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LIMIT-POINT CRITERIA FOR THE MATRIX STURM-LIOUVILLE OPERATOR AND ITS POWERS

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Abstract. We consider matrix Sturm-Liouville operators generated by the formal expression

$$l[y] = -(P(y' - Ry))' - R^* P(y' - Ry) + Qy,$$

in the space $L_n^2(I)$, $I := [0, \infty)$. Let the matrix functions $P := P(x), Q := Q(x)$ and $R := R(x)$ of order n ($n \in \mathbb{N}$) be defined on I, P is a nondegenerate matrix, P and Q are Hermitian matrices for $x \in I$ and the entries of the matrix functions P^{-1}, Q and R are measurable on I and integrable on each of its closed finite subintervals. The main purpose of this paper is to find conditions on the matrices P, Q and R that ensure the realization of the limit-point case for the minimal closed symmetric operator generated by $l^k[y]$ ($k \in \mathbb{N}$). In particular, we obtain limit-point conditions for Sturm-Liouville operators with matrix-valued distributional coefficients.

Keywords: quasi-derivative, quasi-differential operator, matrix Sturm-Liouville operator, deficiency numbers, distributions.

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1. PRELIMINARIES

Let $I := [0, +\infty)$ and let the complex-valued matrix functions $P := P(x), Q := Q(x)$ and $R := R(x)$ of order n ($n \in \mathbb{N}$) be defined on I. Suppose that P is a nondegenerate matrix, P and Q are Hermitian matrices for $x \in I$ and the entries of the matrix functions P^{-1}, Q and R are measurable on I and integrable on each of its closed finite subintervals (i.e. belong to the space $L^1_{loc}(I)$).

1.1. Let us consider the block matrix

$$F = \begin{pmatrix} R & P^{-1} \\ Q & -R^* \end{pmatrix}, \tag{1.1}$$

where * is the conjugation symbol. Let $AC_{n,loc}(I)$ be the space of complex-valued n-vector functions $y(x) = (y_1(x), y_2(x), \ldots, y_n(x))^t$, t is the transposition symbol, with locally absolutely continuous entries on I. Using matrix F, we define quasi-derivatives $y^{[i]}$ $(i = 0, 1, 2)$ of a given vector function $y \in AC_{n,loc}(I)$ by setting

$$y^{[0]} := y, \quad y^{[1]} := P(y' - Ry), \quad y^{[2]} := (y^{[1]})' + R^* y^{[1]} - Qy,$$

provided that $y^{[1]} \in AC_{n,loc}(I)$ and a quasi-differential expression

$$l[y](x) := -y^{[2]}(x), \quad x \in I.$$

Thus,

$$l[y] = -(P(y' - Ry))' - R^* P(y' - Ry) + Qy. \tag{1.2}$$

The set of complex-valued vector functions $\mathcal{D} := \{y(x)|\ y(x), y^{[1]}(x) \in AC_{n,loc}(I)\}$ is the domain of expression (1.2). For $y \in \mathcal{D}$ the expression $l[y]$ exists a.e. on I and locally integrable there.

We note here that for every pair of vector functions $f, g \in \mathcal{D}$ and for every pair of numbers α and β such that $0 \le \alpha \le \beta < \infty$ the following vector analogue of Green's formula holds:

$$\int_\alpha^\beta \{\langle l[f](x), g(x)\rangle - \langle f(x), l[g](x)\rangle\} dx = [f, g](\beta) - [f, g](\alpha), \tag{1.3}$$

where $\langle u, v\rangle = v^* u = \sum_{s=1}^n u_s \overline{v_s}$ is the inner product of vectors u and v and the form $[f, g](x)$ is defined by

$$[f, g](x) := \langle f(x), g^{[1]}(x)\rangle - \langle f^{[1]}(x), g(x)\rangle. \tag{1.4}$$

Let $L_n^2(I)$ be the Hilbert space of equivalence classes of all complex-valued n-vector functions Lebesgue measurable on I for which the sum of the squared absolute values of coordinates is Lebesgue integrable on I.

Let D_0' denote the set of all complex-valued vector functions $y \in \mathcal{D}$ which vanish outside of a compact subinterval of the interior of I (this subinterval may be different for different functions) and such that $l[y] \in L_n^2(I)$. This set is dense in $L_n^2(I)$. By formula $L_0' y = l[y]$ the expression l on the set D_0' defines a symmetric (not necessary closed) operator in $L_n^2(I)$. Let L_0 and D_0 denote the closure of this operator and its domain, respectively. The operator L_0 and operators associated with it are called matrix Sturm-Liouville operators.

