Axial Vibration of Bars Using Fractional Viscoelastic Material Models

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Abstract

This paper presents dynamic analysis of a bar with one end fixed and other free, loaded with force at its free end. The viscoelastic material of the bar is described by fractional models (Scot-Blair, Voigt, Maxwell and Zener models). Rayleigh-Ritz and Laplace transform methods were applied to obtain closed-form solution of the considered problem.

Keywords: fractional calculus, fractional viscoelastic material models, vibration of continuous systems

1. Introduction

Fractional calculus has been successfully applied to modelling of viscoelastic materials [1]. Many authors considered such material models in dynamic analysis of beams, for example [2], [3], [4]. In this paper we consider vibration of a bar with one end fixed and other free, loaded with force on its free end. The most popular fractional models of viscoelastic material (Scott-Blair, Voigt, Maxwell and Zener models) were considered. General theory of the nonhomogeneous fractional differential equations with constant coefficients [5] and Laplace transform were used to solve obtained equations in time domain.

The paper is organized as follows: first some definitions and notations from fractional calculus are introduced and constitutive relations between stress and strain for considered fractional models of viscoelastic material are given. In section 3 we state the problem. In section 4 we solve it and in section 5 we give some numerical examples of the solutions.

2. Preliminaries and notations

There exist many definitions of fractional derivative. The most popular in application to viscoelasticity are Riemann-Liouville and Caputo fractional derivatives [1]. They are defined as follows:

$$D_{RL}^{\alpha}F(t) = \frac{d^n}{d}\left(\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1}\,F(\tau)d\tau\right)$$

$$D_{C}^{\alpha}F(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1}\,F^{(n)}(\tau)d\tau$$

where n – an integer is such that $n\text{-}1 < \alpha \le n$. These definitions are not equivalent. It is well known paradox that Rieman-Liouville derivative of constant function is not equal to zero while Caputo derivative is. There are also some problems with initial conditions for fractional differential equations with Riemann-Liouville derivatives [6], [7].This is the reason why some authors consider only Caputo derivatives, however when $F^{(k)}(0^+)=0$, $k = 0, 1,..., n\text{-}1$, then both derivatives are equivalent [1]:

$$D_{RL}^{\alpha}F(t) = D_{C}^{\alpha}F(t)$$

In this paper we consider problem with zero initial conditions, so, in this case, Rieman-Liouville and Caputo fractional derivatives are equivalent and further we will denote fractional derivative of order α as $D^{\alpha}F(t)$.

Laplace transform of $D^{\alpha}F(t)$ can be evaluated from formula [5]:

$$\mathcal{L}(D^{\alpha}F(t)) = s^{\alpha}\mathcal{L}(F(t)) \tag{1}$$

Below we give constitutive relations between stress $\sigma(t)$ and strain $\varepsilon(t)$ for considered fractional models of viscoelastic material (for each model $0 < \alpha \le 1$):

Scott-Blair model

$$\sigma(t) = \eta D^{\alpha}\varepsilon(t) \tag{2}$$

fractional Voigt model

$$\sigma(t) = E\varepsilon(t) + \eta D^{\alpha}\varepsilon(t) \tag{3}$$

fractional Maxwell model

$$\sigma(t) + aD^{\alpha}\sigma(t) = \eta D^{\alpha}\varepsilon(t) \tag{4}$$

fractional Zener model

$$\sigma(t) + aD^{\alpha}\sigma(t) = E\varepsilon(t) + \eta D^{\alpha}\varepsilon(t) \tag{5}$$

where a, E, η – some constants.

3. Problem formulation

Let us consider a bar with length L and cross section A with one end fixed and other free, loaded with axial force $f(t)$ at its free end, where $f(t)$ is a given time function. Dynamic equation for axial displacement $u(x,t)$ of the bar with boundary and initial conditions are written below (dots denote time derivatives, primes spatial):

$$\varrho\ddot{u}(x,t) = \sigma'(x,t) + \frac{f(t)\delta(x-L)}{A} \quad (6)$$

$$u(0,t) = 0,\ \sigma(L,t) = 0 \quad (7)$$

$$u(x,0) = 0, \dot{u}(x,0) = 0 \quad (8)$$

where δ - Dirac delta and $f(t)$ is defined below:

$$f(t) = \begin{cases} \frac{t}{t_1} f_0 & for\ 0 \le t < t_1 \\ f_0 & for\ t_1 \le t < t_2 \\ 0 & otherwise \end{cases} \quad (9)$$

where t_1, t_2, f_0 - some constants. It means, we apply the force to the bar for some time and keep it constant to the moment t_2, when we remove the force. Using definition of Caputo or Riemann-Liuville fractional derivatives (both give the same result, because $f(0) = 0$), recalled in previous section, one could show that $D^\alpha f(t)$ for $0 < \alpha < 1$ is given by the formula (note that $D^\alpha f(t)$ is indeterminate at t_2):

$$D^\alpha f(t) = \begin{cases} \frac{f_0\, t^{1-\alpha}}{t_1\, \Gamma(2-\alpha)}, & 0 \le t < t_1 \\ \frac{f_0}{t_1 \Gamma(2-\alpha)} \left(t^{1-\alpha} - (t-t_1)^{1-\alpha}\right), & t_1 \le t < t_2 \\ \frac{f_0}{t_1 \Gamma(2-\alpha)} \left(t^{1-\alpha} - (t-t_1)^{1-\alpha}\right) - \frac{f_0 (t-t_2)^{-\alpha}}{\Gamma(1-\alpha)}, & t > t_2 \end{cases} \quad (10)$$

Using relation $\varepsilon(x,t) = u'(x,t)$ equation (6) can be transformed according to the fractional models of the viscoelastic material (2)-(5) (below fractional derivatives refers to time):

Scott-Blair model:

$$\ddot{u}(x,t) = \frac{\eta}{\varrho}\left(D^\alpha u(x,t)\right)'' + \frac{f(t)\delta(x-L)}{A} \quad (11)$$

fractional Voigt model:

$$\ddot{u}(x,t) = \frac{E}{\varrho} u''(x,t) + \frac{\eta}{\varrho}\left(D^\alpha u(x,t)\right)'' + \frac{f(t)\delta(x-L)}{A} \quad (12)$$

fractional Maxwell model:

Let us apply operator $1+aD^\alpha$ on both size of the equation (6). One get the following equation after using constitutive relation (4):

$$D^{2+\alpha} u(x,t) + \frac{1}{a}\ddot{u}(x,t) = \frac{\eta}{a\varrho}\left(D^\alpha u(x,t)\right)'' + \frac{\delta(x-L)}{a\varrho A}\left(f(t) + aD^\alpha f(t)\right) \quad (13)$$

fractional Zener model:
Similarly as for Maxwell model one can get the equation of motion for Zener model:

$$D^{2+\alpha}u(x,t)+\frac{1}{a}\ddot{u}(x,t)=$$

$$=\frac{E}{a\varrho}u''(x,t)+\frac{\eta}{a\varrho}\left(D^{\alpha}u(x,t)\right)''+\frac{\delta(x-L)}{a\varrho A}\left(f(t)+aD^{\alpha}f(t)\right) \tag{14}$$

In all these equations (11)-(14) $0<\alpha\leq 1$ and for $\alpha=1$ we have classical viscoelastic models (Newton model instead of Scott-Blair model etc.)

4. Solution

We use Rayleigh-Ritz method to solve equations (11)-(14) with boundary and initial conditions (7), (8). We search solution in the form:

$$u(x,t)=\sum_{i=0}^{n}w_n(t)\phi_n(x) \tag{15}$$

where $w_n(t)$ - unknown time functions, $\phi_n(x)$ - n-th mode of free vibration of the bar. They are known functions:

$$\phi_n(x)=\sqrt{2/_L}\cdot\sin(\lambda_n x)\,,\qquad \lambda_n=\frac{(2n-1)\pi}{2L}\,,\qquad n=1,2,\ldots$$

They satisfy the orthogonality conditions:

$$\int_0^L\phi_i(x)\,\phi_j(x)dx=\delta_{ij}\,,\quad \int_0^L\phi_i''(x)\,\phi_j(x)dx=-\lambda_i^2\delta_{ij}\,,\quad i,j=1,2,\ldots \tag{16}$$

where δ_{ij} - Kronecker delta. Functions $\phi_n(x)$ fulfil boundary conditions (7) and consequently function (15) also.

