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GEODESY AND CARTOGRAPHY
Vol. 70, no. 2, article no. e05, 2021

DOI: https://doi.org/10.24425/gac.2021.136680

Original paper

Conformal and empirical transformation between the PL-ETRF89 and PL-ETRF2000 reference frames using the new adjustment of the former Polish I class triangulation network

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Received: 2021-05-09 / Accepted: 2021-06-23

Abstract: The European reference frame ETRF2000 was introduced on the territory of Poland on 1 July 2013, named PL-ETRF2000, as a result of the appropriate measurement campaign 2008-2011. The new PL-ETRF2000 reference frame has replaced the previously used PL-ETRF89 frame, which had more than 10 years of history in Poland until 2013, implemented in almost all geodetic and cartographic "products", in geodetic networks, economic map systems and databases. The relationship of the new reference frame with the previously used PL-ETRF89 frame has become an important practical issue. Currently, all position services of the ASG-EUPOS (Active Geodetic Network – EUPOS) system use only the PL-ETRF2000 reference frame, which also results from the relevant legal and technical regulations. The relationships between the frames was considered in two aspects: "theoretical", expressed by conformal (Helmert, 7-parameter) transformation, and "empirical", based on an interpolation grid that allows to take into account local distortions of the PL-ETRF89 frame. The estimation of the parameters of the conformal transformation model was based on 330 points of the POLREF network, while to create an interpolation grid approximately 6500 points of the old triangulation network were additionally used, after new adjustment in PL-ETRF200 reference frame. Basic algorithms for the transformation between two frames and mapping systems are implemented in the new version of the TRANSPOL program, which is available on the web (www.gugik.gov.pl).

Keywords: European reference frames, empirical transformation, interpolation grid, network adjustment, 3D Helmert transformation

1. Introduction

The spatial reference frame currently in force since 2013 in Poland PL-ETRF2000 is a realization of European Reference Frame ETRF2000 at epoch 2011. The frame was introduced as a result of the measurement campaign (2008–2011) integrating the ASG-EUPOS stations with the basic Polish networks (EUREF-POL, POLREF, EUVN and selected points of the former class I, i.e. the astronomical-geodetic network, and the filling triangulation). The network calculations were performed independently by two research teams: the Warsaw University of Technology (WUT) (Liwosz et al., 2011) and the Space Research Center (CBK) (Jaworski, 2011). Formally, the first variant (WUT) was adopted as a basic solution, although the results of the two solutions were very similar. The maximum differences in absolute value in the GNSS vector components were 1.1 cm, and the average differences were less than 1 mm. Based on the cited studies and other analyzes (e.g. Bosy, 2011; Kadaj, 2013), it can be concluded that the determined coordinates of the points are characterized by high, millimetres accuracy, which affects the accuracy of positioning using the ASG-EUPOS system services.

It should be emphasized that the previous reference system PL-ETRF89, represented by the EUREF-POL and POLREF networks established in the 1990s, was implemented in 2000–2012 in almost all geodetic "products" in Poland. First, there were basic, detailed and measurement networks of that time, then economic maps and some of the topographic maps. Simultaneously with the introduction of this system in Poland, analogue geodetic and cartographic resources were computerized and transformed into newly-defined cartographic systems (PL-2000, PL-1992) from the old systems representing the PUŁKOWO'42 reference frame. The new Polish cartographic systems are defined as modified Gauss–Krüger projections with following parameters: PL-2000 system divided into 4 area zones with axial meridians 15°, 18°, 21°, 24° and shrinkage scale 0.999923, PL-1992 one-zone system with axial meridian 19° and shrinkage scale 0.9993 (see Balcerzak, 1994 or e.g. Kadaj, 2001).

As a result of the new measurement campaign (2008-2011) a new reference frame PL-ETRF2000 was introduced, first based on satellite networks, i.e. ASG-EUPOS, POLREF, EUREF-POL, EUVN stations – totalling approx. 500 points. Subsequently, the former class I triangulation network was re-adjusted (Fig. 1 and Table 1). It was a local densification of the PL-ETRF2000 reference frame, necessary in practice when integrating measurement data from different reference systems. Regardless of the issue of standardizing databases in terms of reference frames, the new coordinates of the former I class network, together with satellite networks, allowed to create tools (in the form of an interpolation grid) for the transformations between two frames, with elimination of local deformations of the PL-ETRF89 frame.

2. Adjustment of the primary Polish I class triangle network in PL-ETRF2000 frame

In Table 1 the parameters of the former I class triangulation network (Fig. 1) are given. The network was adjusted earlier (in 1996) in the PL-ETRF89 frame on the GRS80 ellipsoid. In the same geometric structure, but with the assumption of the tie points in the PL-ETRF2000 system and in connection with the introduction of a new frame of reference in Poland, the network was adjusted in 2012. The accuracy statistics in both adjustments were similar (Tables 2 and 3). In both cases, the GEONET program was used with different implementations of the same alignment algorithm (in 1996 in the

WATCOM – FORTRAN language, and in 2012 in the DELPHI language). In 1996, in addition to adjusting the network on the ellipsoid, an analogous adjustment of this network was also performed in the PL-1992 system (Gauss–Krüger mapping in application

Table 1. Characteristic parameters of the network

Number of:	
all network points	$p = 6877$
reference points including direction points	$s = 362$
determined point	$r = 6515$
angles	$k = 45537$
directional station	$dir\text{-}s = 970$
directional observation	$dir\text{-}o = 4302$
classic distances	$d = 1002$
unknown coordinates	$n = 13030$
all independent observation	$m = 49871$
redundance	$m - n = 36841$