Suppose further that $\lambda \in \mathbb{C}$ and $\Im\lambda \ne 0$, $\Im\lambda$ is the imaginary part of the complex number λ. Denote by R_λ and $R_{\overline{\lambda}}$ the ranges of $L_0 - \lambda I_n$ and $L_0 - \overline{\lambda} I_n$, I_n is the $n \times n$ identity matrix, respectively, and by $\mathcal{N}_\lambda$ and $\mathcal{N}_{\overline{\lambda}}$ the orthogonal complements in $L_n^2(I)$ of $R_{\overline{\lambda}}$ and R_λ. The spaces $\mathcal{N}_\lambda$ and $\mathcal{N}_{\overline{\lambda}}$ are called deficiency spaces. The numbers n_+ and n_- $(n_+ = dim\, \mathcal{N}_\lambda$, $n_- = dim\, \mathcal{N}_{\overline{\lambda}})$ are deficiency numbers of the operator L_0 in the upper-half or lower-half of the complex plane, respectively, moreover, the pair (n_+, n_-) is called the deficiency index of L_0.

As it was done, for example, in [1] and [18], it is possible to show that the deficiency numbers n_+ and n_- coincide with the maximum number of linearly independent solutions of the equation

$$l[y] = \lambda y$$

belonging to the space $L^2_n(I)$, when $\Im\lambda > 0$ and $\Im\lambda < 0$, respectively. They also satisfy the double inequality

$$n \le n_+, n_- \le 2n \tag{1.5}$$

and, in addition, $n_+ = 2n$ if and only if $n_- = 2n$. Using the analogy of the spectral theory of scalar Sturm-Liouville operators on the half-axis, one may say that the expression $l[y]$ (the operator L_0) is in the limit-point case if $n_+ = n_- = n$ or in the limit-circle case if $n_+ = n_- = 2n$, (see, for example, [1]).

Let us consider the equation

$$l[y](x) = f(x), \quad a \le x \le b, \tag{1.6}$$

where $[a, b]$ is a finite real interval and $f(x)$ some vector function in $L^1_n[a, b]$, $L^1_n[a, b]$ is the space of integrable n-vector functions on $[a, b]$.

Let vector function $\phi(x)$ be such that

$$\phi(x) \in AC_n[a, b], \quad \phi(a) = \phi(b) = 0. \tag{1.7}$$

If we scalar multiply (1.6) by $\phi(x)$, integrate over $[a, b]$ and integrate by parts on the left, we obtain

$$\int_a^b \{\langle Py', \phi'\rangle - \langle PRy, \phi'\rangle - \langle R^*Py', \phi\rangle + \langle (R^*PR + Q)y, \phi\rangle\} = \int_a^b \langle f, \phi\rangle. \tag{1.8}$$

If the equality (1.8) holds for all such functions $\phi(x)$, then one may say that y is a weak solution of (1.6).

Thus, if y satisfies (1.6), we have (1.8) for all functions $\phi(x)$ with (1.7). Conversely, one might ask whether if y satisfies (1.8) for all such $\phi(x)$, then y satisfies (1.6).

Let P_0, Q_0 and P_1 be Hermitian matrix functions of order n with Lebesgue measurable entries on I such that P_0^{-1} exists and $\|P_0^{-1}\|, \|P_0^{-1}\|\|P_1\|^2$, $\|P_0^{-1}\|\|Q_0\|^2$ are locally Lebesgue integrable. Let also $\Phi := P_1 + iQ_0$ and $\tilde{\Phi} := P_1 - iQ_0$. Assume that the block entries in the matrix (1.1) are represented as $P := P_0$, $Q := -\tilde{\Phi}P_0^{-1}\Phi$ and $R := P_0^{-1}\Phi$, then we obtain the block matrix

$$F = \begin{pmatrix} P_0^{-1}\Phi & P_0^{-1} \\ -\tilde{\Phi}P_0^{-1}\Phi & -\tilde{\Phi}P_0^{-1} \end{pmatrix}.$$

The conditions listed above on the matrix functions P_0, Q_0 and P_1 suggest that all entries of F belong to the space $L^1_{loc}(I)$. Detailed justification of this fact is given in [17].

As above, using the matrix F, we can define the quasi-derivatives of given vector function $y \in AC_{n,loc}(I)$, assuming

$$y^{[0]} := y, \quad y^{[1]} := P_0 y' - \Phi y, \quad y^{[2]} := (y^{[1]})' + \tilde{\Phi} P_0^{-1} y^{[1]} + \tilde{\Phi} P_0^{-1} \Phi y.$$

Suppose further that the elements of matrix function P_0 also belong to $L^1_{loc}(I)$, then the entries of Φ are locally integrable on I. Thus, if we interpret the derivative $'$ in the sense of distributions, then we can remove all the brackets in the expression $y^{[2]}$ and the quasi-differential expression $l[y]$ in terms of distributions can be written as

$$l[y] = -(P_0 y')' + i((Q_0 y)' + Q_0 y') + P_1' y. \tag{1.9}$$

In particular, if $P_0(x) = I$, $Q_0(x) = O$, O is the zero matrix and $P_1(x) = V(x)$, where $V(x)$ is a real-valued symmetric matrix function such that the entries of the matrix $V^2(x)$ are locally integrable on I, then the expression (1.9) takes the form

$$l[y] = -y'' + V' y.$$

Detailed description of scalar quasi-differential expressions of second order with generalized derivatives is given in [14] and matrix expressions in [15–17].

