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DERIVATIVE OF A FUNCTION AT A POINT AND INTEGRAL BY PROFESSOR IGOR KLUVÁNEK

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Abstract
Professor Igor Kluvánek had developed a unique course of calculus (mathematical analysis) to teach students the differential and integral calculus. In the present paper, this concept is briefly outlined. The notion of derivative is introduced via continuity. The definition of integral given in this article applies an idea of Archimedes.

1. Introduction

Professor Igor Kluvánek was an important Slovak mathematician. He prepared a new course of mathematical analysis during his 23 years stay at the Flinders University in Adelaide, South Australia. The goal of his course was to clarify and simplify the calculus teaching for students. In this way, we can explain the notions of calculus for wide scale people. Led by this idea, Professor Kluvánek prepared a course of mathematical analysis which was oriented to explain, to make easier understanding and to develop calculus terms. His course was not published up that time and so the co-workers at the Department of Mathematics at Pedagogical Faculty of The Catholic University in Ružomberok completed compiled sources of Kluvánek. The brother of Igor Kluvánek, Professor Pavol Kluvánek helps to perform this research work. There was published the first and second parts of the course

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of mathematical analysis. The third part will be published at the end of this year. In this article we will show the definitions of derivative and integral used by Professor Igor Kluvánek.

2. Differentiation and derivative of a function at a point

Kluvánek introduces the notion of the differentiation of a function at a point via continuity:

A function f is said to be differentiable at a point x if there exists a function φ, continuous at 0, such that $f(x+u) - f(x) = \varphi(u)u$ for every u in a neighbourhood of 0. The value $\varphi(0)$ is called the derivative of the function f at the point x.

More information about advantages of this type of definition can be found in [11]. The most important advantage of the Kluvánek conception is the unique art of definition of the key notions of the calculus, which the students can understand better. Following [3], we show suitability symbolism for the derivative:

"We can find in the better textbooks on calculus that the symbols dy/dx and $f'(x)$ can change each other. One can often find also the third symbol $Df(x)$.

Equation $f'(x) = Df(x)$ does not produce any problems. Under some restrictions, it is possible to change the operator of the differentiation and in this case we can use the both symbols $f'(x)$ and $Df(x)$. I don't believe that the author of the appendix F (see [3]) knows about what he writes.

This author allows me to write

$$f'(z) = \frac{dy}{dz} = Dz^2 = 2z$$

instead of

$$f'(x) = \frac{dy}{dx} = Dx^2 = 2x.$$

He can have a significant objections if I write

$$f'(3) = \frac{dy}{d3} = D9 = 6.$$

In these examples we see a problem which we can have in different contexts. If there is a difference between $f'(x)$ and dy/dx only in the choice of symbol similar to the difference between $f'(x)$ and $Df(x)$, it will be hard to explain, why some authors in some contexts use the symbol dy/dx, however some of them prefer other symbols."

3. Differentiation of composite function

Kluvánek criticized the proof in the course of pure mathematics of G.H. Hardy (see [1]), because Hardy used the limits instead of continuity. This confusion is copied by some authors in their calculus textbooks. We show now, how it is possible to avert this confusion (see [4]):

Theorem 1. *If a function f is differentiable at a point x and a function g is differentiable at the point $y = f(x)$, then the composite function $h = g \circ f$ is differentiable at the point x and $Dh(x) = Dg(y)Df(x)$.*

Proof. Since f is differentiable at x, there exists a function φ continuous at 0 such that $\varphi(0) = Df(x)$ and

$$f(x+u) - f(x) = \varphi(u)u$$

for all u in a neighbourhood of 0. Since g is differentiable at y, there exists a function ψ continuous at 0 such that $\psi(0) = Dg(y)$ and

$$g(x+v) - g(x) = \psi(v)v$$

for all v in a neighbourhood of 0.

Hence,

$$h(x+u) - h(x) = g\left(f(x+u)\right) - g\left(f(x)\right) =$$

$$= g\left(f(x) + (f(x+u) - f(x))\right) - g\left(f(x)\right) = g\left(f(x) + \varphi(u)u\right) - g\left(y\right) =$$

$$= \psi\left(\varphi(u)u\right)\varphi(u)u$$

for every u in a neighbourhood of 0.

