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PROPERTIES OF THE LEAST ACTION LEVEL AND THE EXISTENCE OF GROUND STATE SOLUTION TO FRACTIONAL ELLIPTIC EQUATION WITH HARMONIC POTENTIAL

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Abstract. In this article we consider the following fractional semilinear elliptic equation

$$(-\Delta)^s u + |x|^2 u = \omega u + |u|^{2\sigma} u \quad \text{in } \mathbb{R}^N,$$

where $s \in (0,1)$, $N > 2s$, $\sigma \in (0, \frac{2s}{N-2s})$ and $\omega \in (0, \lambda_1)$. By using variational methods we show the existence of a symmetric decreasing ground state solution of this equation. Moreover, we study some continuity and differentiability properties of the ground state level. Finally, we consider a bifurcation type result.

Keywords: harmonic potential, fractional Sobolev space, ground state solution, bifurcation result, variational method.

Mathematics Subject Classification: 45G05, 35J60, 35B25.

1. INTRODUCTION

In this article we consider the following fractional semilinear elliptic equation

$$(-\Delta)^s u + |x|^2 u = \omega u + |u|^{2\sigma} u \text{ in } \mathbb{R}^N, \tag{1.1}$$

where $N > 2s$, $s \in (0,1)$, $\sigma \in (0, \frac{2s}{N-2s})$, $\omega \in (0, \lambda_1)$.

The fractional Laplacian is characterized as

$$\mathfrak{F}((-\Delta)^s u(\xi)) = |\xi|^s \widehat{u}(\xi),$$

where $\mathfrak{F}(u) := \widehat{u}$ is the Fourier transform of u and for functions u smooth enough, it can be defined by the principal value of the singular integral

$$(-\Delta)^s u(x) = C_{N,s}\, P.V. \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}} dy.$$

The problem under consideration arises in the study of standing waves to the following time-dependent fractional Schrödinger equation,

$$i\frac{\partial\psi}{\partial t} = (-\Delta)^2\psi + |x|^2\psi - |\psi|^{2\sigma}\psi, \tag{1.2}$$

where a standing wave solution to (1.2) has the following form

$$\psi(t,x) = e^{-i\omega t}u(x), \quad \omega \in \mathbb{R}.$$

This kind of solution reduces (1.1) to the following semi-linear fractional elliptic equation

$$(-\Delta)^s u(x) + |x|^2 u(x) - |u(x)|^{2\sigma} u(x) = \omega u(x) \quad \text{in } \mathbb{R}^N. \tag{1.3}$$

Recently great attention has to paid to the existence of standing wave solutions to equation (1.3). For example, Ding and Hajaiej [7] considered the existence of ground state solutions of equation (1.2). Moreover, they considered the orbital stability of standing waves and provided an interesting numerical result about the dynamics. Guo [11] and Hajaiej and Song [12] have discussed the uniqueness result of the ground state solution of (1.3). To other related results we refer to the readers to [4, 5, 9, 10, 13, 17, 19, 20] and the reference therein.

Motivated by these previous results this paper deal with the existence of ground state solution of problem (1.1). Moreover we study some continuity and differentiability properties of the ground state level. Finally we consider a bifurcation type result.

2. PRELIMINARIES AND EXISTENCE RESULT

We shall work in the Hilbert space

$$H^s(\mathbb{R}^N) := \left\{ u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dydx < \infty \right\},$$

endowed with the norm

$$\|u\| = \left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dydx + \int_{\mathbb{R}^N} |u(x)|^2 dx \right)^{1/2}. \tag{2.1}$$

Note that, if $0 < s < 1$ be such that $2s < N$, then there exists a constant $C_{2_s^*} = C(N,s)$, such that

$$\|u\|_{L^{2_s^*}(\mathbb{R}^N)} \leq C_{2_s^*}\|u\| \tag{2.2}$$

for every $u \in H^s(\mathbb{R}^N)$, where $2_s^* = \frac{2N}{N-2s}$ is the fractional critical exponent. Moreover, the embedding $H^s(\mathbb{R}^N) \subset L^p(\mathbb{R}^N)$ is continuous for any $p \in [2, 2_s^*]$ and is locally compact whenever $p \in [2, 2_s^*)$ (for more details see [6]).

Moreover, we introduce the following fractional Sobolev spaces

$$X^s = \left\{ u \in H^s(\mathbb{R}^N) \,:\, \int\limits_{\mathbb{R}^N} |x|^2 |u(x)|^2 dx < \infty \right\},$$

endowed with the norm

$$\|u\|_s = \left(\int\limits_{\mathbb{R}^N} |x|^2 |u(x)|^2 dx + \int\limits_{\mathbb{R}^N} \int\limits_{\mathbb{R}^N} \frac{|u(x) - u(y)|}{|x-y|^{n+2s}} dy dx \right)^{1/2}. \tag{2.3}$$

Considering this space we have the following embedding.