We note here that in this case the relation (1.8) takes the form

$$\int_a^b \{\langle P_0 y', \phi' \rangle - \langle \Phi y, \phi' \rangle - \langle \tilde{\Phi} y', \phi \rangle\} = \int_a^b \langle f, \phi \rangle.$$

1.2. Let us consider the block matrix F of order $2kn$ ($k \in \mathbb{N}, k > 1$):

$$F = \begin{pmatrix} R & P^{-1} & O & O & O & O & \cdots & O & O \\ Q & -R^* & I_n & O & O & O & \cdots & O & O \\ O & O & R & P^{-1} & O & O & \cdots & O & O \\ O & O & Q & -R^* & I_n & O & \cdots & O & O \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ O & O & O & O & O & O & \cdots & R & P^{-1} \\ O & O & O & O & O & O & \cdots & Q & -R^* \end{pmatrix},$$

where I_n is the $n \times n$ identity matrix and P, Q, R satisfy the conditions listed in Subsection 1.1.

As above, using the matrix F, we define the quasi-derivatives $y^{[i]}$ ($i = 0, 1, \ldots, 2k$) of a given vector function $y \in AC_{n,loc}(I)$ assuming

$$y^{[0]} := y, \quad y^{[1]} := P(y' - Ry), \quad y^{[2]} := (y^{[1]})' + R^* y^{[1]} - Qy,$$

$$y^{[3]} := P((y^{[2]})' - Ry^{[2]}), \quad y^{[4]} := (y^{[3]})' + R^* y^{[3]} - Qy^{[2]}, \ldots,$$

$$y^{[2k-1]} := P((y^{[2k-2]})' - Ry^{[2k-2]}), \quad y^{[2k]} := (y^{[2k-1]})' + R^* y^{[2k-1]} - Qy^{[2k-2]},$$

provided that $y^{[i]} \in AC_{n,loc}(I)$ $(i = 1, \dots, 2k-1)$ and a quasi-differential expression

$$l^k[y](x) := (-1)^k y^{[2k]}(x), \quad x \in I. \tag{1.10}$$

Note that the quasi-differential expression $l^k[y]$ constructed in this way is a formal k-power of (1.2). The explicit form of this expression is too large, because of it we do not present it here.

The set of complex-valued vector functions

$$\mathcal{D} := \{y(x) \mid y(x), y^{[i]}(x) \in AC_{n,loc}(I), i = 1, \dots, 2k-1\}$$

is the domain of (1.10). For $y \in \mathcal{D}$ the expression $l^k[y]$ exists a.e. on I and locally integrable there.

Similarly as in Subsection 1.1, we can define a minimal closed symmetric operator L_0 generated by the expression (1.10) and introduce the concept of the deficiency numbers of this operator. And in this case, the numbers n_+ and n_- coincide with the maximum number of linearly independent solutions of the equation

$$l^k[y] = \lambda y$$

belonging to the space $L_n^2(I)$ when $\Im\lambda > 0$ or $\Im\lambda < 0$. Moreover, they satisfy double inequality $nk \le n_+, n_- \le 2kn$ and $n_+ = 2kn$ if and only if $n_- = 2kn$.

Additionally, assuming that the matrix functions P_0, P_1, Q_0 satisfy the conditions listed in Subsection 1.1, we can define a formal k power of the quasi-differential expression (1.9) where the derivatives are understood in the generalized sense.

As example, we present here the explicit form of $l^2[y]$ if the matrix F takes the form

$$F = \begin{pmatrix} V(x) & I_n & O & O \\ -V^2(x) & -V(x) & I_n & O \\ O & O & V(x) & I_n \\ O & O & -V^2(x) & -V(x) \end{pmatrix},$$

where $V(x)$ is a matrix function with sufficiently smooth entries. In this case the quasi-differential expression $l^2[y]$ has the form