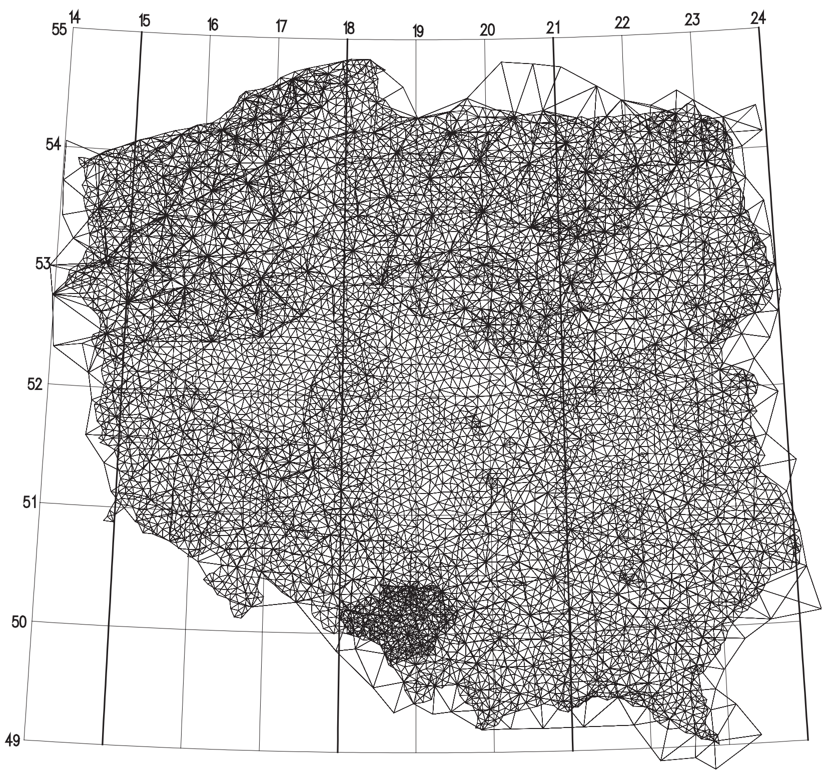

Fig. 1. Classic triangle network in Poland

to territory of Poland), using also another program (Kozakiewicz, 1998). Compatibility of the calculation results from two programs had some control and reliability significance.

Table 2. Statistic of point position errors

Statistics of network point position errors		
parameter	PL-ETRF2000	PL-ETRF89
for points in the territory of Poland:		
number of points	6805	6805
mP (max) [m]	0.081	0.082
mP (average) [m]	0.019	0.019
for foreign points:		
number of points	72	72
mP (max) [m]	0.156	0.155
mP (average) [m]	0.072	0.72

The estimated mean error of the observations with unit weight in the PL-ETRF89 frame was $\mu_0 = 1.014$, and in the PL-ETRF2000 frame was $\mu_0 = 1.013$. The insignificant difference results from the more precise tie points in the PL-ETRF2000 system. The value of this parameter in the unitless form proves the mutual compliance of the assumed errors of mean observations and the parameters of the distribution of deviations obtained as a result of the network adjustment.

Tables 2 and 3, and Figures 2, 3 and 4 show the accuracy statistics after the network adjustment in the PL-ETRF2000 reference frame and their comparison with the analogous statistics from the network adjustment in the PL-ETRF89 frame. As shown in Table 3, in the area of Poland, the average value of the point location error is approx. 2 cm, although the maximum value of this parameter was approx. 8 cm. It can be said that this network represents, on average, an internal accuracy at the level of POLREF

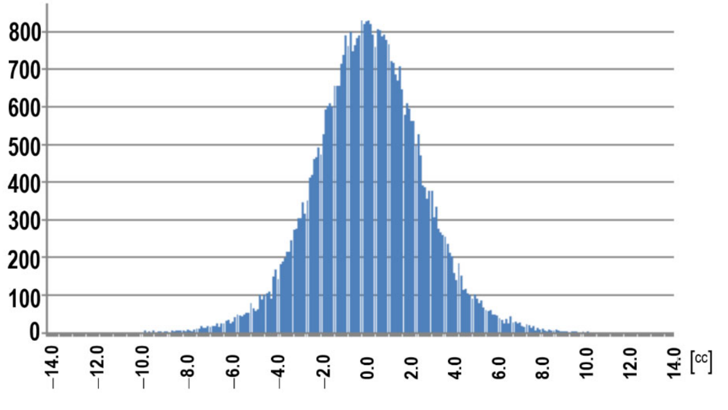

Fig. 2. Distribution of the number of 45537 angular corrections in classes every 0.6^{cc}

Table 3. Statistic of observational corrections

Observation correction statistics		
parameter	PL-ETRF2000	PL-ETRF89
for directional observations [cc]:		
number of corrections	4302	4302
the smallest	–5.00	–5.00
the greatest	5.00	4.90
average	–0.00	0.00
absolute average	0.89	0.89
RMS	1.15	1.15
for angular observations [cc]:		
number of corrections	45537	45537
the smallest	–13.60	–13.60
the greatest	12.00	12.00
average	0.09	0.09
absolute average	1.84	1.84
RMS	2.36	2.36
for distances [m]:		
number of corrections	1002	1002
the smallest	–0.0488	–0.0459
the greatest	0.0511	0.0548
average	–0.0005	–0.0001
absolute average	0.0074	0.0074
RMS	0.0099	0.0099

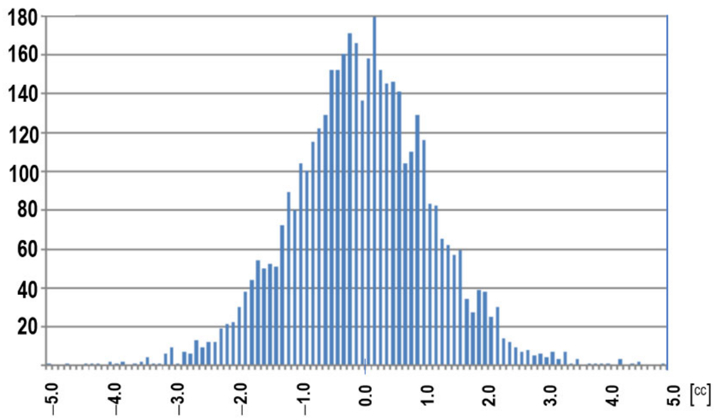

Fig. 3. Distribution of the number of 4302 direction corrections in classes every 0.1^{cc}

(approximately 200 points of this network were reference points). It also proves that the internal accuracy of the old basic network equalled the accuracy of GPS positioning in the 90's. The histograms of corrections for observations in Figures 2, 3 and 4 highlight that the network does not contain significant systematic errors. Positive and negative corrections have comparable frequency of occurrence and the corrections show an error distribution with no tendency to asymmetry. It also proves that the observations were correctly reduced on the GRS80 ellipsoid.