Lemma 2.1. *The embedding $X^s \hookrightarrow H^s(\mathbb{R}^N)$ is continuous.*

Proof. Let $b > 0$ such that the set $\Lambda = \{x \in \mathbb{R}^N \,:\, |x|^2 < b\}$ has finite measure and

$$\operatorname{meas}(\Lambda)^{\frac{2_s^*-2}{2_s^*}} < \frac{1}{C_{2_s^*}^2},$$

where $C_{2_s^*}$ is given by (2.2). Then

$$\begin{aligned} \int\limits_{\mathbb{R}^N} |u(x)|^2 dx &\le \left(\int\limits_{\mathbb{R}^N} |u(x)|^{2_s^*} dx \right)^{\frac{2}{2_s^*}} \operatorname{meas}(\Lambda)^{\frac{2_s^*-2}{2_s^*}} + \frac{1}{b} \int\limits_{\mathbb{R}^N} |x|^2 |u(x)|^2 dx \\ &\le C_{2_s^*}^2 \left(\int\limits_{\mathbb{R}^N} |u(x)|^2 dx + \int\limits_{\mathbb{R}^N} \int\limits_{\mathbb{R}^N} \frac{|u(x)-u(z)|^2}{|x-z|^{N+2s}} dz dx \right) \operatorname{meas}(\Lambda)^{\frac{2_s^*-2}{2_s^*}} \\ &\quad + \frac{1}{b} \int\limits_{\mathbb{R}^N} |x|^2 |u(x)|^2 dx. \end{aligned}$$

This implies that

$$\begin{aligned} &\int\limits_{\mathbb{R}^N} |u(x)|^2 dx \\ &\le \frac{\max\{C_{2_s^*}^2 \operatorname{meas}(\Lambda)^{\frac{2_s^*-2}{2_s^*}}, \frac{1}{b}\}}{1 - C_{2_s^*}^2 \operatorname{meas}(\Lambda)^{\frac{2_s^*-2}{2_s^*}}} \left(\int\limits_{\mathbb{R}^N} \int\limits_{\mathbb{R}^N} \frac{|u(x)-u(z)|^2}{|x-z|^{N+2s}} dz dx + \int\limits_{\mathbb{R}^N} |x|^2 |u(x)|^2 dx \right) \end{aligned}$$

Let

$$\Theta = \frac{1 - C_{2_s^*}^2 \operatorname{meas}(\Lambda)^{\frac{2_s^*-2}{2_s^*}}}{C_{2_s^*}^2 \operatorname{meas}(\Lambda)^{\frac{2_s^*-2}{2_s^*}}}.$$

This shows that

$$\|u\|^2 \leq \left(1+\frac{1}{\Theta}\right)\|u\|_s^2,$$

which yields that the embedding $X^s \hookrightarrow H^s(\mathbb{R}^N)$ is continuous. $\square$

Remark 2.2.

1. Since the embedding $H^s(\mathbb{R}^N) \subset L^p(\mathbb{R}^N)$ is continuous for any $p \in [2, 2_s^*]$ and is locally compact whenever $p \in [2, 2_s^*)$, then by Lemma 2.1, $X^s \subset L^p(\mathbb{R}^N)$ continuously for $p \in [2, 2_s^*]$ and locally compact for $p \in [2, 2_s^*)$.
2. Since the external potential $V(x) = |x|^2$ is coercive, we can show that the embedding $X^s \subset L^p(\mathbb{R}^N)$ is compact for $p \in [2, 2_s^*)$.

Associated to problem (1.1), we have the functional $I_\omega : X^s \to \mathbb{R}$ defined as

$$\begin{aligned} I_\omega(u) = \frac{1}{2}\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(z)|^2}{|x-z|^{N+2s}}dzdx + \int_{\mathbb{R}^N}|x|^2u^2(x)dx\right) - \frac{\omega}{2}\int_{\mathbb{R}^N}u^2(x)dx \\ - \frac{1}{2\sigma+2}\int_{\mathbb{R}^N}|u(x)|^{2\sigma+2}dx. \end{aligned} \tag{2.4}$$

Standard arguments prove that $I_\omega \in C^1(X^s, \mathbb{R})$ and for all $u, v \in X^s$ we have

$$\begin{aligned} I'_\omega(u)v = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{[u(x)-u(z)][v(x)-v(z)]}{|x-z|^{N+2s}}dzdx + \int_{\mathbb{R}^N}|x|^2u(x)v(x)dx \\ - \omega\int_{\mathbb{R}^N}u(x)v(x)dx - \int_{\mathbb{R}^N}|u(x)|^{2\sigma}u(x)v(x)dx. \end{aligned} \tag{2.5}$$

Hence, the critical points of I_ω are weak solutions of problem (1.1).

Let λ_1 defined as

$$\lambda_1 = \inf_{u\in X^s\setminus\{0\}} \frac{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(z)|^2}{|x-z|^{N+2s}}dzdx + \int_{\mathbb{R}^N}|x|^2u^2(x)dx}{\int_{\mathbb{R}^N}u^2(x)dx},$$

which is the simple first eigenvalue of the following linear problem

$$(-\Delta)^s u + |x|^2 u = \lambda u, \quad x \in \mathbb{R}^N.$$

Therefore,

$$\lambda_1\|u\|^2_{L^2(\mathbb{R}^N)} \leq \|u\|_s^2. \tag{2.6}$$

Then, we have the following lemma.

Lemma 2.3. *If $\omega < \lambda_1$, the functional I_ω satisfies the Palais–Smale condition at any level $c \in \mathbb{R}$, namely, any sequence $(u_n) \subset X^s$ such that*

$$I_\omega(u_n) \to c \quad \text{and} \quad I'_\omega(u_n) \to 0 \quad \text{as } n \to \infty \tag{2.7}$$

has a converging subsequence.