$$l^2[y] = y^{(4)} - 2(V'(x)y')' + ((V'(x))^2 - V^{(3)}(x))y.$$

1.3. Let us mention here that one of the important problems in the spectral theory of the matrix Sturm-Liouville operators is to determine the deficiency numbers of the operator L_0. In particular, to find the conditions on the entries of the matrix function F that ensure the realization of the given pair (n_-, n_+). One of the first works in this direction was a paper of V.B. Lidskii [12]. Later this problem for classical matrix Sturm-Liouville operators and operators with generalized coefficients was discussed in many works, see, for instance, [3–5, 9, 11–13, 15–17, 19–22] (and also the references therein). In particular, for example, in [17] the authors obtained the conditions of nonmaximality of deficiency numbers of operator L_0 generated by (1.2). M.S.P. Eastham in [4] investigated the values of the deficiency numbers depending on the indices of power functions which are entries of the matrix coefficient of the second order differential operator. In [19] the method presented in [2] for scalar (quasi) differential operators was generalized to operators generated by the matrix expression $-y''+P(x)y$. In [13] the authors obtained several criteria for a matrix Sturm-Liouville-type equation of special form to have maximal deficiency indices. In [3] it is presented the conditions on the coefficients of the expression (1.2) such that the deficiency numbers of the operator L_0 are defined as the number of roots of a special kind polynomial lying in the left half-plane. The authors of [11] established a relationship between the spectral properties of the matrix Schrödinger operator with point interactions on the half-axis and block Jacobi matrices of certain class. In particular, they constructed examples of such operators with arbitrary possible equal values of the deficiency numbers. We also mention that in [1, 23] the deficiency numbers problem for matrix operators generated by differential expressions of even order higher than the second is considered and in [6–8, 10] this problem was discussed for powers of ordinary (quasi)differential expressions.

The main goal of this work is to obtain new sufficient conditions on the entries of the matrices P, Q and R when the limit-point case can be realized for the expressions $l[y]$ and $l^k[y]$ $(k>1)$ constructed above in Subsections 1.1 and 1.2 (Theorems 2.1 and 2.10). In particular, we apply these results to obtain new interval limit-point criteria (Corollary 2.11 and 2.12) and consider two examples of matrix Sturm-Liouville operators with minimal deficiency numbers. We also note here that our approach is based on the equality (1.8) and generalizes some results of [2] and [8] to the matrix case. This method allows to obtain the limit-point conditions for the operators with distributional coefficients and, in particular, for the matrix Sturm-Liouville operator with point interactions.

2. LIMIT-POINT CONDITIONS

One of the main theorem is the following:

Theorem 2.1. *Let w be a scalar non-negative absolutely continuous function on I, suppose that the $n\times n$ matrix functions P, Q and R satisfy the conditions listed above in Subsection 1.1 and there exist positive constants K_1, K_2, K_3, K_4, K_5 and a, such that for $x \geq a$*

(i) $P \geq K_1\|P\|I_n$,
(ii) $\frac{w^2}{\|P\|} \leq K_2$,
(iii) $\|P\|\left(\frac{w}{\|P\|^{\frac{1}{2}}}\right)'^2 \leq K_3$,
(iv) $w\|PR\| \leq K_4\|P\|$,
(v) $w^2(R^*PR+Q) \geq -K_5\|P\|I_n$,
(vi) $\int_a^\infty \frac{w}{\|P\|} = \infty$,

where $\|\cdot\|$ is the self-adjoint norm. Then the operator L_0 generated by (1.2) *is in the limit-point case.*

The proof of this theorem is established with the help of a few lemmas.

Let us mention that everywhere below the symbols $K, K_1, K_2, \dots$ denote various positive constants and $\epsilon, \epsilon_1, \epsilon_2, \dots$ denote "small" positive constants. These constants will not necessarily be the same on each occurrence. And we write $K(\epsilon)$ when we indicate the dependence of K on ϵ.

Lemma 2.2. *Let w be as in Theorem 2.1 and let v be a scalar non-negative absolutely continuous function with support in a compact $J \subset I$. Suppose that there exist positive constants K_i, $(i = 1, 2, \dots, 7)$ independent of J such that* (i)–(v) *in Theorem 2.1 are satisfied on J and also*

(a) $\|P\|v' \le K_6 w$,
(b) $v \le K_7$.

Let $l[y](x) = f(x)$. Then, given any $\epsilon > 0$, there exists a positive constant $K(\epsilon)$, independent of J, such that

$$\int_J v^{2+\alpha} w^2 \|y'(x)\|^2 dx \le \epsilon \int_J v^{\alpha} \|y(x)\|^2 dx + K(\epsilon) \int_J v^{4+\alpha} \|l^2[y](x)\| dx. \tag{2.1}$$

Proof. The proof involves the use of (1.8) and the simple inequality

$$2|ab| \le \epsilon a^2 + (1/\epsilon) b^2$$

which holds for arbitrary $\epsilon > 0$. All integrals are over J and we omit the dx symbol for brevity.

Using (1.8), we obtain

$$\Re \int \langle Py', \phi' \rangle - \int |\langle PRy, \phi' \rangle + \langle R^* Py', \phi \rangle| + \Re \int \langle (R^* PR + Q) y, \phi \rangle \le \int |\langle f, \phi \rangle|, \tag{2.2}$$

here $\Re f$ is a real part of function f.

