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# Pointwise completeness, pointwise degeneracy and stability of standard and positive linear systems after discretization

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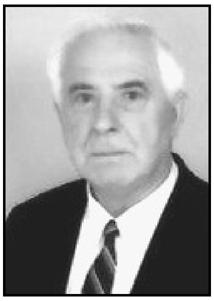

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**Abstract**

Definitions and necessary and sufficient conditions of the pointwise completeness, pointwise degeneracy and stability of standard and positive continuous-time and discrete-time linear systems are given. A problem of influence of the discretization of standard and positive continuous-time linear systems on the pointwise completeness, pointwise degeneracy and stability of standard and positive discrete-time linear systems is analyzed. The derivative is approximated using forward rectangular rule. Considerations are illustrated by numerical examples.

**Keywords**: discretization, forward rectangular rule, pointwise completeness, pointwise degeneracy, stability, standard and positive continuous-time linear system, standard and positive discrete-time linear system.

## Punktowa zupełność, punktowa degeneracja i stabilność standardowych i dodatnich układów liniowych po dyskretyzacji

**Streszczenie**

Standardowy układ dynamiczny, niepoddany wymuszeniu, jest nazywany punktowo zupełnym, jeżeli każdy zadany stan końcowy można osiągnąć poprzez odpowiedni wybór stanu początkowego. Standardowy układ dynamiczny jest punktowo degenerowany w kierunku *v*, jeżeli istnieje stan końcowy, który jest nieosiągalny dla każdego warunku początkowego. W pracy podano definicje oraz warunki konieczne i wystarczające punktowej zupełności, punktowej degeneracji oraz stabilności standardowych i dodatnich liniowych układów ciągłych i dyskretnych. Dokonano analizy wpływu dyskretyzacji standardowego i dodatniego liniowego układu ciągłego na punktową zupełność, punktową degenerację i stabilność standardowego i dodatniego liniowego układu dyskretnego. Pochodna jest aproksymowana przy wykorzystaniu metody prostokątnej w przód. Rozważania zobrazowano przykładami numerycznymi. Praca ma następującą strukturę. W rozdziałach 2-5 podano definicje punktowej zupełności, punktowej degeneracji i stabilności liniowego układu ciągłego oraz liniowego układu dyskretnego. W rozdziałach 6 i 7 dokonano analizy wpływu dyskretyzacji standardowego i dodatniego liniowego układu ciągłego na punktową zupełność, punktową degenerację i stabilność standardowego i dodatniego liniowego układu dyskretnego. Rozdział 8 zawiera przykłady numeryczne, natomiast uwagom końcowym poświęcony jest rozdział 9.

**Slowa kluczowe**: dyskretyzacja, aproksymacja prostokątna w przód, punktowa zupełność, punktowa degeneracja, stabilność, standardowy i dodatni liniowy układ ciągły, standardowy i dodatni liniowy układ dyskretny.

## 1. Introduction

An autonomous dynamical system (without input signal) is called pointwise complete, if every final state can be reached by suitable choice of the initial state. The system is pointwise degenerated in the direction *v*, if there exists at least one final state, which is unreachable for any initial state.

The problem of pointwise completeness and pointwise degeneracy for standard and positive linear systems has been analyzed in [1-3, 7-8, 11-14, 19-22] and in monograph [16]. The problem of stability for standard and positive linear systems has been considered in [4-6, 17] and in monographs [9-10, 15-16].

In this paper we consider the problem of influence of the discretization of standard and positive continuous-time linear systems on the pointwise completeness, pointwise degeneracy and stability of standard and positive discrete-time linear systems. Definitions and necessary and sufficient conditions of the pointwise completeness and pointwise degeneracy of standard and positive continuous-time and discrete-time linear systems are given. The derivative is approximated using forward rectangular rule. Considerations are illustrated by numerical examples.