Proof. Let $(u_n)_{n\in\mathbb{N}} \subset X^s$ be a sequence verifying (2.7). Hence, for n large enough we have

$$\frac{1}{2}\|u_n\|_s^2 - \frac{\omega}{2}\|u_n\|_{L^2(\mathbb{R}^N)}^2 - \frac{1}{2\sigma+2}\|u_n\|_{L^{2\sigma+2}(\mathbb{R}^N)}^{2\sigma+2} = c + o_n(1) \tag{2.8}$$

and

$$\|u_n\|_s^2 - \omega\|u_n\|_{L^2(\mathbb{R}^N)}^2 - \|u_n\|_{L^{2\sigma+2}(\mathbb{R}^N)}^{2\sigma+2} = o_n(1)\|u_n\|_s. \tag{2.9}$$

Multiplying (2.9) by $-\frac{1}{2\sigma+2}$ and adding to (2.8) we have

$$\left(\frac{1}{2} - \frac{1}{2\sigma+2}\right)\|u_n\|_s^2 - \omega\left(\frac{1}{2} - \frac{1}{2\sigma+2}\right)\|u_n\|_{L^2(\mathbb{R}^N)}^2 = c + o_n(1) - \frac{o(1)}{2\sigma+2}\|u_n\|_s.$$

Hence,

$$\left(\frac{1}{2} - \frac{1}{2\sigma+2}\right)\left(1 - \frac{\omega}{\lambda_1}\right)\|u_n\|_s^2 \le c + o_n(1) - \frac{o_n(1)}{2\sigma+2}\|u_n\|_s.$$

Together with $\omega < \lambda_1$, we know that $(u_n)_{n\in\mathbb{N}}$ is bounded in X^s. Consequently, up to a subsequence, $u_n \rightharpoonup u$ in X^s and by Remark 2.2(2),

$$u_n \to u \text{ in } L^2(\mathbb{R}^N) \text{ and in } L^{2\sigma+2}(\mathbb{R}^N).$$

Therefore, by using the following equality

$$\langle I'_\omega(u_n) - I'_\omega(u), u_n - u\rangle + o_n(1) = \|u_n - u\|_s^2 - \omega\|u_n - u\|_{L^2(\mathbb{R}^N)}^2 - \|u_n - u\|_{L^{2\sigma+2}(\mathbb{R}^N)}^{2\sigma+2},$$

we conclude that $u_n \to u$ in X^s. □

Theorem 2.4. *Suppose that $\omega < \lambda_1$. Then, problem* (1.1) *has at least one radially symmetric ground state solution.*

Proof. We divide the proof into three parts. In the first step we use the mountain pass theorem to study the existence of a weak solution. In the second step by using Nehari's manifold we show that this solution is a ground state solution. Finally, by using symmetry rearrangement we show that this solution is radially symmetric.

Step 1. Clearly, $I_\omega(0) = 0$, and, by Lemma 2.3, I_ω verifies the Palais–Smale condition. Now, we claim that I_ω satisfies the geometry mountain pass condition. First, we note that there exists $\omega_0 > 0$ such that for any $u \in X^s$ we have

$$\|u\|_s^2 - \omega\|u\|_{L^2(\mathbb{R}^N)}^2 \ge \omega_0\|u\|_s^2. \tag{2.10}$$

If not, since $\omega < \lambda_1$, for any $\tilde{\omega} \le 0$, there is $u_0 \in X^s$ such that

$$0 < \left(1 - \frac{\omega}{\lambda_1}\right)\|u_0\|_s^2 \le \|u_0\|_s^2 - \omega\|u_0\|_{L^2(\mathbb{R}^N)}^2 \le \tilde{w}\|u_0\|_s^2 \le 0,$$

which is a contradiction. Therefore, (2.10) holds true. Remark 2.2 and (2.10) yield that

$$I_\omega(u) \ge \frac{w_0}{2}\|u\|_s^2 - \frac{C_{2\sigma+2}^{2\sigma+2}}{2\sigma+2}\|u\|_s^{2\sigma+2}.$$

Therefore, there are $\rho > 0$ and $\alpha > 0$ such that, for any $u \in X^s$ with $\|u\|_s = \rho$,

$$I_\omega(u) \geq \alpha.$$

On the other hand, let $\varphi \in C_0^\infty(\mathbb{R}^n)$ such that $\|\varphi\|_s = 1$, then

$$\frac{I_\omega(t\varphi)}{t^2} = \frac{1}{2} - \frac{\omega}{2} \int\limits_{\mathrm{supp}(\varphi)} \varphi^2(x)dx - \frac{t^{2\sigma}}{2\sigma+2} \int\limits_{\mathrm{supp}(\varphi)} |\varphi(x)|^{2\sigma+2}dx,$$

which implies that $I_\omega(t_0\varphi) < 0$ for t_0 large enough. Therefore, by applying the mountain pass theorem [3], there exists $\tilde{u} \in X^s$ such that

$$I_\omega(\tilde{u}) = c \quad \text{and} \quad I'_\omega(\tilde{u}) = 0,$$

where

$$c = \inf_{\gamma \in \Gamma_\omega} \sup_{t \in [0,1]} I_\omega(\gamma(t))$$

and

$$\Gamma_\omega = \{\gamma \in C([0,1], X^s) \,:\, \gamma(0) = 0,\, I_\omega(\gamma(1)) < 0\}.$$