Let

$$\chi(u) = \psi\left(\varphi(u)u\right)\varphi(u)u$$

for every u such that $\varphi(u)u$ belongs to the domain of the function ψ. By properties of continuous functions, the function χ is continuous at 0 and our calculation shows that

$$h(x+u) - h(x) = \chi(u)u$$

for every u in a neighbourhood of 0. Hence, the function h is differentiable at x and $Dh(x) = \chi(0) = \psi(0)\varphi(0) = Dg(y)Df(x)$. □

4. Introduction to the notion of integral

Example 1. During the first 19 weeks of the financial year, the wage of an employee was 186 Euro weekly. Then he was promoted and had 203.50 Euro weekly. A month before the end of the financial year, due to general salaries and wages increase, his wage was increased to 211.30 Euro weekly. This last month represents 4.4 working weeks (four full weeks and two working days, each representing 0.2 of a working week). Indicate how the weekly wage depends on time.

Solution:

If we want to introduce a function indicating how the weekly wage of the employee depended on time, we represent the year by interval $[0; 52]$, taking a week for a unit of time. Then the function f representing the dependence of the wage on time can be defined in the following manner:

$$f(t) = \begin{cases} 186 & \text{for } t \in [0; 19], \\ 203.50 & \text{for } t \in (19; 47.6), \\ 211.30 & \text{for } t \in [47.6; 52]. \end{cases}$$

If $\chi_A(t)$ is a characteristic function of the set A, then we have

$$f(t) = 186 \cdot \chi_{[0;19]}(t) + 203.50 \cdot \chi_{(19;47.6)}(t) + 211.30 \cdot \chi_{[47.6;52]}(t)$$

for every $t \in [0; 52]$.

Now we can ask what was the average (mean) wage of that employee during the year or what was his total income from wages that year? Clearly, his total income was

$$186 \cdot 19 + 203.50 \cdot (47.6 - 19) + 211.30 \cdot (52 - 47.6) = 10\,283.82 \text{ Euro.}$$

His average wage was

$$\frac{10283.82}{52} = 197.76 \text{ Euro}$$

per week (rounded to whole cents). In this example, it is easy to see that the function f is a step function and it does not matter, if we use open and bounded intervals for calculating the total income.

Here we have defined $c_1 = 186$; $c_2 = 203.50$; $c_3 = 211.30$; $J_1 = [0; 19]$, $J_2 = [19; 47.6]$, $J_3 = [47.6; 52]$. If the number $b - a = \lambda(J)$ is the length of the interval $J = [a; b]$, then the total income has the form

$$c_1\lambda(J_1) + c_2\lambda(J_2) + c_3\lambda(J_3) = \sum_{j=1}^{3} c_j\lambda(J_j).$$

This number is also the area of the set $S = \{(t, y) : t \in [0; 52],\ 0 \leq y \leq f(t)\}$.

Therefore, it is possible to express the step function by the formula

$$f(x) = \sum_{j=1}^{n} c_j \chi_{J_j}(x)$$

for every x in an interval I, where n is a positive integer, c_j are arbitrary numbers and J_j some bounded intervals $\left(\bigcup_{j=1}^{n} J_j = I \right)$ for every $j = 1, 2, 3, \dots, n$. In each case, the number

$$\sum_{j=1}^{n} c_j \lambda(J_j)$$

is called the integral of the function f.

Example 2. Now, we try to calculate the area of the set

$$S = \{(x, y) : x \in I,\ 0 \leq y \leq f(x)\},$$

where f is some continuous function and non-negative in the (compact) interval I.

Solution: If the function f is not constant in the interval I, then the set S is not equal to the union of a finite number of rectangles. Nevertheless, with the exception of some points on the boundary, which may be disregarded from the point of view when calculating the area, this set can be covered by an infinite sequence of non-overlapping rectangles as illustrated in Figure 1. The sum of the areas of these rectangles is equal to the area of S.

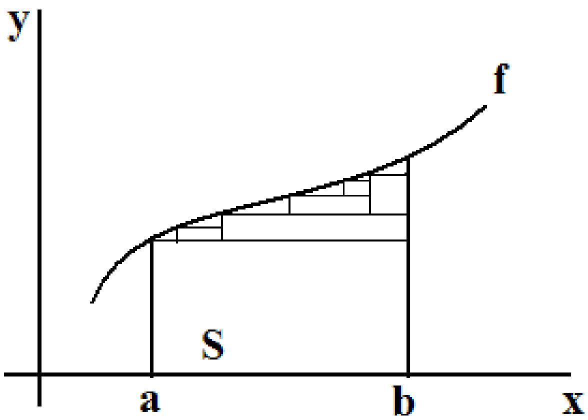

Figure 1

That is, there exist intervals $J_j \subset I$ and numbers c_j, $j = 1, 2, 3, \ldots$, such that