The paper is organized as follows. In sections 2-5 some definitions and theorems concerning pointwise completeness, pointwise degeneracy and stability of standard and positive continuous-time and discrete-time linear systems are given. The main result of the paper is presented in sections 6 and 7 where influence of the discretization of standard and positive continuous-time linear systems on the pointwise completeness, pointwise degeneracy and stability of standard and positive discrete-time linear systems is analyzed. Examples illustrating these considerations are presented in section 8. Concluding remarks are given in section 9.

The following notation will be used: $\Re$ - the set of real numbers, $\Re^{n\times m}$ - the set of $n\times m$ real matrices and $\Re^n = \Re^{n\times 1}$, $Z_+$ - the set of nonnegative integers, $M_n$ - the set of $n\times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), $I_n$ - the $n\times n$ identity matrix.

## 2. Pointwise completeness, pointwise degeneracy and stability of standard continuous-time linear systems

Consider the autonomous continuous-time linear system described by the equation

$$\dot{x}(t) = A_c x(t)\,, \tag{1}$$

where $x(t)\in\Re^n$ is the state vector and $A_c\in\Re^{n\times n}$.

**Theorem 1.** [16] Solution of the equation (1) has the form

$$x(t) = e^{A_c t}x_0\,. \tag{2}$$

From expansion of $e^{A_c t}$ it follows that $\det e^{A_c t}\neq 0$ for every matrix $A_c$ and time $t$.

**Definition 1.** [16] The standard continuous-time linear system (1) is called pointwise complete for $t = t_f$ if for every final state $x_f \in \Re^n$ there exists an initial condition $x(0) = x_0$ such that $x(t_f) = x_f$.

**Theorem 2.** [16] The standard continuous-time linear system (1) is pointwise complete for $t = t_f$ for every matrix $A_c$.

**Definition 2.** [16] The standard continuous-time linear system (1) is called pointwise degenerated in the direction $v$ and time $t = t_f$, if there exists a nonzero vector $v \in \Re^n$ such that for every initial state $x_0 \in \Re^n$ the solution (2) for $t = t_f$ satisfies the condition $v^T x_f = 0$.

**Theorem 3.** [16] The standard continuous-time linear system is not pointwise degenerated for every matrix $A_c$.

**Definition 3.** [10] The standard continuous-time linear system (1) is called asymptotically stable if $\lim_{t\to\infty} x(t) = 0$ for all $x_0 \in \Re^n$.

**Theorem 4.** [10] The standard continuous-time linear system (1) is asymptotically stable if and only if $\mathrm{Re}\, s_k < 0$ for $k = 1,2,...,n$, where $s_k$, $k = 1,2,...,n$ are the eigenvalues of the matrix $A_c$.

## 3. Pointwise completeness, pointwise degeneracy and stability of positive continuous-time linear systems

**Definition 4.** [9, 15] The continuous-time linear system (1) is called (internally) positive if $x(t) \in \Re_+^n$, $t \ge 0$ for any initial conditions $x_0(t) \in \Re_+^n$.

**Theorem 5.** [9, 15] The continuous-time linear system (1) is positive if and only if $A_c \in M_n$.

**Definition 5**. [16] The positive continuous-time system (1) is called pointwise complete for $t = t_f$ if for every final state $x_f \in \Re_+^n$ there exists an initial condition $x_0 \in \Re_+^n$ such that $x(t_f) = x_f$.

**Theorem 6.** [16] The positive continuous-time linear system (1) is pointwise complete for $t = t_f$ if and only if the matrix $A_c$ is diagonal.

**Definition 6.** [16] The positive continuous-time system (1) is called pointwise degenerated in the direction $v$ for $t = t_f$ if there exists at least one final state $x_f \in \Re_+^n$, which is unreachable for $t = t_f$ from any initial state $x_0 \in \Re_+^n$, or equivalently the equality $x(t_f) = x_f$ is not satisfied for any $x_0 \in \Re_+^n$.