Assume that $\phi = v^{2+\alpha} \frac{w^2}{\|P\|} y$.

Next, we note that

$$\begin{aligned} \Re \int \Big\langle Py', \Big(v^{2+\alpha} \frac{w^2}{\|P\|} y\Big)' \Big\rangle \ge \int \{ \langle P \Big(v^{1+\alpha/2} \frac{w}{\|P\|^{1/2}} y\Big)', \Big(v^{1+\alpha/2} \frac{w}{\|P\|^{1/2}} y\Big)' \rangle \\ - \Big| \Big\langle P \Big(v^{1+\alpha/2} \frac{w}{\|P\|^{1/2}} y\Big)', \Big(v^{1+\alpha/2} \frac{w}{\|P\|^{1/2}}\Big)' y \Big\rangle \\ - \Big\langle P \Big(v^{1+\alpha/2} \frac{w}{\|P\|^{1/2}}\Big)' y, \Big(v^{1+\alpha/2} \frac{w}{\|P\|^{1/2}} y\Big)' \Big\rangle \Big| \\ - \Big| \Big\langle P \Big(v^{1+\alpha/2} \frac{w}{\|P\|^{1/2}}\Big)' y, \Big(v^{1+\alpha/2} \frac{w}{\|P\|^{1/2}}\Big)' y \Big\rangle \Big|. \end{aligned}$$

Furthermore, using (i)–(iii) of Theorem 2.1, the Cauchy-Schwarz inequality and that P is Hermitian matrix, we get

$$\Re \int \Big\langle Py', \Big(v^{2+\alpha} \frac{w^2}{\|P\|} y\Big)' \Big\rangle \ge K_1 \int \|P\| \Big\| \Big(v^{1+\alpha/2} \frac{w}{\|P\|^{1/2}} y\Big)' \Big\|^2 - K(\epsilon_1) \int v^{\alpha} \|y\|^2. \tag{2.3}$$

Next, we estimate the expression

$$\int \Big| \Big\langle PRy, \Big(v^{2+\alpha} \frac{w^2}{\|P\|} y\Big)' \Big\rangle + \Big\langle R^* P y', \Big(v^{2+\alpha} \frac{w^2}{\|P\|}\Big) y \Big\rangle \Big|.$$

Since the norm $\|\cdot\|$ is self-adjoint, then $\|PR\| = \|R^*P\|$. Using also the properties of inner products, norms and the condition (ii)–(iv) of Theorem 2.1 and (a),(b) of Lemma 2.2 we obtain

$$\begin{aligned} &\Big| \Big\langle PRy, \Big(v^{2+\alpha} \frac{w^2}{\|P\|} y\Big)' \Big\rangle + \Big\langle R^* P y', \Big(v^{2+\alpha} \frac{w^2}{\|P\|}\Big) y \Big\rangle \Big| \\ &\leq \|PR\| \left(v^{1+\alpha/2} \frac{w}{\|P\|^{1/2}}\right) \left(v^{1+\alpha/2} \frac{w}{\|P\|^{1/2}} y\right)' \|y\| \\ &\quad + \|PR\| \left(v^{1+\alpha/2} \frac{w}{\|P\|^{1/2}}\right)' \|y\|^2 + \|PR\| \left(v^{2+\alpha} \frac{w^2}{\|P\|}\right) \|y'\| \|y\| \\ &\leq \frac{\epsilon_1 K_3}{2} \|P\| \left(v^{1+\alpha/2} \frac{w}{\|P\|^{1/2}} y\right)'^2 + \frac{1}{2} \epsilon_2 v^{2+\alpha} w^2 \|y'\|^2 + K(\epsilon_1, \epsilon_2) v^\alpha \|y\|^2. \end{aligned} \tag{2.4}$$

Furthermore, using (v), we obtain

$$\Re \int \Big\langle -(R^*PR + Q)y, v^{2+\alpha} \frac{w^2}{\|P\|} y \Big\rangle \leq K \int v^\alpha \|y\|^2. \tag{2.5}$$

Also we shall need the estimate

$$\frac{1}{1+\epsilon_3} v^{2+\alpha} w^2 \|y'\|^2 \leq \|P\| \left\| \left(v^{1+\alpha/2} \frac{w}{\|P\|^{1/2}} y\right)' \right\|^2 + K(\epsilon_3, \epsilon_4) v^\alpha \|y\|^2. \tag{2.6}$$

This inequality immediately follows from the product rule for $\left(v^{1+\frac{\alpha}{2}} \frac{w}{\|P\|^{\frac{1}{2}}} y\right)'$ and the conditions (ii), (iii) of Theorem 2.1 and (a), (b) of Lemma 2.2.