$$f(x) = \sum_{j=1}^{\infty} c_j \chi_{J_j}(x) \tag{1}$$

for every $x \in I$, and the area of set S is equal to the number

$$\sum_{j=1}^{\infty} c_j \lambda(J_j). \tag{2}$$

The class of functions to which the procedure can be applied is much larger than in the case when $c_j \geq 0$ for every $j = 1, 2, 3, \ldots$. In particular, we now may consider functions with both positive and negative values. Consequently, we can also calculate the integral (2) of a function f in a host of situations, when it has an interpretation different from that of the area of a planar figure. Of course, if so desired, the integral of a function in an interval I can always be interpreted "geometrically" as a difference of the areas of the sets
$S^+ = \{(x, y) : x \in I,\ 0 \leq y \leq f(x)\}$ and $S^- = \{(x, y) : x \in I,\ f(x) \leq y \leq 0\}$.

5. Definition of the integral

To obtain a workable definition of integral for a large enough class of functions, it suffices to require the existence of the sum (2) and to note that this sum is then independent on the particular choice of the numbers c_j and intervals J_j, $j = 1, 2, 3, \ldots$, used in the representation (1) of the function f.

Definition 1. *A function f is said to be integrable on the interval I, whenever there exist numbers c_j and bounded intervals $J_j \subset I$, $j = 1, 2, 3, \ldots$ such that*

$$\sum_{j=1}^{\infty} |c_j| \lambda(J_j) < \infty \tag{3}$$

and the equality

$$f(x) = \sum_{j=1}^{\infty} c_j \chi_{J_j}(x)$$

holds for every $x \in I$ such that

$$\sum_{j=1}^{\infty} |c_j| \chi_{J_j}(x) < \infty\,. \tag{4}$$

Now we shall introduce the notions of a virtually primitive function.

We shall use the term *a condition $\mathcal{P}$ is fulfilled nearly everywhere*. This means that the set of points for which the condition $\mathcal{P}$ is not fulfilled is at most countable.

Definition 2. *A function F is said to be virtually primitive to a function f in an interval E if the function F is continuous in the interval E and $F'(x) = f(x)$ nearly everywhere in E.*

In this definition, we do not require E to be a compact interval, it can also be an unbounded interval.

We shall prove that if a function f is integrable on the interval I, then the sum (2) is the same for every choice of the numbers c_j and intervals J_j, $j = 1, 2, 3, \ldots$, satisfying the condition (3) such that (1) holds for every $x \in I$ for which the inequality (4) does hold.

The next three theorems, which are technical ones, are useful in the proof that the definition of the Kluvánek integral is correct.

Theorem 2. *Let n be a positive integer, c_j non-negative numbers, J_j bounded subintervals of I, $j = 1, 2, 3, \ldots, n$, d_k non-negative numbers and K_k bounded intervals, $k = 1, 2, 3, \ldots$, such that*

$$\sum_{j=1}^{n} c_j \chi_{J_j}(x) \le \sum_{k=1}^{\infty} d_k \chi_{K_k}(x) \tag{5}$$

for every $x \in (-\infty, \infty)$. Then

$$\sum_{j=1}^{n} c_j \lambda(J_j) \le \sum_{k=1}^{\infty} |d_k| \lambda(K_k). \tag{6}$$

Proof. It follows from the assumptions that a is a number not greater than the left end-point and b is a number not less than the right end-point of each of the intervals J_j, $j = 1, 2, 3, \ldots, n$. Let F_j be a function virtually primitive in $(-\infty, \infty)$ to the function $c_j \chi_{J_j}$ such that $F_j(a) = 0$, $j = 1, 2, 3, \ldots, n$, and G_k the function virtually primitive to $d_k \chi_{K_k}$ such that $G_k(a) = 0$, $k = 1, 2, 3, \ldots$ Since $c_j \lambda(J_j) = F_j(b)$, $j = 1, 2, 3, \ldots, n$, if we prove that

$$\sum_{j=1}^{n} F_j(b) \le \sum_{k=1}^{\infty} G_k(b),$$

then (6) will follow.

Suppose to the contrary that

$$\sum_{k=1}^{\infty} G_k(b) < \sum_{j=1}^{n} F_j(b). \tag{7}$$

We shall obtain a contradiction.