**Theorem 7.** [16] The positive continuous-time linear system (1) is pointwise degenerated in the direction $v$ for $t = t_f$ if and only if the matrix $A_c$ is not diagonal.

**Definition 7.** [9, 15] The positive continuous-time linear system (1) is called asymptotically stable if $\lim_{t\to\infty} x(t) = 0$ for all $x_0 \in \Re_+^n$.

**Theorem 8.** [9, 15] The positive continuous-time linear system (1) is asymptotically stable if and only if all coefficients of the polynomial

$$\det[I_n s - A_c] = s^n + a_{n-1}s^{n-1} + ... + a_1 s + a_0 \tag{3}$$

are positive, i.e. $a_k > 0$, $k = 1,2,...,n-1$.

## 4. Pointwise completeness, pointwise degeneracy and stability of standard discrete-time linear systems

Consider the autonomous discrete-time linear system described by the equation

$$x_{i+1} = A_d x_i, \tag{4}$$

where $x_i \in \Re^n$ is the state vector and $A_d \in \Re^{n\times n}$.

**Theorem 9.** [16] Solution of the equation (3) has the form

$$x_i = A_d^i x_0, \quad i \in Z_+. \tag{5}$$

**Definition 8.** [16] The standard discrete-time linear system (4) is called pointwise complete for $i = q$ if for every final state $x_f \in \Re^n$ there exists an initial condition $x_0$ such that $x_q = x_f$.

**Theorem 10.** [16] The standard discrete-time linear system (4) is pointwise complete if and only if the matrix $A_d$ is nonsingular.

**Definition 9.** [16] The standard discrete-time linear system (4) is called pointwise degenerated in the direction $v$ for $i = q$ if there exists a nonzero vector $v \in \Re^n$ such that for all initial conditions $x_0 \in \Re^n$ the solution (4) for $i = q$ satisfies the condition $v^T x_q = 0$.

**Theorem 11.** [16] The standard discrete-time linear system (4) is pointwise degenerated in the direction $v$ for $i = q$ if and only if the matrix $A_d$ is singular. The vector $v$ can be found from $v^T A_d = 0$.

**Definition 10.** [10] The standard discrete-time linear system (4) is called asymptotically stable if $\lim_{i\to\infty} x_i = 0$ for all $x_0 \in \Re^n$.

**Theorem 12.** [10] The standard discrete-time linear system (4) is asymptotically stable if and only if $|z_k| < 1$ for $k = 1,2,...,n$, where $z_k$, $k = 1,2,...,n$ are the eigenvalues of the matrix $A_d$.

## 5. Pointwise completeness, pointwise degeneracy and stability of positive discrete-time linear systems

**Definition 11.** [9, 15] The discrete-time linear system (4) is called (internally) positive if $x_i \in \Re_+^n$, $i \in Z_+$ for any initial conditions $x_0(t) \in \Re_+^n$.

**Theorem 13.** [9, 15] The discrete-time linear system (4) is positive if and only if $A_d \in \Re_+^{n\times n}$.

**Definition 12**. [16] The positive discrete-time system (4) is called pointwise complete for $i = q$ if for every final state $x_f \in \Re_+^n$ there exists an initial condition $x_0 \in \Re_+^n$ such that $x_q = x_f$.

**Theorem 14**. [16] The positive discrete-time linear system (4) is pointwise complete for $i = q$ if and only if the matrix $A_d$ is monomial (i.e. every its row and every its column contains only one positive entry and the remaining entries are zero).

**Definition 13.** [16] The positive discrete-time system (4) is called pointwise degenerated in the direction $v$ for $i = q$ if there exists at least one final state $x_f \in \Re_+^n$, which is unreachable in $q$ steps from any initial state $x_0 \in \Re_+^n$, or equivalently the equality $x_q = x_f$ is not satisfied for any $x_0 \in \Re_+^n$.

**Theorem 15.** [16] The positive discrete-time linear system (4) is pointwise degenerated for $i = q$ if and only if the matrix $A_d$ is not monomial.