Step 2. We claim that $\tilde{u}$ is a ground state solution. In fact, let

$$\mathcal{N}_\omega = \{u \in X^s \setminus \{0\} \,:\, I'_\omega(u)u = 0\}.$$

Note that $\mathcal{N}_\omega \neq \emptyset$, since $\tilde{u} \in \mathcal{N}_\omega$. Furthermore, I_ω is bounded from below on $\mathcal{N}_\omega$ and there exists $\Lambda > 0$ such that

$$I_\omega(u) > \Lambda \ \text{ for every } \ u \in \mathcal{N}_\omega.$$

Hence, there exists $\tilde{c} > 0$ such that

$$\tilde{c} = \inf_{u \in \mathcal{N}_\omega} I_\omega(u).$$

Clearly $\tilde{c} \leq c$. Let $w_n \subset \mathcal{N}_\omega$ be a minimizing sequence for $\tilde{c}$, so $(w_n)_{n\in\mathbb{N}}$ is a (PS)-sequence. As in the proof of Lemma 2.3, $(w_n)_{n\in\mathbb{N}}$ is bounded in X^s and up to a subsequence $w_n \to w$ in X^s. Furthermore, we can show that

$$I'_\omega(w_n)\varphi \to I'_\omega(w)\varphi, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^N), \tag{2.11}$$

and, since w_n is a (PS)-sequence, we obtain

$$I'_\omega(w)\varphi = 0, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^N).$$

Therefore w is a nontrivial weak solution of (1.1). Since $w \in \mathcal{N}_\omega$, we know that

$$I'_\omega(w)w^- = -\|w^-\|_s^2 = 0,$$

where $w^- = \max\{-w, 0\}$. Therefore, w is a ground state solution, that is, a non-negative solution with lowest energy and by [11, Theorem 1.1], $\tilde{u} = w$.

Step 3. $\tilde{u}$ is radially symmetric. Setting a minimizing sequence $(u_n)_{n\in\mathbb{N}} \subset X^s$ such that

$$I_\omega(u_n) \to \tilde{c} = \inf_{u\in\mathcal{N}_\omega} I_\omega(u).$$

Let $v_n = (u_n)^*$ the symmetric rearrangement of u_n. Since symmetric rearrangements are continuous in X^s (see [2]), then $v_n \in X^s$. Moreover, it is well known that

$$\int_{\mathbb{R}^N} |u|^{2\sigma+2}dx = \int_{\mathbb{R}^N} |u^*|^{2\sigma+2}dx, \quad \int_{\mathbb{R}^N} |u|^2 dx = \int_{\mathbb{R}^N} |u^*|^2 dx \tag{2.12}$$

and

$$\begin{gathered}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u^*(x)-u^*(y)|^2}{|x-y|^{N+2s}}dydx \le \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dydx, \\ \int_{\mathbb{R}^N} |x|^2|u^*|^2dx \le \int_{\mathbb{R}^N} |x|^2|u|^2dx.\end{gathered} \tag{2.13}$$

Therefore, for n large enough, we get

$$\tilde{c} \le I_\omega(v_n) \le I_\omega(u_n) \le \tilde{c} + \frac{1}{n}, \tag{2.14}$$

and by Ekeland's variational principle there is a sequence $(z_n)_{n\in\mathbb{N}} \subset X^s$ such that

$$I_\omega(z_n) \to \tilde{c}, \quad I'_\omega(z_n) \to 0 \quad \text{and} \quad \|z_n - v_n\|_s \to 0.$$

Consequently, by the continuity of symmetric rearrangement and Lemma 2.3, there exists $z \in X^s$ such that $z_n \to z$ in X^s, $I_\omega(z) = \tilde{c}$, $I'_\omega(z) = 0$ and

$$\lim_{n\to\infty} \|z - v_n\|_s = 0.$$

The last equality shows that $z = \tilde{u}$. □

3. PROPERTIES OF THE LEAST ACTION LEVEL

By the previous section, the energy levels

$$\tilde{C}(\omega) = \inf_{u\in\mathcal{N}_\omega} I_\omega(u) \quad \text{and} \quad C(\omega) = \inf_{\gamma\in\Gamma_\omega} \sup_{t\in[0,1]} I_\omega(\gamma(t)),$$

are well defined. Moreover as in [15, Lemma 4.2] we can show that

$$\tilde{C}(\omega) = \inf_{u\in X^s\setminus\{0\}} \max_{t\ge 0} I_\omega(tu) = C(\omega). \tag{3.1}$$

Now we will study their properties such as continuity and differentiability. Note that, in the case $s = 1$, these properties where studied in [8, 14, 16]. We start our analysis with the following result.

Lemma 3.1. *For every $\omega < \lambda_1$, the function $\omega \mapsto \tilde{C}(\omega)$ is continuous.*

Proof. To show this result we borrowed some ideas of [1]. Set $(\omega_n)_{n\in\mathbb{N}} \subset (0, \lambda_1)$ and $\omega_0 \in (0, \lambda_1)$ such that

$$\omega_n \to \omega_0.$$

The previous analysis implies that

$$\liminf_{\omega_n \in (0,\lambda_1)} \tilde{C}(\omega_n) > 0 \quad \text{and} \quad \limsup_{\omega_n \in (0,\lambda_1)} \tilde{C}(\omega_n) < +\infty.$$

Next, let $u_n \in X^s$ the function which satisfy

$$I_{\omega_n}(u_n) = \tilde{C}(\omega_n) \quad \text{and} \quad I'_{\omega_n}(u_n) = 0. \tag{3.2}$$

In what follows, we will consider two sequences $\{\omega_{n_j}\}$ and $\{\omega_{n_k}\}$ such that

$$\tilde{C}(\omega_{n_j}) \geq \tilde{C}(\omega_0), \quad \forall n_j \tag{3.3}$$

and

$$\tilde{C}(\omega_{n_k}) \leq \tilde{C}(\omega_0), \quad \forall n_k. \tag{3.4}$$