First, note that $0 \leq G_k(x) \leq G_k(b)$ for every $x \in [a,b]$ and every $k = 1,2,3,\ldots$ Hence, by (7), the sequence of functions $\{G_k\}_{n=1}^{\infty}$ is uniformly convergent in the interval $[a,b]$. Let

$$F(x) = \sum_{j=1}^{n} F_j(x) \text{ and } G(x) = \sum_{k=1}^{\infty} G_k(x)$$

for every $x \in [a,b]$. The functions $F_j(x)$, $j = 1,2,3,\ldots,n$, and $G_k(x)$, $k = 1,2,3,\ldots$, on the interval $[a,b]$ are continuous. Therefore, the functions $F(x)$ and $G(x)$ are also continuous in the interval $[a,b]$ and, of course, $F(a) = G(a) = 0$. Let

$$k = \frac{F(b) - G(b)}{2(b-a)} \text{ and } q = \frac{F(b) - G(b)}{2}.$$

By (7), $k > 0$ and $q > 0$. If $t \in (0,k)$, let

$$h_t(x) = F(x) - G(x) - t(x-a) - q$$

for every $x \in [a,b]$. Then, for every $t \in (0,k)$, h_t is a function continuous in the interval $[a,b]$ such that $h_t(a) < 0$ and $h_t(b) > 0$. Let $\xi(t)$ be its maximal root in the interval (a,b). That is $h_t(\xi(t)) = 0$ and $h_t(y) > 0$ for every $y \in (\xi(t), b)$.

The function $\xi(t)$, $t \in (0,k)$, is (strictly) increasing, because, if $0 < t < s < k$, then

$$h_s(\xi(t)) = h_s(\xi(t)) - h_t(\xi(t)) = (t-s)(\xi(t) - a) < 0$$

and, hence, the largest root, $\xi(s)$, of the function h_s is greater than $\xi(t)$. So, this function is injective. Since its domain, $(0,k)$, is not a countable set, the set of its values $\{\xi(t); t \in (0,k)\}$ is not countable either. But the set of end-points of all intervals J_j, $j = 1,2,3,\ldots,n$ and $K_k, k = 1,2,3,\ldots$, is countable. So, there is a number $t \in (0,k)$ such that $\xi(t)$ is not an end-point of any of intervals J_j, $j = 1,2,3,\ldots,n$ and $K_k, k = 1,2,3,\ldots$ Let t be such a number and $x = \xi(t)$ the corresponding point of the interval (a,b). Then $h_t(x) = 0$ and $h_t(y) > 0$ for every $y \in (x,b)$. That is,

$$F(x) - G(x) = t(x-a) - q \text{ and } F(y) - G(y) > t(y-a) - q$$

for every $y \in (x, b)$. Consequently,

$$\frac{F(y) - F(x)}{y - x} - \frac{G(y) - G(x)}{y - x} > t \tag{8}$$

for every $y \in (x, b]$.

On the other hand, since x is not an end-point of any of the intervals J_j and K_k, each function F_j and G_k is differentiable at x and $F_j'(x) = c_j \chi_{J_j}(x)$ for $j = 1, 2, 3, \ldots, n$, and $G_k'(x) = d_k \chi_{K_k}(x)$ for $k = 1, 2, 3, \ldots$ So, by (5),

$$F'(x) = \sum_{j=1}^{n} F_j'(x) \leq \sum_{k=1}^{\infty} G_k'(x).$$

Since $t > 0$, there exists a positive integer m such that

$$F'(x) \leq \sum_{k=1}^{\infty} G_k'(x) < \sum_{k=1}^{m} G_k'(x) + t.$$

Therefore,

$$\lim_{y \to x^+} \left(\frac{F(y) - F(x)}{y - x} - \sum_{k=1}^{m} \frac{G_k(y) - G_k(x)}{y - x} \right) < t.$$

From the properties of limits, we have that there exists a point y in the interval $[x, b]$ such that

$$\frac{F(y) - F(x)}{y - x} - \sum_{k=1}^{m} \frac{G_k(y) - G_k(x)}{y - x} < t. \tag{9}$$

Now, $G_k(y) - G_k(x) > 0$ for every $k = m+1, m+2, \ldots$, because the functions G_k are non-decreasing. Hence,

$$\frac{G(y) - G(x)}{y - x} = \sum_{k=1}^{\infty} \frac{G_k(y) - G_k(x)}{y - x} \geq \sum_{k=1}^{m} \frac{G_k(y) - G_k(x)}{y - x}.$$