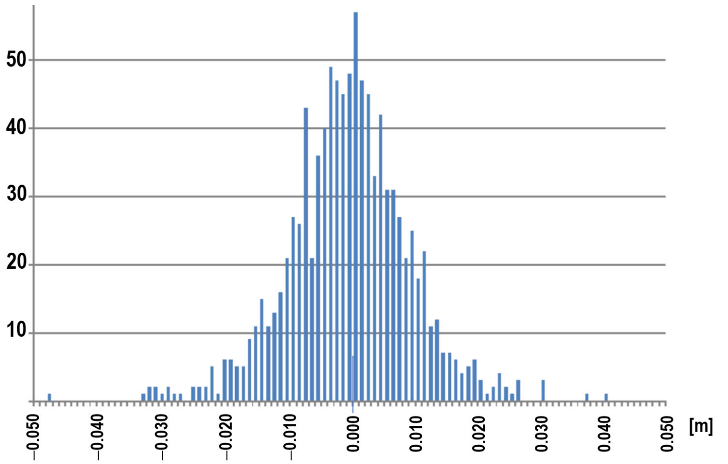

Fig. 4. Distribution of the number of 1002 distance corrections in classes every 0.001 m

The network adjustments in the PL-ETRF89 and PL-ETRF2000 frames significantly increases the set of reference points from 330 POLREF points by approx. 6500 of the former triangulation network to create optimal transformation formulas between the frames (Fig. 5). Unfortunately, unlike the POLREF network, the triangulation network, adjusted on the ellipsoid, does not allow for the application of a three-dimensional transformation. Therefore, we assume that the mathematical model of the three-dimensional conformal (7-parameter Helmert) transformation is based solely on the POLREF network points, while the triangulation network (no height component), with a significantly higher density, contains useful information, but only in terms of coordinates latitude and longitude. The mathematical transformation itself does not take into account local deformations of the reference systems. We dealt with a similar problem in the transformations between the PL-1965 cartographic system in the PUŁKOWO'42 system and the PL-2000 system in new frames of reference. In addition to transformations according to mathematical transformation laws, additionally post-transformational corrections were applied, constituting the distributions of transformation deviations formed on the reference points to all transformed points.

A similar issue concerns the transformation between the PL-ETRF89 and PL-ETRF2000 frames. In this case, instead of applying post-transformational corrections that require constant access to reference points, we will use an interpolation grid with the resolution of $0.01^\circ \times 0.01^\circ$. At all grid nodes, the differences of the coordinates δB,

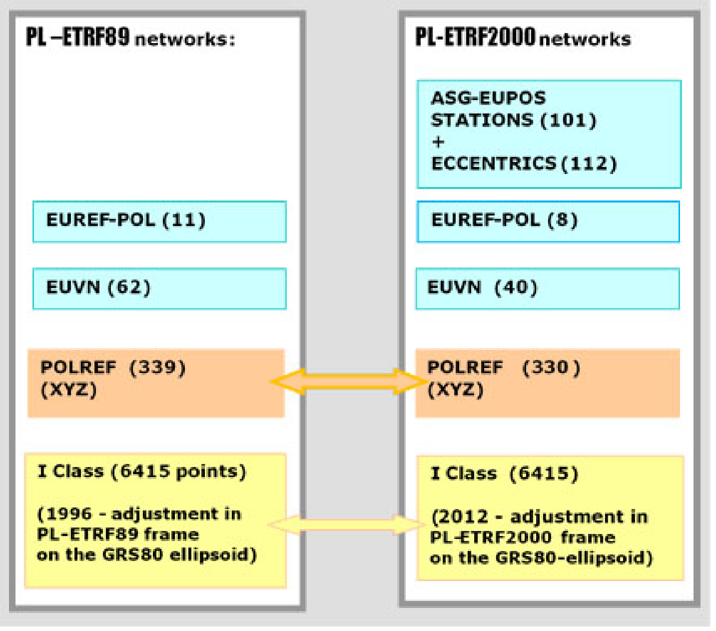

Fig. 5. Geodetic networks representing two references. POLREF (330 points) used theoretical transformation between two frames. For empirical method (interpolation grid) also primary I class network was added

δL, δh between the systems of interest are determined (interpolated on the basis of reference points). For the determination of the increments δB, δL both the POLREF network and the adjusted triangulation network were taken into account, while for the determination of the ellipsoidal height differences, only POLREF points were taken. For the purpose of control and qualitative assessment, analogous values that would result from the conformal transformation were compared in the grid nodes. Therefore, we will consider a 3D transformation based on POLREF points and a qualitative assessment of this computation.

3. Conformal (Helmert) transformation between PL-ETRF89 and PL-ETRF2000 on the basis of POLREF network

The parameters of the 3D conformal transformation between the PL-ETRF89 and PL-ETRF2000 reference frames were determined on the basis of 330 POLREF network points, which, as a result of inventory and control analysis, showed stability and good physical condition. In addition, as reference points, they had to be points that have coordinates in both systems (in the PL-ETRF2000 system, determined as part of the 2008–2011 campaign). A conformal (7-parameter) transformation, without introducing additional local corrections, is conventionally called a "theoretical" or "mathematical" transformation. By the term "empirical" transformation we understand a conversion that takes into account the local deviations of the reference points from the conformal mathematical model.