Analysis of (3.3)*:* From the above results, we know that $\{\tilde{C}(\omega_{n_j})\}$ is bounded. Consequently, there are a subsequence $\{\omega_{n_{j_i}}\} \subset \{\omega_{n_j}\}$ and $C_0 > 0$ such that

$$C(\omega_{n_{j_i}}) \to C_0.$$

In the sequel, we will use the following notations:

$$u_i = u_{n_{j_i}} \quad \text{and} \quad \omega_i = \omega_{n_{j_i}}.$$

Thereby,

$$\omega_i \to \omega_0 \quad \text{and} \quad \tilde{C}(\omega_i) \to C_0.$$

We claim that $C_0 = \tilde{C}(\omega_0)$. In fact, from (3.3),

$$C_0 = \lim_i \tilde{C}(\omega_i) \geq \tilde{C}(\omega_0). \tag{3.5}$$

Let $w_0 \in X^s$ be such that

$$I_{\omega_0}(w_0) = \tilde{C}(\omega_0) \quad \text{and} \quad I'_{\omega_0}(w_0) = 0.$$

Moreover, we denote by $t_i > 0$ the real number which verifies

$$I_{\omega_i}(t_i w_0) = \max_{t\geq 0} I_{\omega_i}(t w_0).$$

Thus, by definition of $\tilde{C}(\omega_0)$,

$$\tilde{C}(\omega_i) \leq I_{\omega_i}(t_i w_0).$$

It is possible to prove that $\{t_i\}$ is a bounded sequence. Then without loss of generality we can assume that $t_i \to t_0$. Now, the Lebesgue theorem gives

$$\lim_i I_{\omega_i}(t_i w_0) = I_{\omega_0}(t_0 w_0) \le I_{\omega_0}(w_0) = \tilde{C}(\omega_0),$$

leading to

$$C_0 \le \tilde{C}(\omega_0). \tag{3.6}$$

From (3.5)–(3.6),

$$\tilde{C}(\omega_0) = C_0.$$

The above study implies that

$$\lim_i C(\omega_{n_{j_i}}) = C(\omega_0).$$

Analysis of (3.4)*:* Note that (3.2) implies that $\{u_n\}$ is a bounded sequence in X^s. Consequently, there is $u_0 \in X^s$ such that up to a subsequence

$$u_n \rightharpoonup u_0 \text{ in } \quad X^s.$$

The above information permits to conclude that u_0 is a nontrivial solution of the problem

$$(-\Delta)^s u + |x|^2 u = \omega_0 u + |u|^{2\sigma} u \quad \text{in } \mathbb{R}^N, \quad u \in X^s. \tag{3.7}$$

By Fatous' lemma, it is possible to prove that

$$\liminf_n I_{\omega_n}(u_n) \ge I_{\omega_0}(u_0). \tag{3.8}$$

On the other hand, there is $\zeta_n > 0$ such that

$$\tilde{C}(\omega_n) \le I_{\omega_n}(\zeta_n u_0), \quad \forall n.$$

So

$$\limsup_n I_{\omega_n}(u_n) = \limsup_n \tilde{C}(\omega_n) \le \limsup_n I_{\omega_n}(\zeta_n u_0) = I_{\omega_0}(u_0). \tag{3.9}$$

From (3.8)–(3.9),

$$\lim_n I_{\omega_n}(u_n) = I_{\omega_0}(u_0).$$

The last limit yields

$$u_n \to u_0 \text{ in } X^s.$$

Since $\{\tilde{C}(\omega_{n_{j_k}})\}$ is bounded, there are a subsequence $\{\omega_{n_{j_k}}\} \subset \{\omega_{n_j}\}$ and $C_* > 0$ such that

$$\tilde{C}(\omega_{n_{j_k}}) \to C_*.$$

In the sequel, we will use the following notations:

$$u_k = u_{n_{j_k}} \quad \text{and} \quad \omega_k = \omega_{n_{j_k}}.$$

Thus,

$$u_k \to u_0, \quad \omega_k \to \omega_0 \quad \text{and} \quad \tilde{C}(\omega_k) \to C_*.$$

In what follows, we denote by $t_k > 0$ the real number which verifies

$$I_{\omega_0}(t_k u_k) = \max_{t \geq 0} I_{\omega_0}(t u_k).$$

Thus, by definition of $\tilde{C}(\omega_0)$,

$$\tilde{C}(\omega_0) \leq I_{\omega_0}(t_k u_k).$$

It is possible to prove that $\{t_k\}$ is a bounded sequence. Then without loss of generality we can assume that $t_k \to t_*$. Now, the Lebesgue theorem gives

$$\lim_k I_{\omega_0}(t_k u_k) = I_{\omega_0}(t_* u_0) = \lim_k I_{\omega_k}(t_k u_k) \leq \lim_k \tilde{C}(\omega_k) = C_*.$$

Thereby,

$$\tilde{C}(\omega_0) \leq C_*. \tag{3.10}$$

On the other hand, from (3.4),

$$\lim_k \tilde{C}(\omega_k) \leq \tilde{C}(\omega_0)$$

leading to

$$C_* \geq \tilde{C}(\omega_0). \tag{3.11}$$

From (3.10)–(3.11),

$$C_* = \tilde{C}(\omega_0).$$

The above study implies that

$$\lim_k \tilde{C}(\omega_{n_{j_k}}) = \tilde{C}(\omega_0).$$

From (3.3) and (3.4),

$$\lim_n \tilde{C}(\omega_n) = \tilde{C}(\omega_0).$$

□

Lemma 3.2. *The function $w \mapsto \tilde{C}(w)$ is a decreasing function for every $w < \lambda_1$.*