**Definition 14.** [9, 15] The positive discrete-time linear system (4) is called asymptotically stable if $\lim_{i\to\infty} x_i = 0$ for all $x_0 \in \Re_+^n$.

**Theorem 16.** [9, 15] The positive discrete-time linear system (4) is asymptotically stable if and only if all coefficients of the polynomial

$$\det[I_n(z+1) - A_d] = z^n + \bar{a}_{n-1}z^{n-1} + ... + \bar{a}_1 z + a_0, \tag{6}$$

are positive, i.e. $\bar{a}_k > 0$, $k = 1,2,...,n-1$.

## 6. Influence of the discretization on the pointwise completeness, pointwise degeneracy and stability for standard systems

The derivative will be approximated using forward rectangular rule

$$\dot{x}(t) \approx \frac{x_{i+1} - x_i}{\Delta t}, \tag{7}$$

where $x(t) \in \Re^n$ is the state vector of continuous-time linear system, $x_i \in \Re^n$ is the state vector of discrete-time linear system and $\Delta t$ is the discretization period.

The approximation of derivative (7) based on forward rectangular rule is the simplest method and the algorithm is easy to implement in a digital controller or a computer. More advanced methods of approximation are given in [18].

From (1) and (7) we have

$$x_{i+1} = A_d x_i, \tag{8}$$

where $A_d = (I + \Delta t A_c) \in \Re^{n\times n}$.

The characteristic polynomial of the matrix $A_d$ has the form

$$p_{A_d} = \det[I_n z - A_d] = \det[I_n(z-1) - \Delta t A_c] =$$
$$\det\left\{\Delta t\left[I_n\frac{z-1}{\Delta t} - A_c\right]\right\} = (\Delta t)^n \det\left[I_n\frac{z-1}{\Delta t} - A_c\right]. \tag{9}$$

The characteristic polynomial of the matrix $A_c$ has the form

$$p_{A_c} = \det[I_n s - A_c]. \tag{10}$$

From (9) and (10) we obtain

$$s = \frac{z-1}{\Delta t} \tag{11}$$

and therefore

$$z = s\Delta t + 1. \tag{12}$$

**Theorem 17.** The discretized continuous-time linear system (7) is pointwise complete for $i = q$ if and only if the matrix $A_d$ is nonsingular (all eigenvalues $z_1$, $z_2$,…, $z_n$ are different than 0, i.e. $z_k = s_k\Delta t + 1 \neq 0$ for $k = 1,...,n$).

**Proof.** For $i = q$ we have $x_f = x_q = A_d^q x_0$. From this equation it is possible to find $x_0$ for any given vector $x_f$ if and only if $\det A_d^q \neq 0$. Note that $\det A_d^q = (\det A_d)^q$. Therefore, $x_0 = A_d^{-q} x_f$ if and only if $\det A_d \neq 0$. □

**Theorem 18.** The discretized continuous-time linear system (7) is pointwise degenerated in the direction $v$ for $i = q$ if and only if the matrix $A_d$ is singular. The vector $v$ can be found from $v^T A_d = 0$.

**Proof.** There exists a vector $v$ such that $v^T A_d^q = 0$, if and only if the matrix $A_d$ is singular. In this case premultiplying the equation $x_f = A_d^q x_0$ by $v^T$ we obtain $v^T x_f = v^T A_d^q x_0 = 0$ and $v^T A_d = 0$. □

**Theorem 19.** The discretized continuous-time linear system (7) is asymptotically stable if and only if the standard continuous-time linear system (1) is asymptotically stable and the discretization period satisfies the condition

$$0 < \Delta t < \min_{1\le k\le n} \frac{2\alpha_k}{\alpha_k^2 + \beta_k^2}, \tag{13}$$

where $s_k = -\alpha_k + j\beta_k$, $k = 1,2,...,n$ are the eigenvalues of the matrix $A_c$.