3.1. Used method for 3D conformal transformation

The conformal transformation model is expressed in the known form

$$\boldsymbol{X} = \kappa \cdot \boldsymbol{R} \cdot \boldsymbol{x} + \boldsymbol{X}_0\,, \tag{1}$$

where (*col* – column vector):

$\boldsymbol{x} = col[x, y, z]$ – vector in the conventional old frame (e.g. PL-ETRF89),
$\boldsymbol{X} = col[X, Y, Z]$ – vector in a conventional current frame (e.g. PL-ETRF2000),
$\boldsymbol{X}_0 = col[X_0, Y_0, Z_0]$ – translation vector in the current frame,
κ – scale factor (similarity scale),
$\boldsymbol{R}$ – orthonormal matrix of rotation, i.e. satisfying the condition:

$$\boldsymbol{R}^T \cdot \boldsymbol{R} = \boldsymbol{I}(identity\ matrix) \Rightarrow \boldsymbol{R}^{-1} = \boldsymbol{R}^T\,. \tag{2}$$

At the same time, with respect to (2), the inverse transformation to (1) will be:

$$\boldsymbol{x} = (1/\kappa) \cdot \boldsymbol{R}^T \cdot (\boldsymbol{X} - \boldsymbol{X}_0)\,. \tag{3}$$

The transformation (1) or (2) has 7 parameters: 1 scale factor, 3 components of a translation vector and 3 axial rotation angles (α, β, γ), defining the matrix elements:

$$\boldsymbol{R} = \boldsymbol{R}(\alpha,\beta,\gamma) = \boldsymbol{R}_\gamma \cdot \boldsymbol{R}_\beta \cdot \boldsymbol{R}_\alpha \Rightarrow \boldsymbol{X} = \kappa \cdot \boldsymbol{R}_\gamma \cdot \left(\boldsymbol{R}_\beta \cdot (\boldsymbol{R}_\alpha \cdot \boldsymbol{x})\right) + \boldsymbol{X}_0\,, \tag{4}$$

where $\boldsymbol{R}_\alpha$, $\boldsymbol{R}_\beta$, $\boldsymbol{R}_\gamma$ – orthonormal matrices of single rotation:

$$\boldsymbol{R}_\alpha = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & \sin(\alpha) \\ 0 & -\sin(\alpha) & \cos(\alpha) \end{bmatrix}, \quad \boldsymbol{R}_\beta = \begin{bmatrix} \cos(\beta) & 0 & -\sin(\beta) \\ 0 & 1 & 0 \\ \sin(\beta) & 0 & \cos(\beta) \end{bmatrix}, \quad \boldsymbol{R}_\gamma = \begin{bmatrix} \cos(\gamma) & \sin(\gamma) & 0 \\ -\sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{5}$$

In the case of small angles (α, β, γ) (if it is possible to simplify $\alpha \approx \sin(\alpha)$, $\cos(\alpha) \approx 1$ and analogously for the angles (β, γ)) then model (1) is reduced to a linear form (e.g. Deakin, 1998; Watson, 2006). Although the resulting transformation matrix is not strictly orthonormal, it is assumed that the resulting distortions are practically negligible. In any case, however, we must make sure that this simplification does not significantly affect the result.

To determine the parameters of conformal transformation between the PL-ETRF89 and PL-ETRF2000 systems, the GEONET program (www.geonet.net.pl) was used, in which an algorithm allowing any values of rotation angles and scale changes was implemented. We briefly present the basic issues for this implementation.

Formula (1) is non-linear with respect to the axial rotation angles and the scale factor. In order to estimate the parameters, we can use the linearization of equations (1) with respect to the 7-element vector of unknowns (e.g. Deakin, 1998; Watson, 2006), which requires knowledge of their approximate values. An alternative to this approach is to formulate a parametric problem of the least squares method with additional conditions for the unknowns but without using the angular quantities (α, β, γ). The parameter estimation task on the basis of a set of reference points can be formulated as follows:

$$\boldsymbol{V}_i = \boldsymbol{S} \cdot \boldsymbol{x}_i + \boldsymbol{X}_0 - \boldsymbol{X}_i \quad \text{for } \; i = 1, 2, \ldots, p, \tag{6}$$

$$\boldsymbol{S}^T \cdot \boldsymbol{S} = \kappa^2 \cdot \boldsymbol{I}, \quad \boldsymbol{S} = \kappa \cdot \boldsymbol{R}, \tag{7}$$

$$\sum \boldsymbol{V}_i^T \cdot \boldsymbol{V}_i = \min, \tag{8}$$

where $\boldsymbol{V}_i$ – vector of residuals for the i-th reference point, p – number of reference points.

We assume that each vector equation (6) has 12 unknown parameters (9 elements of the matrix S and 3 translation parameters), but with additional conditions for these unknown parameters: 6 conditions for the isometric model $(\kappa = 1)$ or 5 conditions (after eliminating the scale) for the conformal transformation model. Of course, due to the nonlinearity of the condition (7), the least squares problem (8) requires linearization and determination of approximate values of the unknowns. These approximate values are obtained by performing the task (8) with reference to equations (6) only, i.e. with the "temporary" assumption of the affine transformation.

The above estimation model can be simplified using the known property of the conformal transformation (7-parameter, Helmert transformation) that if we shift the original system to the center of mass of this system (the point with mean values of the coordinates of the matching points), in the sense of the least squares condition (8), the translation vector $\boldsymbol{X}_0$ in the new frame will be the center of mass of the set of matching points in this system. Thus, the equation of corrections (6) can be written as:

$$\boldsymbol{V}_i = \boldsymbol{S} \cdot \underline{\boldsymbol{x}_i} - \underline{\boldsymbol{X}_i}, \tag{9}$$

$$\underline{\boldsymbol{x}_i} = \boldsymbol{x}_i - \boldsymbol{x}_s, \quad \underline{\boldsymbol{X}_i} = \boldsymbol{X}_i - \boldsymbol{X}_s,$$

$$\boldsymbol{x}_s = \left(\frac{1}{p}\right) \cdot \sum \boldsymbol{x}_i, \quad \boldsymbol{X}_s = \left(\frac{1}{p}\right) \cdot \sum \boldsymbol{X}_i. \tag{10}$$

The final parameters are obtained by linearization with respect to the elements of the $\boldsymbol{S}$ matrix and the iterative Gauss–Newton procedure (Deutsch, 1965), implemented in the program GEONET (www.geonet.net.pl).