Proof. Let $\omega_1 \leq \omega_2 < \lambda_1$, then

$$\begin{aligned} I_{\omega_2}(u) &= \frac{1}{2}\|u\|_s^2 - \frac{\omega_2}{2}\|u\|^2_{L^2(\mathbb{R}^N)} - \frac{1}{2\sigma+2}\|u\|^{2\sigma+2}_{L^{2\sigma+2}(\mathbb{R}^N)} \\ &\leq \frac{1}{2}\|u\|_s^2 - \frac{\omega_1}{2}\|u\|^2_{L^2(\mathbb{R}^N)} - \frac{1}{2\sigma+2}\|u\|^{2\sigma+2}_{L^{2\sigma+2}(\mathbb{R}^N)} = I_{\omega_1}(u), \end{aligned}$$

which implies

$$I_{\omega_2}(tu) \leq I_{\omega_1}(tu), \quad \forall t \geq 0 \ \text{and} \ u \in X^s,$$

and

$$\max_{t \geq 0} I_{\omega_2}(tu) \leq \max_{t \geq 0} I_{\omega_1}(tu), \ \ \forall u \in X^s.$$

Therefore, by (3.1), we get

$$\tilde{C}(\omega_2) \leq \tilde{C}(\omega_1). \tag{3.12}$$

□

Remark 3.3. Note that by (3.12) we have that $\tilde{C}$ is a decreasing function. Furthermore, we can show that the function $\tilde{C}$ is strictly decreasing, i.e.

$$\omega_1 < \omega_2 \quad \text{implies that} \quad \tilde{C}(\omega_2) < \tilde{C}(\omega_1). \tag{3.13}$$

In fact, let u_{ω_1} be a critical point with critical value $\tilde{C}(\omega_1)$. Then for any $t > 0$ we have

$$\tilde{C}(\omega_1) = I_{\omega_1}(u_{\omega_1}) \geq I_{\omega_1}(tu_{\omega_1}) > I_{\omega_2}(tu_{\omega_1}).$$

Let $t^* > 0$ be such that $t^* u_{\omega_1} \in \mathcal{N}_{\omega_2}$ and

$$I_{\omega_2}(t^* u_{\omega_1}) = \sup_{t>0} I_{\omega_2}(tu_{\omega_1}).$$

Consequently,

$$\tilde{C}(\omega_1) > I_{\omega_2}(t^* u_{\omega_1}) \geq \inf_{\mathcal{N}_{\omega_2}} I_{\omega_2}(u) = \tilde{C}(\omega_2).$$

Now, let $u_\omega \in X^s$ be the ground state solution given by Theorem 2.4, that is,

$$I_\omega(u_\omega) = \tilde{C}(\omega) \quad \text{and} \quad I'_\omega(u_\omega)u_\omega = 0. \tag{3.14}$$

Then

$$\tilde{C}(\omega) = I_\omega(u_\omega) - \frac{1}{2} I'_\omega(u_\omega)u_\omega = \left(\frac{1}{2} - \frac{1}{2\sigma+2}\right) \|u_\omega\|^{2\sigma+2}_{L^{2\sigma+2}(\mathbb{R}^N)}. \tag{3.15}$$

Theorem 3.4. *The ground state level $\tilde{C}(\omega)$ is differentiable at almost everywhere $\omega < \lambda_1$. Moreover,*

$$\tilde{C}'(\omega) = -\frac{1}{2}\|u_\omega\|^2_{L^2(\mathbb{R}^N)}.$$

Proof. We borrowed some ideas from [18]. Consider the ground state level

$$\tilde{C}(\omega) = \inf_{u \in \mathcal{N}_\omega} I_\omega(u) \quad \text{and} \quad \tilde{C}(\eta) = \inf_{u \in \mathcal{N}_\eta} I_\eta(u), \tag{3.16}$$

where

$$\begin{aligned} I_\omega(u) = \frac{1}{2}\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(z)|^2}{|x-z|^{N+2s}} dz dx + \int_{\mathbb{R}^N} |x|^2 u^2 dx\right) \\ - \frac{\omega}{2}\int_{\mathbb{R}^N} u^2 dx - \frac{1}{2\sigma+2}\int_{\mathbb{R}^N} |u|^{2\sigma+2} dx \end{aligned} \tag{3.17}$$

and

$$\begin{aligned} I_\eta(u) = \frac{1}{2}\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(z)|^2}{|x-z|^{N+2s}} dz dx + \int_{\mathbb{R}^N} |x|^2 u^2 dx\right) \\ - \frac{\eta}{2}\int_{\mathbb{R}^N} u^2 dx - \frac{1}{2\sigma+2}\int_{\mathbb{R}^N} |u|^{2\sigma+2} dx. \end{aligned} \tag{3.18}$$

Note that, for any $\eta < \lambda_1$, we can show that there exists $u_\eta \in X^s \setminus \{0\}$ such that

$$I_\eta(u_\eta) = \tilde{C}(\eta) \quad \text{and} \quad I'_\eta(u_\eta)u_\eta = 0.$$

Furthermore, there exists $t(\omega,\eta) > 0$ such that $t(\omega,\eta)u_\eta \in \mathcal{N}_\omega$, namely

$$\begin{aligned} &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_\eta(x)-u_\eta(y)|^2}{|x-y|^{n+2s}}dydx + \int_{\mathbb{R}^N} |x|^2 u_\eta^2 dx \\ &= \omega \int_{\mathbb{R}^N} u_\eta^2 dx + t^{2\sigma}(\omega,\eta) \int_{\mathbb{R}^N} |u_\eta|^{2\sigma+2} dx \end{aligned} \tag{3.19}$$

and $t(\eta,\eta) = 1$. By the implicit function theorem, $t(\omega,\eta)$ is differentiable with respect to variable $\omega < \lambda_1$.