**Proof.** Eigenvalues of the matrices $A_c$ and $A_d$ are related by (12). The discretized continuous-time linear system is asymptotically stable if and only if its eigenvalues have moduli less than 1, i.e.

$$|z_k| = |s_k\Delta t + 1| = |1 - \Delta t\alpha_k + j\Delta t\beta_k| < 1, \; k = 1,2,...,n. \tag{14}$$

From (14) we have

$$(1 - \Delta t\alpha_k)^2 + (\Delta t\beta_k)^2 < 1. \tag{15}$$

Solving (15) with respect to $\Delta t$ we obtain (13). □

## 7. Influence of the discretization on the pointwise completeness, pointwise degeneracy and stability for positive systems

**Theorem 20.** The discretized continuous-time linear system (7) is positive if and only if the discretization period $\Delta t$ satisfies the condition

$$0 < \Delta t \le \min_{1\le i\le n} \frac{-1}{a_{ii}}, \quad a_{ii} < 0, \tag{16a}$$

or

$$\Delta t > 0 , \quad a_{ii} \geq 0 , \tag{16b}$$

where $a_{ii}$ , $i = 1,2,...,n$ are diagonal entries of the matrix $A_c \in M_n$ .

**Proof.** From (8) and Theorem 13 we have $A_d = (I + \Delta t A_c) \in \Re_+^{n\times n}$ . If $A_c \in M_n$ and $A_d \in \Re_+^{n\times n}$ then

$$1 + a_{ii}\Delta t \geq 0 . \tag{17}$$

Solving (17) with respect to $\Delta t$ we obtain (16). □

**Theorem 21.** The discretized positive continuous-time linear system (7) is pointwise complete if and only if the positive continuous-time linear system (1) is pointwise complete and the discretization period $\Delta t$ satisfies the condition (16).

**Proof.** If the matrix $A_c \in M_n$ is diagonal then the matrix $A_d = (I + \Delta t A_c)$ is also diagonal and $A_d \in \Re_+^{n\times n}$ if the discretization period $\Delta t$ satisfies (16). A diagonal matrix with its positive entries is also a monomial matrix. □

**Theorem 22.** The discretized positive continuous-time linear system (7) is asymptotically stable for any $\Delta t > 0$ if and only if the positive continuous-time system (1) is asymptotically stable.

**Proof.** From comparison of polynomials (3) and (6) we have

$$\det[I_n s - A_c] = s^n + a_{n-1}s^{n-1} + ... + a_1 s + a_0 , \tag{18a}$$

$$\begin{aligned} &\det[I_n(z+1) - A_d] = \det[I_n z + I_n - I_n - \Delta t A_c] = \\ &\det[I_n z - \Delta t A_c] = z^n + \overline{a}_{n-1}z^{n-1} + ... + \overline{a}_1 z + a_0 . \end{aligned} \tag{18b}$$

Taking into account (12) it can be shown that the coefficients of polynomials (18) are related by

$$\overline{a}_{n-1} = a_{n-1}\Delta t , \ \overline{a}_{n-2} = a_{n-2}(\Delta t)^2 , \ \ldots , \ \overline{a}_0 = a_0(\Delta t)^n . \tag{19}$$

It follows that $\overline{a}_k > 0$ , $k = 1,2,...,n-1$ for any $\Delta t > 0$ if $a_k > 0$ , $k = 1,2,...,n-1$ . □

## 8. Numerical examples

**Example 1.**

Consider the standard continuous-time linear system (1) with state matrix

$$A_c = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} . \tag{20}$$

By Theorem 2 every standard continuous-time linear system is pointwise complete.

The characteristic polynomial of the matrix $A_c$ has the form

$$p_{A_c} = \det[I_2 s - A_c] = s^2 + 3s + 2 . \tag{21}$$

There are two asymptotically stable eigenvalues $s_1 = -1$ and $s_2 = -2$ .