Next, we note here that

$$\int |\langle f, \phi \rangle| = \int \Big| \Big\langle f, v^{2+\alpha} \frac{w^2}{\|P\|} y \Big\rangle \Big| \leq \epsilon \int v^{4+\alpha} \|f\|^2 + K(\epsilon) \int v^\alpha \|y\|^2. \tag{2.7}$$

Substitute now (2.3)–(2.7) into (2.2) and choose $\epsilon_1, \epsilon_2, \epsilon_3$ sufficiently small so that $(K_1 - \epsilon_1 K_3/2)(1+\epsilon_3)^{-1} - \epsilon_2/2 > 0$ we obtain the inequality (2.1). □

From the Green's formula (1.3) we obtain the following lemma.

Lemma 2.3. *If y_1, y_2 are solutions of*

$$l[y_1](x) = f_1(x), \quad l[y_2](x) = f_2(x) \tag{2.8}$$

and $y_1, y_2, f_1, f_2 \in L^2_n(I)$ then the form $[y_1, y_2](x)$ (see (1.4)) tends to a finite limit as $x \to \infty$.

Moreover, we get the ensuing lemma.

Lemma 2.4. *If f_1, f_2 in $L_n^2(I)$ and for every pair of solutions $y_1, y_2 \in L_n^2(I)$ of (2.8)*

$$[y_1, y_2](x) \to 0, \quad x \to \infty,$$

then the set of such solutions has dimension at most n.

Proof of Theorem 2.1. Here we apply the ideas of [8] to the matrix case.
From (vi) it follows that, for some $b > a$, $w(b) > 0$ and hence, since w is continuous, there is a $\delta > 0$ such that $\frac{w}{\|P\|} > 0$ on $[b, b+\delta]$. Define

$$\theta(x) = \int_b^x \frac{w}{\|P\|}, \quad x \geq b,$$

$$v(x) = \begin{cases} 1 - exp(\theta(x) - \theta(X)), & b + \delta \leq x \leq X, \\ 0, & x \geq X, \end{cases}$$

and in $[b, b+\delta)$ choose v such that it vanishes in a right neighborhood of b, $0 \leq v(x) \leq 1$ and v has a continuous derivative in $[b, b+\delta]$. Then from (ii)

$$v' = O\left(\frac{w}{\|P\|}\right).$$

We also choose X such that $\theta(X) > \ln 2$ and T such that $\theta(T) = \theta(X) - \ln 2$. Then

$$v(x) \geq \frac{1}{2}, \quad b + \delta \leq x \leq T. \tag{2.9}$$

Let us consider

$$\left| \int_b^X \frac{vw}{\|P\|}[f, g] \right| \leq \int_b^X \frac{vw}{\|P\|} \left\{ |\langle f, g^{[1]} \rangle| + |\langle f^{[1]}, g \rangle| \right\}.$$

Using now the properties of inner products, norms and (2.1) we obtain that

$$\left| \int_b^X \frac{vw}{\|P\|}[f, g] \right| \leq K \int_b^X \|f\|^2 + \|g\|^2 + \|l[f]\|^2 + \|l[g]\|^2. \tag{2.10}$$

By Lemma 2.3, we know that $[f, g]$ tends to a finite limit. Assume that this limit is $c \neq 0$ and show that this leads to a contradiction with (vi).

Supposing that $[f, g](x) \geq c$ for large x, say $x \geq \gamma$ and choosing $a > \gamma$. For f, g satisfying (2.8) of Lemma 2.3 we have from (2.9) and (2.10) that

$$\frac{c}{2} \int_{b+\delta}^T \frac{w}{\|P\|} \leq \int_b^X \frac{vw}{\|P\|}[f, g] \leq K.$$

It leads to a contradiction with (vi). Therefore, $[f,g] \to 0$ when $x \to \infty$. Using now Lemma 2.4 and the inequality (1.5) we obtain that the operator L_0 generated by (1.2) is in the limit-point case. □

Corollary 2.5. *Let w be a scalar non-negative absolutely continuous function on I, suppose that the $n \times n$ matrix functions P_0, P_1 and Q_0 satisfy the conditions listed above in Subsection 1.1 and there exist positive constants K_1, K_2, K_3, K_4 and a, such that for $x \geq a$*

(i) $P_0 \geq K_1 \|P_0\| I_n$,
(ii) $\frac{w^2}{\|P_0\|} \leq K_2$,
(iii) $\|P_0\| \left(\frac{w}{\|P_0\|^{\frac{1}{2}}} \right)'^2 \leq K_3$,
(iv) $w\|P_1 + iQ_0\| \leq K_4 \|P_0\|$,
(v) $\int_a^\infty \frac{w}{\|P_0\|} = \infty$.

where $\|\cdot\|$ is the self-adjoint norm. Then the operator L_0 generated by (1.9) *is in the limit-point case.*

To prove the theorem about deficiency numbers of the operator generated by $l^k[y]$, $k > 1$ we need some additional lemma.