Define the function

$$\begin{aligned} F(\omega,\eta) &= I_\omega(t(\omega,\eta)u_\eta) \\ &= \frac{t^2(\omega,\eta)}{2}\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_\eta(x)-u_\eta(y)|^2}{|x-y|^{n+2s}}dydx + \int_{\mathbb{R}^N} |x|^2 u_\eta^2 dx\right) \\ &\quad - \frac{t^2(\omega,\eta)\omega}{2}\int_{\mathbb{R}^N} u_\eta^2 dx - \frac{t^{2\sigma+2}(\omega,\eta)}{2\sigma+2}\int_{\mathbb{R}^N} |u_\eta|^{2\sigma+2} dx. \end{aligned}$$

So

$$\begin{aligned} \frac{\partial}{\partial\omega}F(\omega,\eta) &= \frac{\partial}{\partial\omega} I_\omega(t(\omega,\eta)u_\eta) \\ &= t(\omega,\eta)\frac{\partial}{\partial\omega}t(\omega,\eta)\|u_\eta\|_{X^s}^2 - t^{2\sigma+1}(\omega,\eta)\frac{\partial}{\partial\omega}t(\omega,\eta)\|u_\eta\|_{L^{2\sigma+2}(\mathbb{R}^N)}^{2\sigma+2} \\ &\quad - t(\omega,\eta)\frac{\partial}{\partial\omega}t(\omega,\eta)\omega\|u_\eta\|_{L^2(\mathbb{R}^N)}^2 - \frac{t^2(\omega,\eta)}{2}\|u_\eta\|_{L^2(\mathbb{R}^N)}^2 \\ &= t(\omega,\eta)\frac{\partial}{\partial\omega}t(\omega,\eta)\left(\|u_\eta\|_{X^s}^2 - \omega\|u_\eta\|_{L^2(\mathbb{R}^N)}^2 - t^{2\sigma}(\omega,\eta)\|u_\eta\|_{L^{2\sigma+2}(\mathbb{R}^N)}^{2\sigma+2}\right) \\ &\quad - \frac{t^2(\omega,\eta)}{2}\|u_\eta\|_{L^2(\mathbb{R}^N)}^2 \\ &= -\frac{t^2(\omega,\eta)}{2}\|u_\eta\|_{L^2(\mathbb{R}^N)}^2. \end{aligned}$$

Therefore,

$$\begin{aligned} \tilde{C}(\omega) - \tilde{C}(\eta) \le I_\omega(t(\omega,\eta)u_\eta) - I_\eta(u_\eta) &= F(\omega,\eta) - F(\eta,\eta) \\ &= (\omega-\eta)\frac{\partial}{\partial\omega}F(\xi,\eta)\Big|_{\xi\in[\omega,\eta]} \\ &= -(\omega-\eta)\frac{t^2(\xi,\eta)}{2}\|u_\eta\|_{L^2(\mathbb{R}^N)}^2, \quad \xi \in [\omega,\eta]. \end{aligned}$$

From this we get

$$\limsup_{\omega\to\eta} \frac{\tilde{C}(\omega)-\tilde{C}(\eta)}{\omega-\eta} \le -\frac{\|u_\eta\|^2_{L^2(\mathbb{R}^N)}}{2}. \tag{3.20}$$

We claim that the map $\omega \mapsto u_\omega$ from $(0,\lambda_1)$ to X^s is continuous. In fact, by contradiction, suppose that there is $\omega_0 < \lambda_1$, a sequence $(\omega_n) \subset (-\infty, \lambda_1)$ with $\omega_n \to \omega_0$ and $\delta > 0$ such that

$$\|u_{\omega_n} - u_{\omega_0}\|_s \ge \delta.$$

Note that $(u_{\omega_n}) \subset X^s$ is a bounded Palais–Smale sequence for I_{ω_0} at the level $\tilde{C}(\omega_0)$. In fact, by Lemma 3.1, the sequence $(\tilde{C}(\omega_n)) \subset \mathbb{R}$ is bounded. Now we have

$$\begin{aligned} I_{\omega_0}(u_n) &= I_{\omega_n}(u_n) + \frac{\omega_n-\omega_0}{2}\|u_n\|^2_{L^2(\mathbb{R}^N)},\\ I'_{\omega_0}(u_n)\varphi &= I'_{\omega_n}(u_n)\varphi + (\omega_n-\omega_0)\int\limits_{\mathbb{R}^N} u_n\varphi dx. \end{aligned}$$