Using (8) with $\Delta t = 1$ we obtain

$$A_d = I_2 + A = \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} . \tag{22}$$

The pointwise completeness of the discretized continuous-time linear system with state matrix (20) can be checked by calculating the determinant

$$\det A_d = \det\begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} = 0 . \tag{23}$$

The system (20) after discretization is pointwise degenerated in the direction defined by $v^T A_d = 0$ .

The characteristic polynomial of the matrix $A_d$ has the form

$$p_{A_d} = \det[I_2 z - A_d] = \det\begin{bmatrix} z-1 & -1 \\ 2 & z+2 \end{bmatrix} = z^2 + z . \tag{24}$$

There are two eigenvalues $z_1 = 0$ and $z_2 = -1$ . The system (20) after discretization is stable but not asymptotically since the condition (13) is not satisfied:

$$0 < \Delta t < 1 . \tag{25}$$

**Example 2.**

Consider the standard continuous-time linear system (1) with state matrix (20). The system is pointwise complete and asymptotically stable since the eigenvalues are $s_1 = -1$ and $s_2 = -2$ .

Using (8) with $\Delta t = 0,9$ we obtain

$$A_d = I_2 + 0,9A = \begin{bmatrix} 1 & 0,9 \\ -1,8 & -1,7 \end{bmatrix} . \tag{26}$$

The pointwise completeness of the discretized continuous-time linear system with state matrix (26) can be checked by calculating the determinant

$$\det A_d = \det\begin{bmatrix} 1 & 0,9 \\ -1,8 & -1,7 \end{bmatrix} = -0,08 \neq 0 . \tag{27}$$

The system (20) after discretization is still pointwise complete.

The characteristic polynomial of the matrix $A_d$ has the form

$$p_{A_d} = \det[I_2 z - A_d] = \det\begin{bmatrix} z-1 & -0,9 \\ 1,8 & z+1,7 \end{bmatrix} = z^2 + 0,7z - 0,08 . \tag{28}$$

There are two eigenvalues $z_1 = 0,1$ and $z_2 = -0,8$ . The system (20) after discretization is still asymptotically stable since the condition (13) is satisfied.

**Example 3.**

Consider the positive continuous-time linear system (1) with state matrix

$$A_c = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} . \tag{29}$$

The system (29) is pointwise complete and asymptotically stable since the matrix $A_c$ is diagonal with eigenvalues $s_1 = -2$ and $s_2 = -1$.

Taking into account (16) we obtain

$$0 < \Delta t \le 0,5 . \tag{30}$$

Using (8) with $\Delta t = 0,3$ we obtain

$$A_d = I_2 + 0,3A = \begin{bmatrix} 0,4 & 0 \\ 0 & 0,7 \end{bmatrix}. \tag{31}$$

The discretized continuous-time linear system (31) is positive since condition (16) is satisfied.

The system (31) is also pointwise complete and asymptotically stable since the matrix $A_d$ is monomial with the eigenvalues $z_1 = 0,4$ and $z_2 = 0,7$ .

## 9. Concluding remarks

The influence of discretization of standard and positive continuous-time linear systems on the pointwise completeness, pointwise degeneracy and stability of standard and positive discrete-time linear systems has been investigated. It has been shown that the choice of discretization period can change the location of eigenvalues (12) of the discretized standard continuous-time linear system (7) and therefore its pointwise completeness (Theorem 17), pointwise degeneracy (Theorem 18) and stability (Theorem 19). The discretized positive continuous-time linear system (7) is pointwise complete if the standard continuous-time linear system is also pointwise complete and the discretization period satisfies the condition (16). Asymptotic stability of the discretized positive continuous-time linear system (7) has been investigated (Theorem 22). Necessary and sufficient conditions of the pointwise completeness, pointwise degeneracy and stability of standard and positive continuous-time linear systems and discrete-time linear systems have been given. The considerations have been illustrated by numerical examples.

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