We transform the transformation model for the local area of Poland into a numerically more favourable form, which results in determining small differences in coordinates between the two systems:

$$\underline{\boldsymbol{X}} = \underline{\boldsymbol{x}} - \underline{\boldsymbol{x}} + \boldsymbol{S} \cdot \underline{\boldsymbol{x}} = \underline{\boldsymbol{x}} + (\boldsymbol{S} - \boldsymbol{I}) \cdot \underline{\boldsymbol{x}}, \tag{11}$$

or

$$\boldsymbol{X} - \boldsymbol{X}_s = \boldsymbol{x} - \boldsymbol{x}_s + (\boldsymbol{S} - \boldsymbol{I}) \cdot \underline{\boldsymbol{x}}.$$

From here we get

$$\boldsymbol{X} = \boldsymbol{x} + \boldsymbol{\Delta}, \quad \boldsymbol{\Delta} = \boldsymbol{\delta}_0 + \boldsymbol{\delta}, \quad \boldsymbol{\delta}_0 = \boldsymbol{X}_s - \boldsymbol{x}_s, \quad \boldsymbol{\delta} = (\boldsymbol{S} - \mathbf{1}) \cdot \underline{\boldsymbol{x}}. \tag{12}$$

The obtained form of the transformation model is particularly advantageous for reference frames similar to each other, such as of PL-ETRF89 and PL-ETRF2000 frames. Then $\boldsymbol{\Delta}$ will be a vector – an amendment to the original coordinates, the components of which are absolutely small numbers. The $\boldsymbol{\delta}_0$ component determines the mutual shift of the center of mass of the primary system to the center of mass of the secondary system, while $\boldsymbol{\delta}$ is the result of slight rotations around the center of mass. The above properties are illustrated by the result of the PL-ETRF89 ⇒ PL-ETRF2000 transformation parameter estimation. As we will show, for the precise determination of the increment $\boldsymbol{\delta}$ (with an error less than 0.0001 m), the components of the vector of increments $\underline{\boldsymbol{x}}$ in formula (12) can be rounded up to hundreds of meters.

3.2. Parameter estimation results

Parameters of the formula (12) for the conformal transformation PL-ETRF89 ⇒ PL-ETRF2000 (epoch 2011.0) were estimated on the basis of 330 reference points of the POLREF network. The formula has the following "incremental" numerical form:

$$X = x + \Delta x, \quad Y = y + \Delta y, \quad Z = z + \Delta z, \tag{13}$$

where:

$$\begin{aligned} \Delta x &= (-0.0322) + (-0.00000005102) \cdot \underline{x} + (-0.00000000746) \cdot \underline{y} \\ &\qquad + (0.00000004804) \cdot \underline{z}, \\ \Delta y &= (-0.0347) + (0.00000000746) \cdot \underline{x} + (-0.00000005102) \cdot \underline{y} \\ &\qquad + (0.00000006152) \cdot \underline{z}, \\ \Delta z &= (-0.0507) + (-0.00000004804) \cdot \underline{x} + (-0.00000006152) \cdot \underline{y} \\ &\qquad + (-0.00000005102) \cdot \underline{z}, \end{aligned} \tag{13a}$$

$$\underline{x} = x - x_s, \qquad \underline{y} = y - y_s, \qquad \underline{z} = z - z_s \tag{13b}$$

(increments of coordinates relative to the center of mass),

$$x_s = 3696570.6591, \qquad y_s = 1297521.5905, \qquad z_s = 5011111.1273 \tag{13c}$$

(coordinates of the center of mass in the primary reference frame).

In this formula as in (12), we read directly the $*$ components of the vector of the shift of the center of mass of the original system as a vector $(-0.0322, -0.0347, -0.0507)$. We also note that without compromising the result, the differences $\underline{x}$, $\underline{y}$, $\underline{z}$ can be rounded up to hundreds of meters.

How does this affect the values Δx, Δy, Δz? Let us denote the rounding errors of the quantities $\underline{x}$, $\underline{y}$, $\underline{z}$ by ε_x, ε_y, ε_z and the errors of the resulting values $e_{\Delta x}$, $e_{\Delta y}$, $e_{\Delta z}$.

Let δ denote the limit error of rounding the increments $\underline{x}$, $\underline{y}$, $\underline{z}$:

$$|\varepsilon_x|,\ |\varepsilon_y|,\ |\varepsilon_z| \leq \delta\ [\mathrm{m}]. \tag{14}$$

Then from (13a) is:

$$\begin{aligned} |e_{\Delta x}| &= |(-0.00000005102)\cdot\varepsilon_x + (-0.00000000746)\cdot\varepsilon_y + (0.00000004804)\cdot\varepsilon_z| \\ &\leq 0.00000010648\cdot\delta\,, \\ |e_{\Delta y}| &\leq 0.00000012000\cdot\delta\,, \\ |e_{\Delta z}| &\leq 0.00000016058\cdot\delta\,. \end{aligned} \tag{15}$$

Hence, it can be seen that rounding the increments $\underline{x}$, $\underline{y}$, $\underline{z}$ to the position of meters, tens or hundreds of meters does not significantly affect the resulting coordinates. Only rounding these increments to kilometres affects the results at the level of fractions of a millimetre. If the length of the vector of increments $d = \sqrt{(x^2+y^2+z^2)} < 500$ m, and the required rounding error of the resulting coordinates $\leq (1/2)\cdot 10^4$, then instead of (13a) we can assume $\delta x = \delta y = \delta z = 0$, and the transformation only takes into account the translation of the center of mass of the system of reference points.

From formulas (7), (13) and (13a) results the scale factor or its change $d\kappa = \kappa - 1$ in relation to the isometric transformation. This change after conversion of measurement units is –0.051 mm/km.

Estimated parameters and formulas of the inverse mathematical transformation between the reference frames PL-ETRF2000 and PL-ETRF89 are as follows:

$$x = X + \Delta X, \quad y = Y + \Delta Y, \quad z = Z + \Delta Z, \tag{16}$$

$$\begin{aligned} \Delta X &= (0.0322) + (0.00000005102)\cdot\underline{X} + (0.00000000746)\cdot\underline{Y} \\ &\qquad + (-0.00000004804)\cdot\underline{Z}, \\ \Delta Y &= (0.0347) + (-0.00000000746)\cdot\underline{X} + (0.00000005102)\cdot\underline{Y} \\ &\qquad + (-0.00000006152)\cdot\underline{Z}, \\ \Delta Z &= (0.0507) + (0.00000004804)\cdot\underline{X} + (0.00000006152)\cdot\underline{Y} \\ &\qquad + (0.00000005102)\cdot\underline{Z}, \end{aligned} \tag{16a}$$

$$\underline{X} = X - X_0\,, \qquad \underline{Y} = Y - Y_0\,, \qquad \underline{Z} = Z - Z_0\,, \tag{16b}$$

$$X_0 = 3696570.6268, \qquad Y_0 = 1297521.5559, \qquad Z_0 = 5011111.0767\,. \tag{16c}$$

We notice that the center of mass shift vector (0.0322, 0.0347, 0.0507) now has opposite algebraic sign of the components.