So, (9) contradicts (8). □

Theorem 3. *Let c_j and d_j be non-negative numbers and let J_j and K_j be subintervals of I, $j = 1, 2, 3, \ldots$ such that*

$$\sum_{j=1}^{\infty} c_j \lambda(J_j) < \infty, \quad \sum_{j=1}^{\infty} d_j \lambda(K_j) < \infty$$

and

$$\sum_{j=1}^{\infty} c_j \chi_{J_j}(x) = \sum_{j=1}^{\infty} d_j \chi_{K_j}(x) \tag{10}$$

for every x for which

$$\sum_{j=1}^{\infty} c_j \chi_{J_j}(x) < \infty \quad \text{and} \quad \sum_{j=1}^{\infty} d_j \chi_{K_j}(x) < \infty.$$

Then

$$\sum_{j=1}^{\infty} c_j \lambda(J_j) = \sum_{j=1}^{\infty} d_j \lambda(K_j). \tag{11}$$

Proof. Let ε be an arbitrary positive number. Let n be a positive integer such that

$$\sum_{j=n+1}^{\infty} c_j \lambda(J_j) < \frac{\varepsilon}{2}.$$

Then

$$\sum_{j=1}^{n} c_j \chi_{J_j}(x) \leq \sum_{j=1}^{\infty} d_j \chi_{K_j}(x) + \sum_{j=n+1}^{\infty} c_j \chi_{J_j}(x)$$

for every $x \in (-\infty, \infty)$ with no exception.

By Theorem 2,

$$\sum_{j=1}^{n} c_j \lambda(J_j) \leq \sum_{j=1}^{\infty} d_j \lambda(K_j) + \sum_{j=n+1}^{\infty} c_j \lambda(J_j) < \sum_{j=1}^{\infty} d_j \lambda(K_j) + \frac{\varepsilon}{2}.$$

Hence,

$$\sum_{j=1}^{\infty} c_j \lambda(J_j) = \sum_{j=1}^{n} c_j \lambda(J_j) + \sum_{j=n+1}^{\infty} c_j \lambda(J_j) <$$

$$< \sum_{j=1}^{\infty} d_j \lambda(K_j) + \frac{\varepsilon}{2} + \sum_{j=n+1}^{\infty} c_j \lambda(J_j) < \sum_{j=1}^{\infty} d_j \lambda(K_j) + \varepsilon.$$

Because the inequality between the first and the last terms holds for every positive ε, we have

$$\sum_{j=1}^{\infty} c_j \lambda(J_j) \leq \sum_{j=1}^{\infty} d_j \lambda(K_j).$$

The reverse inequality can be proved by a symmetric argument. Hence (11) holds. □

Recall that nonnegative x^+ and nonpositive x^- parts of a number x are defined by

$$x^+ = \begin{cases} x & \text{if} \quad x \geq 0, \\ 0 & \text{if} \quad x < 0; \end{cases}$$

$$x^- = \begin{cases} -x & \text{if} \quad x < 0, \\ 0 & \text{if} \quad x \geq 0. \end{cases}$$

Then: $x^+ \geq 0$, $x^- \geq 0$, $x = x^+ - x^-$ and $|x| = x^+ + x^-$ for any number x.

Theorem 4. *Let c_j and d_j be numbers and let J_j and K_j be subintervals of I for every $j = 1, 2, 3, \ldots$ such that*

$$\sum_{j=1}^{\infty} |c_j| \lambda(J_j) < \infty, \quad \sum_{j=1}^{\infty} |d_j| \lambda(K_j) < \infty. \tag{12}$$

If

$$\sum_{j=1}^{\infty} c_j \chi_{J_j}(x) = \sum_{j=1}^{\infty} d_j \chi_{K_j}(x)$$

for every $x \in I$ for which

$$\sum_{j=1}^{\infty} |c_j| \chi_{J_j}(x) < \infty \quad \text{and} \quad \sum_{j=1}^{\infty} |d_j| \chi_{K_j}(x) < \infty,$$

then

$$\sum_{j=1}^{\infty} c_j \lambda(J_j) = \sum_{j=1}^{\infty} d_j \lambda(K_j).$$

Proof. The conditions (12) imply:

$$\sum_{j=1}^{\infty} c_j^+ \lambda(J_j) < \infty, \; \sum_{j=1}^{\infty} c_j^- \lambda(J_j) < \infty, \; \sum_{j=1}^{\infty} d_j^+ \lambda(K_j) < \infty, \; \sum_{j=1}^{\infty} d_j^- \lambda(J_j) < \infty.$$