Lemma 2.6. *Suppose that all hypothesis of Lemma 2.2 are satisfied. Then, given any $\epsilon > 0$, there exists a positive constant $K(\epsilon)$, independent of J, such that*

$$\int_J v^{4j} \|l^j[y]\|^2 dx \leq \epsilon \int_J v^{4(j+1)} \|l^{j+1}[y]\|^2 dx + K(\epsilon) \int_J v^{4(j-1)} \|l^{j-1}[y]\|^2 dx. \tag{2.11}$$

Proof. In the proof all integrals are over J and we omit dx symbol for brevity.
Put $f = l^{j-1}[y]$, $g = l[f] = l^j[y]$. Then

$$\begin{aligned} \int v^{4j} \langle l^{j-1}[y], l^{j+1}[y] \rangle &= \int v^{4j} \langle f, l[g] \rangle \\ &= \int v^{4j} \langle l[f], g \rangle + \int (v^{4j})' \langle Pf, g' \rangle \\ &\quad - \int (v^{4j})' \langle R^* Pf, g \rangle - \int (v^{4j})' \langle Pf', g \rangle + \int (v^{4j})' \langle PRf, g \rangle. \end{aligned} \tag{2.12}$$

Using (a) of Lemma 2.2, we note that

$$(v^{4j})' \leq K v^{4j-1} \frac{w}{\|P\|}.$$

Therefore, we obtain

$$\left| \int (v^{4j})' \langle Pf, g' \rangle \right| \leq \int |(v^{4j})'| \|P\| \|f\| \|g'\| \leq K \int v^{4j-1} w \|f\| \|g'\|.$$

From (2.1) with $\alpha = 4(j-1)$ we have

$$\left|\int (v^{4j})'\langle Pf, g'\rangle\right| \le K_1(\epsilon_1, \epsilon_2)\int v^{4(j+1)}\|l^2[f]\|^2 + K_2(\epsilon_1,\epsilon_2)\int v^{4j}\|l[f]\|^2 + K_3(\epsilon_1)\int v^{4(j-1)}\|f\|^2. \tag{2.13}$$

And

$$\left|\int (v^{4j})'\langle Pf', g\rangle\right| \le K_4(\epsilon_3)\int v^{4j}\|l[f]\|^2 + K_5(\epsilon_3)\int v^{4(j-1)}\|f\|^2. \tag{2.14}$$

Similarly, using (iv) of Theorem 2.1, we get

$$\left|\int (v^{4j})'\langle R^* Pf, g\rangle\right| \le K_6(\epsilon_4)\int v^{4(j-1)}\|f\|^2 + K_7\int v^{4j}\|l[f]\|^2. \tag{2.15}$$

Therefore, substituting (2.13)–(2.15) into (2.12), we obtain (2.11). □

Lemma 2.7. *Under the hypothesis of Lemma 2.2, given $\epsilon > 0$ there exists a $K(\epsilon) > 0$, independent of J, such that*

$$\int_J v^{4j}\|l^j[y]\|^2 dx \le \epsilon\int_J v^{4k}\|l^k[y]\|^2 dx + K(\epsilon)\int_J \|y\|^2 dx \tag{2.16}$$

for $j = 1, 2, \ldots, k-1$.

Proof. The proof is by induction on k and almost exactly the same as the proof of Lemma 2.4 in [10, p. 91]. □

Definition 2.8 (see [10])**.** Let $l[y]$ be a symmetric differential expression and let $k \in \mathbb{N}, k > 1$. We say that $l^k[y]$ is partially separated if y and $l^k[y]$ in $L^2_n(I)$ together imply that $l^r[y]$ is in $L^2_n(I)$ for $r = 1, 2, \ldots, k-1$.

The next lemma follows from [10, Corollary 5.3.6].

Lemma 2.9. *If $l[y]$ is limit-point then $l^k[y]$, $k > 1$ is limit-point if and only if $l^k[y]$ is partially separated.*

Theorem 2.10. *Suppose the hypothesis of Theorem 2.1 hold. Then $l^k[y]$ is limit-point for any $k \in \mathbb{N}$.*

Proof. Let us show that the expression $l^k[y]$ is partially separated.