Scott-Blair model:
When we substitute (15) into (11) we obtain:

$$\sum_{n=1}^{\infty}\left(\ddot{w}_n(t)\phi_n(x)-\frac{\eta}{\varrho}\phi_n''(x)D^{\alpha}w_n(t)\right)=\frac{f(t)\delta(x-L)}{\varrho A}$$

Let us multiply last equation by $\phi_i(x)$ for some $i=1, 2, \ldots$ and integrate both sides of it in interval $0\leq x\leq L$. Using orthogonality conditions (16) we obtain equation in time domain:

$$\ddot{w}_i(t)+a_iD^{\alpha}w_i(t)=\tilde{f}_i(t)\,,\quad i=1,2,\ldots \tag{17}$$

where

$$a_i=\frac{\eta\lambda_i^2}{\varrho}\,,\quad \tilde{f}_i(t)=\frac{(-1)^{i-1}f(t)}{\varrho A}$$

Using (1) we can get Laplace transform of the solution of (17) in the form:

$$\mathcal{L}\big(w_i(t)\big) = \frac{\mathcal{L}\left(\tilde{f}_i(t)\right)}{s^2 + a_i s^\alpha}$$

Applying Theorem 5.5 from [5] we get the following solution of the equation (17):

$$w_i(t) = \int_0^t (t-\tau)\, G(t-\tau)\tilde{f}_i(t)d\tau \tag{18}$$

where

$$G(t) = E_{2-\alpha,2}(-a_i t^{2-\alpha}) \tag{19}$$

$E_{\alpha\beta}(t)$ is a two-parameter Mittag-Leffler function defined as:

$$E_{\alpha,\beta}(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(n\alpha+\beta)}$$

<u>fractional Voigt model</u>:

Similarly as for Scott-Blair model, for fractional Voigt model (equation (12)) we get following equation in time domain:

$$\ddot{w}_i(t) + a_i D^\alpha w_i(t) + b_i w_i(t) = \tilde{f}_i(t)\,, \;\; i = 1, 2, \ldots \tag{20}$$

where

$$b_i = \frac{E_i \lambda_i^2}{\varrho}$$

As an application of the theorem 5.5 from [5], we get the solution of equation (20) in the form (18). It has the following shape:

$$G(t) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-a_i)^k(-b_i)^n\binom{n+k}{k}}{\Gamma(2n+(2-\alpha)k+2)} t^{2n+(2-\alpha)k} \tag{21}$$

<u>fractional Maxwell model</u>:

Equation of motion in time domain for fractional Maxwell model (equation (13)) is given below:

$$D^{2+\alpha} w_i(t) + c_i \ddot{w}_i(t) + d_i D^\alpha w_i(t) = g(t)\,, \;\; i = 1, 2, \ldots \tag{22}$$

where

$$c_i = \frac{1}{a}\,, \qquad d_i = \frac{\eta\lambda_i^2}{a\varrho}\,, \quad g(t) = \frac{(-1)^{i-1}}{a\varrho A}\big(f(t) + aD^\alpha f(t)\big)$$

Application of the theorem 5.6 (and example 5.13) from [5] gives us the solution of equation (22):

$$w_i(t) = \int_0^t (t-\tau)^{\alpha+1}\, G(t-\tau) g(\tau) d\tau \tag{23}$$

where now

$$G(t) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-c_i)^k \, (-d_i)^n \binom{n+k}{k}}{\Gamma(2n+2+\alpha+\alpha k)} t^{2n+\alpha k} \tag{24}$$

<u>fractional Zener model</u>:
Similarly as for fractional Maxwell model we get the equation of motion for fractional Zener model (equation (14)):

$$D^{2+\alpha} w_i(t) + c_i \ddot{w}_i(t) + d_i D^{\alpha} w_i(t) + e_i w_i(t) = g(t)\,, \quad i = 1, 2, \ldots \tag{25}$$

where

$$e_i = \frac{E\lambda_i^2}{a\varrho}$$

Again, solution of equation of motion (25) we get as an application of theorem 5.6 from [5] in the form (23), where:

$$G(t) = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \sum_{m=0}^{n} X_{k,n,m}\,(t) \tag{26}$$

and

$$X_{k,n,m}(t) = \frac{(-e_i)^m \, (-d_i)^{m-n} \, (-c_i)^k \, (n+k)! \; t^{(n+k)\alpha+2m+(2-\alpha)(n-m)}}{m! \; (n-m)! \; k! \; \Gamma(\alpha(n+k) + 2m + (2-\alpha)(n-m) + 2 + \alpha)}$$