From condition

$$\sum_{j=1}^{\infty} c_j \chi_{J_j}(x) = \sum_{j=1}^{\infty} d_j \chi_{K_j}(x),$$

we have

$$\sum_{j=1}^{\infty} c_j^+ \chi_{J_j}(x) - \sum_{j=1}^{\infty} c_j^- \chi_{J_j}(x) = \sum_{j=1}^{\infty} d_j^+ \chi_{K_j}(x) - \sum_{j=1}^{\infty} d_j^- \chi_{K_j}(x).$$

That is

$$\sum_{j=1}^{\infty} c_j^+ \chi_{J_j}(x) + \sum_{j=1}^{\infty} d_j^- \chi_{K_j}(x) = \sum_{j=1}^{\infty} d_j^+ \chi_{K_j}(x) + \sum_{j=1}^{\infty} c_j^- \chi_{J_j}(x)$$

for every x such that both sides represent a number (not ∞). By Theorem 3,

$$\sum_{j=1}^{\infty} c_j^+ \lambda(J_j) + \sum_{j=1}^{\infty} d_j^- \lambda(K_j) = \sum_{j=1}^{\infty} d_j^+ \lambda(K_j) + \sum_{j=1}^{\infty} c_j^- \lambda(J_j);$$

$$\sum_{j=1}^{\infty} c_j^+ \lambda(J_j) - \sum_{j=1}^{\infty} c_j^- \lambda(J_j) = \sum_{j=1}^{\infty} d_j^+ \lambda(K_j) - \sum_{j=1}^{\infty} d_j^- \lambda(K_j);$$

$$\sum_{j=1}^{\infty} c_j \lambda(J_j) = \sum_{j=1}^{\infty} d_j \lambda(K_j).$$

□

Now we are able to proceed with the definition of integral:

Definition 3. *Let f be a function integrable in the interval I. Let c_j be numbers and let $J_j \subset I$ be intervals, $j = 1, 2, 3, \dots$, satisfying the condition*

$$\sum_{j=1}^{\infty} |c_j| \lambda(J_j) < \infty,$$

such that the equality

$$f(x) = \sum_{j=1}^{\infty} c_j \chi_{J_j}(x)$$

holds for every $x \in I$ meeting the condition

$$\sum_{j=1}^{\infty} |c_j| \chi_{J_j}(x) < \infty.$$

Then the number

$$\sum_{j=1}^{\infty} c_j \lambda(J_j)$$

is called the integral of f in the interval I; it will be denoted by $\int_I f(x)\, dx$.

Clearly, for every constant function $f(x) = \beta$ in the interval $[a, b]$, we have

$$\int_a^b f(x)\, dx = \beta(b - a).$$

6. Conclusions

This article illustrates only a small part of the calculus concept which was prepared by Professor Igor Kluvánek. His method, from the viewpoint of the notion technique, is unique and this method is not a copy of some existing calculus textbooks. He tries to remove the failings by the introduction of calculus notions in the courses for students. This confirms his words: *"The process of explaining, simplifying the notions from the beginning of differential calculus is stopped at some level, however there were connected only some aspects of differential calculus. According to general meaning, all things in this matter had been realised by Cauchy, Bolzano, Weierstrass and their contemporary mathematicians."* This is only partially right as it shows the calculus concept prepared by Professor Igor Kluvánek.

The definition of integral given in this article applies an idea of Archimedes. The most effective method for the calculation of integrals is the one which is based on the differential and integral calculus.

It is well known that the Dirichlet function (the characteristic function of the set of rational numbers) is not integrable in the Riemann sense. It is possible to show that this function is integrable in the sense of Kluvánek and the value of this integral is zero. In fact, let $\mathbb{Q} \cap [a, b] = \{q_j : j \in \mathbb{N}\}$. Let further

$$J_{2j} = \{q_j\}$$

and let J_{2j-1} be any of subintervals of $[0, 1]$. Therefore, the Dirichlet function $D\colon [0, 1] \longrightarrow \mathbb{R}$ can be represented in the form

$$D(x) = \sum_{j=1}^{\infty} c_j \cdot \chi_{J_j}(x),$$

where $c_{2j} = 1$ and $c_{2j-1} = 0$. Hence, the integral of it equals 0.

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