Since $I'_{\omega_n}(u_n) = o_n(1)$ and $(u_n) \subset X^s$ is bounded, we have

$$I'_{\omega_0}(u_n) \to 0, \quad \text{as } n \to \infty.$$

Also

$$\frac{\omega_n-\omega_0}{2}\|u_n\|^2_{L^2(\mathbb{R}^N)} \to 0,$$

and we deduce that $(I_{\omega_0}(u_n))_{n\in\mathbb{N}} \subset \mathbb{R}$ is a bounded sequence. This proves that $(u_{\omega_n}) \subset X^s$ is a bounded Palais–Smale sequence for I_{ω_0}. As in Lemma 2.3, we deduce that $u_{\omega_n} \to u_0$ with $u_0 \in X^s$ a critical point of I_{ω_0}. At this point, using the uniqueness, we deduce that $u_{\omega_n} \to u_{\omega_0}$ and this contradiction concludes the proof. Consequently, since

$$\begin{aligned} \tilde{C}(\omega)-\tilde{C}(\eta) &\ge I_\omega(t(\omega,\omega)u_\omega) - I_\eta(t(\eta,\omega)u_\omega)\\ &= F(\omega,\omega) - F(\eta,\omega)\\ &= (\omega-\eta)\frac{\partial}{\partial\omega}F(\xi,\omega) \;\; \xi\in[\omega,\eta]\\ &= (\omega-\eta)\left(-\frac{t^2(\xi,\omega)}{2}\|u_\omega\|^2_{L^2(\mathbb{R}^N)}\right) \end{aligned}$$

by the previous analysis, we obtain

$$\liminf_{\omega\to\eta} \frac{\tilde{C}(\omega)-\tilde{C}(\eta)}{\omega-\eta} \ge -\frac{\|u_\eta\|^2_{L^2(\mathbb{R}^N)}}{2}. \tag{3.21}$$

Therefore, by (3.20) and (3.21) we obtain

$$\tilde{C}'(\eta) = -\frac{1}{2}\|u_\eta\|^2_{L^2(\mathbb{R}^N)}. \tag{3.22}$$

□

4. BIFURCATION TYPE RESULT

In this section we will prove a bifurcation type result via variational methods. More precisely, we have the following result.

Theorem 4.1. *For every $\omega \in (0, \lambda_1)$ there is a nonnegative and nontrivial solution u_ω of* (1.1) *such that*

$$\|u_\omega\|_s \to 0 \quad \text{as } \omega \to \lambda_1.$$

Proof. Let $\varphi \in X^s$ be such that

$$\|\varphi\|_s^2 = \lambda_1 \|\varphi\|^2_{L^2(\mathbb{R}^N)}.$$

Next,

$$0 < \tilde{C}(\omega) \leq \max_{t\in[0,1]} I_\omega(t\varphi).$$

For every $t \in [0,1]$, we have

$$I_\omega(t\varphi) = \frac{(\lambda_1 - \omega)\|\varphi\|^2_{L^2(\mathbb{R}^N)}}{2} t^2 - \frac{t^{2\sigma+2}}{2\sigma+2}\|\varphi\|^{2\sigma+2}_{L^{2\sigma+2}(\mathbb{R}^N)}.$$

Let

$$g(t) = \frac{(\lambda_1 - \omega)}{2}\|\varphi\|^2_{L^2(\mathbb{R}^N)} t^2 - \frac{\|\varphi\|^{2\sigma+2}_{L^{2\sigma+2}(\mathbb{R}^N)}}{2\sigma+2} t^{2\sigma+2}.$$

By elementary computations, we can show that g attains its maximum at the point

$$t_0 = \left(\frac{(\lambda_1 - \omega)\|\varphi\|^2_{L^2(\mathbb{R}^N)}}{\|\varphi\|^{2\sigma+2}_{L^{2\sigma+2}(\mathbb{R}^N)}} \right)^{\frac{1}{2\sigma}}.$$

Hence

$$0 < \tilde{C}(\omega) \leq g(t_0) = \left(\frac{1}{2} - \frac{1}{2\sigma+2}\right) \frac{\left[(\lambda_1 - \omega)\|\varphi\|^2_{L^2(\mathbb{R}^N)}\right]^{\frac{2\sigma+2}{2\sigma}}}{\|\varphi\|^{\frac{2\sigma+2}{\sigma}}_{L^{2\sigma+2}(\mathbb{R}^N)}},$$

so that

$$\lim_{\omega\to\lambda_1} \tilde{C}(\omega) = 0.$$

From this it is easy to deduce that

$$\lim_{\omega\to\lambda_1} \|u_\omega\|_s = 0,$$

where u_ω is given by Theorem 2.4. Indeed, since

$$\tilde{C}(\omega) = I_\omega(u_\omega) = \frac{1}{2}\|u_\omega\|_s^2 - \omega \int\limits_{\mathbb{R}^N} F(u_\omega)dx$$

and

$$0 = I'_\omega(u_\omega)u_\omega = \|u_\omega\|_s^2 - \omega \int\limits_{\mathbb{R}^N} f(u_\omega)u_\omega dx,$$

where

$$f(t) = |t| + \frac{1}{\omega}|t|^{2\sigma}t \quad \text{and} \quad F(t) = \int\limits_0^t f(\tau)d\tau = \frac{|t|^2}{2} + \frac{1}{\omega(2\sigma+2)}|t|^{2\sigma+2}.$$

Moreover, this function satisfies the Ambrosetti–Rabinowitz condition with $\theta = 2\sigma + 2$. Hence,

$$\begin{aligned}\tilde{C}(\omega) &= I_\omega(u_\omega) - \frac{1}{2\sigma+2}I'_\omega(u_\omega)u_\omega \\ &= \left(\frac{1}{2} - \frac{1}{2\sigma+2}\right)\|u_\omega\|_s^2 + \omega \int\limits_{\mathbb{R}^N} \left(\frac{1}{2\sigma+2}f(u_\omega)u_\omega - F(u_\omega)\right)dx \\ &\geq \left(\frac{1}{2} - \frac{1}{2\sigma+2}\right)\|u_\omega\|_s^2.\end{aligned}$$

This show that $\|u_\omega\|_s \to 0$ as $\omega \to \lambda_1$. □

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