5. Results

Here we give some numerical examples of the solutions of the equations (11)-(14) given by formulas (18) and (23) for functions $G(t)$ defined in equations (19), (21), (24) and (26) for Scott-Blair, Voigt, Maxwell and Zener models respectively. Let us put $a_i = 1$, $b_i = 1$, $c_i = 1$, $d_i = 1$, $e_i = 1$, $a = 1$, $t_1 = 1$ and $t_2 = 2$ (Fig. 2), $t_2 = 10$ (Fig. 3).

Infinite sums in formulas (21), (24) and (26) were replaced by partial sums. Let us denote by $S_{K,N}$ or $S_{N,K,N,M}$ partial sums for series (21), (24) and (26) (capital letters denote higher bounds for sums with respect to index denoted by the same small letter). Higher bounds K,N,M were set in such a way, that

$$\max\left(\left|\frac{S_{N+1,K} - S_{N,K}}{S_{N,K}}\right|, \left|\frac{S_{N,K+1} - S_{N,K}}{S_{N,K}}\right|\right) < 0{,}01$$

and analogously for finite sum $S_{N,\,K,\,N,\,M}$. We have observed that higher bounds N, K, M, computed as described above, increase with t, and for 'big' t results could not been achieved in reasonable time. We are going to overcome this problem in the future by using more sophisticated methods to compute infinite sums numerically [8]. Integrals (18) and (23) were computed by trapezoidal rule with constant time step, sufficiently small.

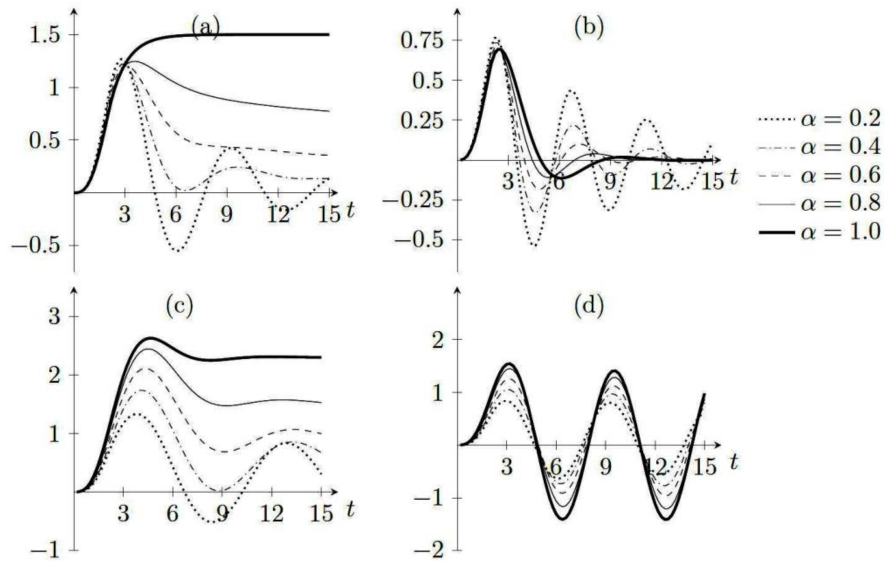

Figure 1. Functions $w_i(t)$ for t_2 = 2 for: (a) Scott-Blair model, (b) fractional Voigt model, (c) fractional Maxwell model, (d) fractional Zener model