Using the definition of v given in the proof of Theorem 2.1, Lemma 2.7 and (2.16) we get

$$\left(\frac{1}{2}\right)^{4j}\int_{b+\delta}^{t}\|l^j[y]\|^2 \le \int_{b+\delta}^{X} v^{4j}\|l^j[y]\|^2 \le K\int_0^\infty \{\|l^k[y]\|^2 + \|y\|^2\}.$$

Since $t \to \infty$ as $X \to \infty$ we can conclude that $l^j[y]$ is in $L^2_n(I)$ for $j = 1, 2, \ldots, k-1$ and that $l^k[y]$ is partially separated. Therefore, the statement of Theorem 2.10 follows from Lemma 2.9. □

Now we give some applications of Theorems 2.1 and 2.10.

Corollary 2.11. *Let*

$$[a_m, b_m], \quad m = 1, 2, \ldots$$

be a sequence of intervals such that

$$0 \le a_1 < b_1 \le a_2 < b_2 \le \ldots$$

and $M_1, M_2, \ldots$ *a sequence of positive numbers such that*

$$\sum_{m=1}^{\infty} \frac{(b_m - a_m)^2}{M_m} = \infty. \tag{2.17}$$

For some fixed $K > 0$ *suppose that in each* $[a_m, b_m]$ *we have*

(i) $P(x) \ge M_m I_n, \quad \|P(x)\| \le K M_m,$
(ii) $(b_m - a_m)\|PR\| \le K M_m,$
(iii) $(b_m - a_m)^2 (R^* P R + Q) \ge -K M_m I_n,$

Then the operator L_0 *generated by* (1.2) *and all its powers* $l^k[y]$, $k = 2, 3, \ldots$ *are in the limit-point case.*

Proof. Taking

$$w(x) = \begin{cases} x - a_m, & a_m \le x \le (a_m + b_m)/2, \\ b_m - x, & (a_m + b_m)/2 \le x \le b_m, \\ 0, & \text{otherwise} \end{cases}$$

in Theorem 2.1 and applying Theorem 2.10 we get the corollary. □

Corollary 2.12. *Let* $[a_m, b_m]$ *and* M_m, $m = 1, 2, \ldots$ *be sequences of intervals and positive numbers satisfying* (2.17) *as in Corollary 2.11. And for some fixed* $K > 0$ *suppose that in each* $[a_m, b_m]$ *we have*

(i) $P_0(x) \ge M_m I_n, \quad \|P_0\| \le K M_m,$
(ii) $(b_m - a_m)\|P_1 + iQ_0\| \le K M_m,$

Then the operator L_0 *generated by* (1.9) *and all its powers* $l^k[y]$, $k = 2, 3, \ldots$, *are in the limit-point case.*

3. EXAMPLES

3.1. Let us consider the differential expression

$$l[y] = -(P_0 y')' + P_1' y \tag{3.1}$$

on $I := [a, +\infty)$, $a > 0$, where $P_0 = x^{\alpha} I_n$, $P_1 = x^{-\beta} Q(x^{\gamma})$, $\alpha \in [0, 2]$, $\beta \ge 0$ and $Q(x^{\gamma})$ is $n \times n$ periodic matrix function with continuous entries. Applying Corollary 2.5 with

$w = x^{\alpha-1}$ to this expression and observing that $x^{-\beta-1}Q(x^\gamma)y$ is a boundary operator, we obtain that the operator, generated by

$$-(x^\alpha y')' + x^\delta Q'(x^\gamma)y, \quad \delta \leq \gamma$$

is in the limit-point case and all its powers are also limit-point.

Remark 3.1. We note here that the expression $-y'' + x^\delta Q(x^\gamma)y$, Q is $n \times n$ periodic matrix function with continuous entries is discussed in detail in [19].

3.2. Let us consider the differential expression (3.1). Suppose that $0 = x_0 < x_1 < x_2 < \ldots$ and $\lim_{m\to\infty} x_m = \infty$. Assume that $P_1(x)$ is a piecewise continuously differentiable matrix function on I and x_m $(m = 0, 1, 2\ldots)$ are points of discontinuity of the first kind of $P_1(x)$. Suppose also that $P_1(x) = Q_m(x)$, $(x_m - x_{m-1})\|Q_m\| \leq k$ $(k > 0)$ on $(x_{m-1}, x_m]$ and

$$\mathcal{H}_m = (h_{ij}^m)_{i,j=1}^n := Q_{m+1}(x_m + 0) - Q_m(x_m - 0)$$

is a jump of the matrix function $P_1(x)$ in x_m. Assume also

$$\sum_{m=1}^{\infty} (x_m - x_{m-1})^2 = \infty.$$

Then, applying Corollary 2.12, we obtain that the operator, generated by

$$-y'' + (P_1'(x) + \sum_{k=1}^{\infty} \mathcal{H}_m \delta(x - x_m))y,$$

here $\delta(x)$ is the Dirac δ-function and $P_1'(x)$ is a derivative of $P_1(x)$ when $x \neq x_m$ $(m = 0, 1, 2\ldots)$ is in the limit-point case and all its powers are also limit-point.

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