As a result of parameter estimation, mean square deviations of point coordinates from the conformal transformation model were calculated as: $s_x = 0.0107$, $s_y = 0.0083$, $s_z = 0.0133$ and the mean absolute deviation $s_p = 0.019$ m.

Table 4 shows the statistics of the deviations between the two reference systems after converting the (B, L) geodetic coordinates to planar (x, y) coordinates (in PL-1992), and Table 5 shows the statistics of differences between the actual coordinates of points in PL-ETRF2000 (after conversion to planar (x, y) in the PL-1992 system) and analogous coordinates obtained from the conformal transformation. Particularly noticeable are the relatively large shifts of the ellipsoidal heights (see Fig. 6), with mean –6.6 cm ($h_{(\text{PL-ETRF2000})}$ minus $h_{(\text{PL-ETRF89})}$).

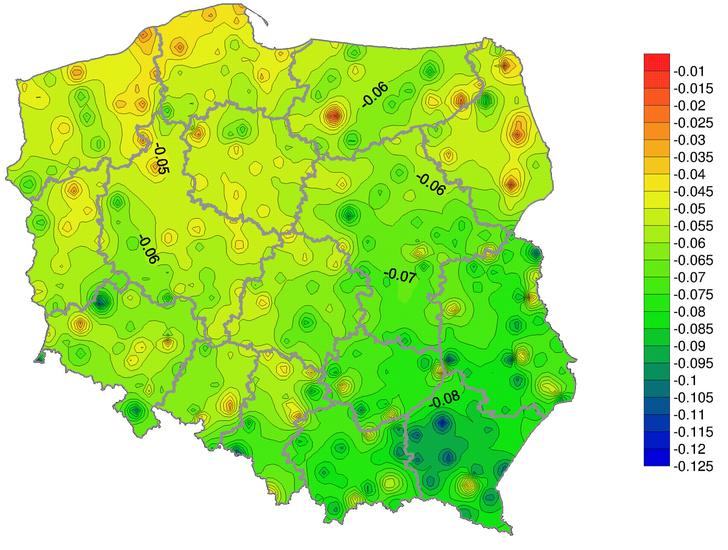

Fig. 6. Distribution of ellipsoidal height differences $\delta h = h_{(\text{PL-ETRF2000})} - h_{(\text{PL-ETRF89})}$ in the area of Poland

Table 4. Statistics of the coordinate differences between the PL-ETRF89 and PL-ETRF2000 reference systems for 330 POLREF network points, where the differences Δx, Δy are expressed in plane coordinates of the PL-1992 cartographic system, and Δh as differences of ellipsoidal heights

parameter	Δx [m]	Δy [m]	Δh [m]
minimum	–0.0442	–0.0508	–0.1287
maximum	0.0290	0.0202	–0.0046
mean	0.0017	–0.0218	–0.0661
RMS	0.0105	0.0244	0.0693

By transforming the Cartesian coordinates in both systems (XYZ) PL-ETRF89 and (XYZ) PL-ETRF2000 into corresponding geodetic coordinates (BLH) PL-ETRF89 and (BLH) PL-ETRF2000, we can determine empirical relationships between the ellipsoidal heights of both systems. Then, by transforming the geodetic coordinates (B, L) in both

Table 5. Statistics of the coordinate deviations for 330 POLREF network points in the PL-ETRF2000 frame from the analogous coordinates obtained as a result of the mathematical transformation from the PL-ETRF89 frame. Deviations δx, δy are expressed in plane coordinates of the PL-1992 cartographic system, and δh as deviations of ellipsoidal heights

parameter	δx [m]	δy [m]	δh [m]
minimum	–0.0375	–0.0342	–0.0642
maximum	0.0229	0.0392	0.0531
mean	–0.0000	–0.0000	–0.0000
RMS	0.0074	0.0079	0.0156

reference systems into plane coordinates (x, y), we will define analogous relationships in the horizontal. The following approximate linear relationships occur between the planar coordinates $xy92$ (as defined in PL-1992) determined in the PL-ETRF89 (indexed 1) and PL-ETRF2000 (indexed 2) reference systems:

$$x(2) = x(1) + \Delta x, \qquad y(2) = y(1) + \Delta y, \qquad h(2) = h(1) + \Delta h, \tag{17}$$

$$\begin{aligned} \Delta x &= (0.0017) + (0.00000004052) \cdot p + (-0.00000001992) \cdot q, \\ \Delta y &= (-0.0218) + (0.00000001992) \cdot p + (-0.00000004052) \cdot q, \\ \Delta h &= (-0.0661) + (0.00000006575) \cdot p + (-0.00000004170) \cdot q, \end{aligned} \tag{17a}$$

$$p = x - 478097.0 \text{ [m]}, \qquad \mathrm{q} = \mathrm{y} - 523344.0 \text{ [m]}. \tag{17b}$$

The conversion inverse to (17) can be performed analogously to formulas (13) and (16).

The values p, q are then the increments of the coordinates relative to the approximate center of the system. It follows from the above that the vector of displacement is (0.0017 m, –0.0218 m), but in the entire territory of Poland, the differences in coordinates may change as in Table 4. The mean vertical shift between two reference frame is –0.0661 m (the GRS80 ellipsoid in the new PL-ETRF2000 system is located 6.6 cm higher than in PL-ETRF89), but the ellipsoidal height differences for 330 points vary in the interval: $\langle -0.129 \text{ m}, -0.005 \text{ m} \rangle$ (see Table 4).