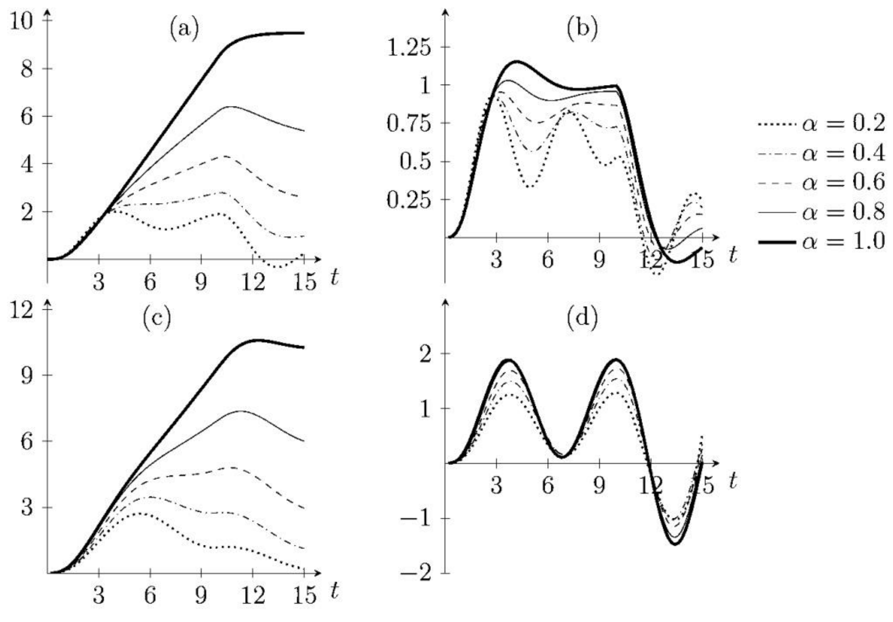

Figure 2. Functions $w_i(t)$ for t_2=10 for: (a) Scott-Blair model, (b) fractional Voigt model, (c) fractional Maxwell model, (d) fractional Zener model

One observed that:

- Bolded curves in Fig. 1 and Fig. 2 represent $w_i(t)$ for classical viscoelastic models. Values of $w_i(t)$ for Newton and Maxwell models are presented in a and c graphs from these figures, for Voigt and Zener models in b and d. After unloading stage $w_i(t)$ stabilize on some level for Newton and Maxwell models, while for Voigt and Zener models after unloading stage $w_i(t)$ oscillate around position of equilibrium, so classical viscoelastic models behave as expected [9].
- With increase of the order of fractional derivative α decrease amplitude of vibration, which is expected behaviour (α = 0 means no damping in the system, α = 1 means classical damping in the system).
- Solutions $w_i(t)$ correctly reflect change in the moment t_2 when load is removed (compare graphs in Fig. 1 and in Fig. 2).

6. Conclusions

Dynamic analysis of a bar with one end fixed and other free, loaded with force on its free end was done. Viscoelastic material of the bar was modelled by fractional models (Scot-Blair, Voigt, Maxwell and Zener fractional models). Using Laplace transform and Rayleigh-Ritz methods, closed-form solutions were obtained and some numerical examples have been presented in this paper.

References

1. F. Mainardi, *Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models*, World Scientific, 2010.
2. M. Di Paola, R. Heuer, A. Pirrotta, *Fractional visco-elastic Euler-Bernoulli beam*, International Journal of Solids and Structures, **50.22-23** (2013) 3505 – 3510.
3. R. Lewandowski, P. Wielentejczyk, *Nonlinear vibration of viscoelastic beams described using fractional order derivatives*, Journal of Sound and Vibration, **399** (2017) 228 – 243.
4. O. Martin, *Nonlinear dynamic analysis of viscoelastic beams using a fractional rheological model*, Applied Mathematical Modelling, **43** (2017) 351 – 359.
5. H. M. Srivastava, A. A. Kilbas, J. J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies, vol. 204, Elsevier, Amsterdam 2006.
6. R. Bagley, *On the equivalence of the Riemann-Liouville and the Caputo fractional order derivatives in modelling of linear viscoelastic materials*, Fractional Calculus and Applied Analysis, **10.2** (2007) 123 – 126.
7. N. Haymans, I. Podlubny, *Physical interpretation of initial conditions for fractional differential equations with Riemann-Liouville fractional derivatives*, Rheologica Acta, **45.5** (2006) 765 – 771.
8. H. Cohen, F. Rodriguez Villegas, D. Zaiger, *Convergence acceleration of alternating series*, Experimental mathematics, **9.1** (2000) 3 – 12.
9. A. S. Novick, B. S. Berry *Anelastic relaxation in crystalline solids*, Academic Press, London, 1972.