}{l_k(j)}\right)^2\right\}\\
&\ge \mu\sum_{j=n_1}^{n-1}\left(\prod_{i=0}^{k}\frac{1}{l_i(j+1)}\right)-\frac{\mu^2}{2}\sum_{j=n_1}^{n-1}\left(\prod_{i=0}^{k}\frac{1}{l_i(j)}\right)^2\\
&\ge \mu\{\log_{k+1}(n)-\log_{k+1}(n_1;C)\}-\frac{\mu^2}{2}\sum_{j=n_1}^{n-1}\frac{1}{l_0(j)^2}.
\end{aligned}$$

Hence, from the boundedness of the function $\sum_{j=n_1}^{n-1} 1/l_0(j)^2$, there exists $C_k > 0$ such that

$$C_k(\log_k(n))^{\mu} \le \prod_{j=n_1}^{n-1}\left(1+\frac{\mu\Delta l_k(j)}{l_k(j)}\right) \le (\log_k(n))^{\mu}$$

for $n \ge n_1$, and therefore, combining (4.2) with these inequalities, we obtain (4.1). $\square$

Using this lemma, we have the following oscillation criteria.

Corollary 4.2. *Equation* (3.1) *can be classified into two types as follows:*

(i) *if* $\lambda > 1/4$, *then all nontrivial solutions of equation* (3.1) *are oscillatory;*
(ii) *if* $0 < \lambda \le 1/4$, *then all nontrivial solutions of equation* (3.1) *are nonoscillatory.*

Proof. Let $\lambda > 1/4$. Then, using Corollary 3.2, we see that the general solution of equation (3.1) with $\lambda > 1/4$ is of the form

$$x(n) = \left(\prod_{j=n_0}^{n-1} r(j)\right)\left\{K_5 \sin\left(\sum_{j=n_0}^{n-1}\theta(j)\right) + K_6 \cos\left(\sum_{j=n_0}^{n-1}\theta(j)\right)\right\},$$

where $r(n)$ and $\theta(n)$ satisfy (3.3). Let $(K_5, K_6) \neq (0,0)$. Then we can rewrite as

$$x(n) = K_7 \left(\prod_{j=n_0}^{n-1} r(j)\right) \sin\left(\sum_{j=n_0}^{n-1}\theta(j) + K_8\right),$$

where $K_7 = \sqrt{K_5^2 + K_6^2}$, $\sin K_8 = K_5/K_7$ and $\cos K_8 = K_6/K_7$. Hence, together with (2.2), we get

$$\tan\theta(n) = \frac{\sqrt{4\lambda-1}}{2L_m(n)}\left(1 + \frac{1}{2}\sum_{i=1}^{m}\frac{1}{L_i(n)}\right)^{-1} = \frac{\sqrt{4\lambda-1}}{2}\Delta l_m(n) \to 0$$

as $n \to \infty$. Thus, there exists $n_1 \ge n_0$ such that $\tan\theta(n)/2 < \theta(n) < \pi/2$ for $n \ge n_1$. Hence we obtain

$$\sum_{j=n_1}^{n-1}\theta(j) \ge \frac{1}{2}\sum_{j=n_1}^{n-1}\tan\theta(j) = \frac{\sqrt{4\lambda-1}}{4}(l_m(n) - l_m(n_1)) \to \infty$$

as $n \to \infty$. We also see that

$$\left|\sum_{j=n_0}^{n}\theta(j) - \sum_{j=n_0}^{n-1}\theta(j)\right| = \theta(n) < \frac{\pi}{2},$$

that is, for any sufficiently large $p \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that

$$p\pi \le \sum_{j=n_0}^{n-1}\theta(j) + K_8 < (p+1)\pi.$$

Hence $x(n)$ is oscillatory, that is, all nontrivial solutions of equation (3.1) are oscillatory.

Let $\lambda = 1/4$. Then it is easy to check that all nontrivial solutions of equation (3.1) are nonoscillatory because $\xi_m(n, 1/2)$ is positive and $l_m(n) \to \infty$ as $n \to \infty$.

Let $\lambda < 1/4$. Then, without loss of generality, we may assume that $z > 1/2$. In fact, if $z < 1/2$, another root of characteristic equation (1.3) is greater than $1/2$. Since $\xi_m(n,z)$ is positive, we have

$$x(n) = K_1\xi_m(n,z) + K_2\xi_m(n,1-z) = \xi_m(n,z)\left(K_1 + K_2\frac{\xi_m(n,1-z)}{\xi_m(n,z)}\right).$$

Hence, from Lemma 4.1, there exists $C > 0$ such that

$$\frac{\xi_m(n,1-z)}{\xi_m(n,z)} \le C(\log_{m-1}(n))^{1-2z} \to 0$$

as $n \to \infty$. Thus $x(n)$ is nonoscillatory. The proof is now complete. □

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