4. Construction of the interpolation grid for empirical transformation between PL-ETRF89 and PL-ETRF2000 and comparison results with conformal transformation model

One of the methods of empirical transformation is the use of local post-transformational corrections, similar to the so-called Hausbrandt corrections, used in plane coordinate transformations (Hausbrandt, 1971 – old edition 1956). The correction value of the coordinate of the transformed point is simply a weighted average calculated from the transformation deviations occurring at the nearest fit points. The weight being a number inversely proportional to the square or other power of the point distance. The idea of such formulated corrections is a special case of the currently used IDW (inverse distance weighting) methodology (e.g. Zawadzki, 2011). This type of transformation requires constant access to the reference point database because the coordinates of these points are needed to calculate post-transformational corrections for each transformed points.

An alternative way in a class of empirical transformations is to use an interpolation grid, made once on the basis of all given reference points. In the TRANSPOL program, a base grid with resolution of $0.01^\circ \times 0.01^\circ$ (in latitude and longitude) was adopted for empirical transformations. In the nodes of the grid, coordinate differences between two systems are determined (for a given pair of systems), which are used to perform the so-called bilinear interpolation.

The use of interpolation grids is commonly known in various spatial problems, in numerical terrain models, geoid or quasi-geoid models, in tasks of transformation between similar coordinate systems. In our case, an important issue will be the integration of various data for the creation of an interpolation grid, i.e. determining the fixed values of post-transformational corrections in the grid nodes.

When designing mathematical and empirical transformation algorithms and their implementations in TRANSPOL program, the following assumptions were made:

– the mathematical transformation was based solely on 330 points of the POLREF network, which as a result of the inventory in the new measurement campaign 2008-2011, considered physically stable and not displaced. It was not possible to include points of a class I aligned triangulation network, because this network is only two-dimensional (the points have only geodetic coordinates B, L, with no ellipsoidal heights).
– the empirical transformation was based on the POLREF network points and also on the points of the adjusted first class triangulation network, only with B, L coordinates in two reference frames (height increments are interpolated only on the basis of POLREF network).

The interpolation grid for the transformation PL-ETRF89 $\Rightarrow$ PL-ETRF2000 was created on the base grid with the resolution of $0.01^\circ \times 0.01^\circ$ (Fig. 7) and the following geodetic coordinates of grid nodes:

$$\begin{aligned} B_i &= 48^\circ + i \cdot 0.01^\circ, \quad i = 0, 1, \ldots, 800; \\ L_j &= 13^\circ + j \cdot 0.01^\circ, \quad j = 0, 1, 2, \ldots, 1200. \end{aligned} \tag{18}$$

As you can see, the original base grid that was used in empirical studies extends (with a band of approx. 1°) beyond the territory of Poland. However, the grid implemented in the TRANSPOL program was "cut" to the borders of Poland.

For each grid node, the values of the coordinate differences between the PL-ETRF2000 and PL-ETRF89 systems were determined in the form of data records:

$$[i, j, (\Delta B, \Delta L, \Delta h)_{ij}] \quad \text{for } i = 0, 1, 2, \ldots, 800; \quad j = 0, 1, 2, \ldots, 1200. \tag{19}$$

The interpolation of coordinate increments ΔB, ΔL, Δh for each grid nodes (Fig. 8) was realized by the IDW (Inverse distance weighting) method. It is one of the most

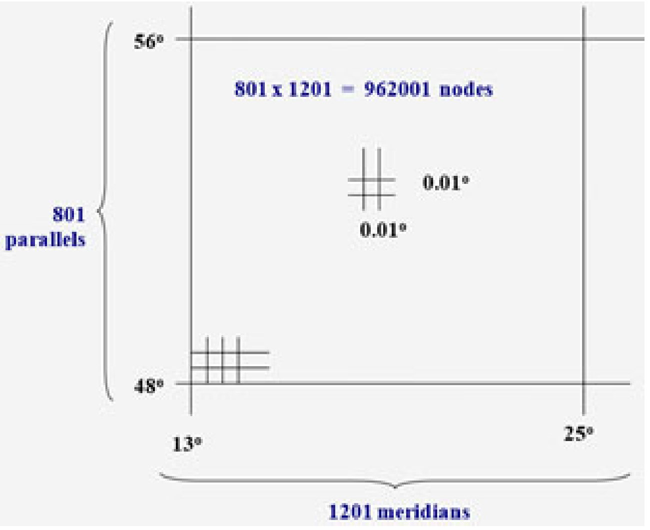

Fig. 7. The structure of the base grid for the area of Poland.
The number of all grid nodes is exact $801 \times 1200 = 962001$

popular methods adopted by geoscientists and has been implemented in many surface interpolation problems (see e.g. Zawadzki, 2011). The increments ΔB and ΔL were interpolated from all geodetic points (POLREF + adjusted triangulation first class).

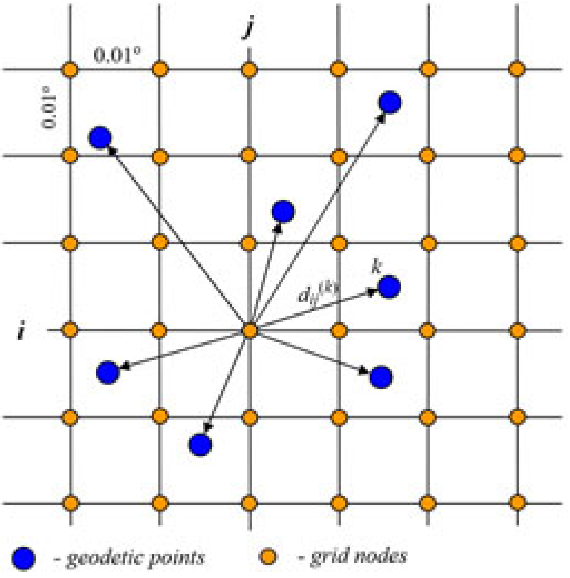

Fig. 8. Interpolation of grid nodes by the IDW (inverse distance weighting) method, with the weight equal $\frac{1}{\left(d_{ij}^{(k)}\right)^q}$, $q = 2$. $d_{ij}^{(k)}$ is the distance between interpolated grid nodes (i, j) and geodetic k-th point

Interpolation within a single cell of the grid (Fig. 9) is performed according to the generally known bilinear interpolation rules (e.g. https://en.wikipedia.org/wiki/Bilinear_interpolation.

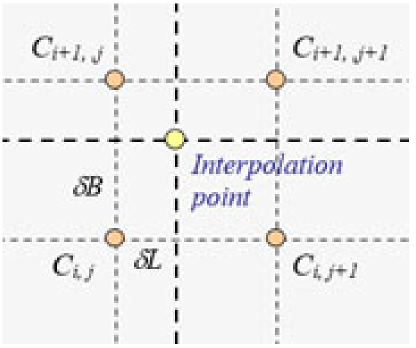

Fig. 9. Interpolation in the cell using the bilinear method. The names $C_{i,j}$, $C_{i+j,j}$, $C_{i+1,j+1}$, $C_{i,j+1}$ generally denote the interpolated features (coordinate increments ΔB, ΔL, Δh)

Table 6 shows the record structure of the interpolation grid implemented in the TRANSPOL program.

Table 6. Fragments of the numerical record of the interpolation grid for the transformation PL-ETRF89 ⇒ PL-ETRF2000

PL-ETRF89 => PL-ETRF2000

B [o]	L [o]	dB [o]	dL [o]	dH [m]
1	2	3	4	5
49.00	14.00	0.000000216	-0.000000213	-0.0837
49.00	14.01	0.000000216	-0.000000215	-0.0837
49.00	14.02	0.000000216	-0.000000215	-0.0838
.....				
52.50	14.98	0.000000016	-0.000000191	-0.0602
52.50	14.99	0.000000017	-0.000000189	-0.0596
52.50	15.00	0.000000018	-0.000000191	-0.0591
.....				
55.00	24.18	-0.000000131	-0.000000503	-0.0496
55.00	24.19	-0.000000132	-0.000000504	-0.0495
55.00	24.20	-0.000000131	-0.000000504	-0.0496

Finally, a comparison was made between the empirical and mathematical (7-parameter) transformation (Table 7). The differences in the B and L coordinates were converted into vector components on the PL-1992 mapping plane.

The parameters given in Table 7 show that the conformal (mathematical) transformation does not fully model the real relationship between the two frames of reference represented by the POLREF network points. Local deviations from the mathematical model are probably the result of errors in the determination of points in the 1990s (the

Table 7. Results of research on differences between empirical and theoretical transformation (data in meters)

parameter	δx [m]	δy [m]	δh [m]
minimum	–0.0364	–0.0468	–0.0561
maximum	0.0294	0.0446	0.0454
mean	0.0001	–0.0002	0.0001
RMS	0.0034	0.0037	0.0025

error in the position of a point in the GPS network was estimated at about 2.5 cm). The empirical transformation is important here, as it eliminates to a certain extent the local deformations of the PL-ETRF89 system in the process of transforming the points into the PL-ETRF2000 system through the interpolation grid.

5. Conclusions

The publication presents theoretical (generalized) and empirical relationships (taking into account local irregularities of coordinate differences) between the three-dimensional reference frames: PL-ETRF89 and PL-ETRF2000. The relations between the above-mentioned reference frames are the result of the author's research carried out in 2012 on the basis of source data provided by GUGiK (Head Office of Geodesy and Cartography). The main aim of the research was to estimate the parameters of the three-dimensional conformal transformation using the set of POLREF network points, determined in both frames as the satellite vector network, and to develop an empirical interpolation grid. In the second case, an important task was to re-level the former class I network, now in the PL-ETRF2000 system, using the results for the construction of the interpolation grid.

As follows from the conducted analyses, horizontal differences (e.g. in the PL-1992 or PL-2000 systems) are in the ranges $\langle -0.044\text{ m}, 0.029\text{ m}\rangle$, on average 0.002 m for the N–S direction, and $\langle -0.051\text{ m}, 0.020\text{ m}\rangle$ on average –0.022 m for the O–W direction, while the differences in the ellipsoidal heights are in the range $\langle -0.128\text{ m}, -0.005\text{ m}\rangle$, average –0.0661 m. As the actual coordinates of the points show different local deviations from the conformal (7-parameter) transformation model, an empirical transformation based on an interpolation grid is generally recommended in practice, eliminating in a sense the real deformations of the PL-ETRF89 system. For example, it will be relevant when we integrate new GNSS observations determined in the PL-ETRF2000 frame with existing points defined earlier in the PL-ETRF89 frame.

When will we use a strictly mathematical (conformal, 7-parameter) transformation? Well, it can take place when a local network with high internal accuracy (e.g. a control network for displacement and deformation measurements) is to be transformed into a different system but without internal deformation (without changing its shape). In such a situation, the transformation consists of rotations and (possibly) a change of scale (although we can also assume the invariability of the scale, i.e. the isometric transformation).

The presented transformation formulas were implemented in the TRANSPOL program (version 2.06) (Kadaj and Świętoń, 2012, www.gugik.gov.pl), but with the assumed area limitation to the Polish borders. Other implementations can be found at www.geonet.net.pl.

One of the practical goals of this paper was to present numerical transformation formulas, useful in integrating geodetic data from different reference systems. Currently, geodetic measurements are performed using satellite services in the PL-ETRF2000 system, while in the previous years most maps and databases (e.g. databases of detailed networks) were made in the PL-ETRF89 system. Apart from the transformation tasks, a statistical evaluation of the coordinate differences between the two systems represented by the geodetic networks was performed, as well as the evaluation of point deviations from the regular conformal transformation model. The statistics of these deviations can be used to assess the accuracy of coordinate transformation in the different problems of integrated geodetic networks.

Data availability statement

Source data for processing is available at the Head Office of Geodesy and Cartography and on the website www.asgeupos.pl. Processed data is available on the website http://www.gugik.gov.pl/bip/prawo/modele-danych including spatial data models implemented in the TRANSPOL v.2.06 program.

Acknowledgements

The work has been elaborated under the statutory research in Department of Geodesy and Geotechnics at Rzeszów University of Technology.

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