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# DIFFERENT KINDS OF BOUNDARY CONDITIONS FOR TIME-FRACTIONAL HEAT CONDUCTION EQUATION

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**Abstract.** The time-fractional heat conduction equation with the Caputo derivative of the order $0 < \alpha < 2$ is considered in a bounded domain. For this equation different types of boundary conditions can be given. The Dirichlet boundary condition prescribes the temperature over the surface of the body. In the case of mathematical Neumann boundary condition the boundary values of the normal derivative are set, the physical Neumann boundary condition specifies the boundary values of the heat flux. In the case of the classical heat conduction equation ($\alpha = 1$), these two types of boundary conditions are identical, but for fractional heat conduction they are essentially different. The mathematical Robin boundary condition is a specification of a linear combination of the values of temperature and the values of its normal derivative at the boundary of the domain, while the physical Robin boundary condition prescribes a linear combination of the values of temperature and the values of the heat flux at the surface of a body.

## 1. Introduction

The conventional theory of heat conduction is based on the classical (local) Fourier law, which relates the heat flux vector $\mathbf{q}$ to the temperature gradient

$$\mathbf{q} = -k \operatorname{grad} T, \tag{1}$$

where $k$ is the thermal conductivity of a solid. In combination with a law of conservation of energy,

$$\rho C \frac{\partial T}{\partial t} = -\mathrm{div}\, \mathbf{q} \tag{2}$$

with $\rho$ being the mass density, $C$ the specific heat capacity, the Fourier law leads to the parabolic heat conduction equation

$$\frac{\partial T}{\partial t} = a\, \Delta T, \tag{3}$$

where $a$ is the thermal diffusivity coefficient.

It should be noted that Eq. (1) is a phenomenological law which states the proportionality of the flux to the gradient of the transported quantity. It is met in several physical phenomena with different names.

For example, it is well known that from mathematical viewpoint the Fourier law (1) in the theory of heat conduction and the Fick law in the theory of diffusion,

$$\mathbf{J} = -k_c\, \mathrm{grad}\, c, \tag{4}$$

where $\mathbf{J}$ is the matter flux, $c$ is the concentration, $k_c$ is the diffusion conductivity, are identical. In combination with the balance equation for mass,

$$\rho \frac{\partial c}{\partial t} = -\mathrm{div}\, \mathbf{J}, \tag{5}$$

the Fick law leads to the classical diffusion equation

$$\frac{\partial c}{\partial t} = a_c\, \Delta c. \tag{6}$$

Here $a_c$ is the diffusivity coefficient.

Similarly, the classical empirical Darcy law, describing the flow of fluid through a porous medium, states proportionality between the fluid mass flux $\mathbf{J}$ and the gradient of the pore pressure $p$,

$$\mathbf{J} = -k_p\, \mathrm{grad}\, p, \tag{7}$$

and leads to the parabolic diffusion equation for the pressure

$$\frac{\partial p}{\partial t} = a_p\, \Delta p. \tag{8}$$

Though we will consider heat conduction, it obvious that the discussion concerns also diffusion as well as the theory of fluid flow through the porous solid.

Nonclassical theories of heat conduction in which the Fourier law and the standard heat conduction equation are replaced by more general equations, constantly attract the attention of the researchers. For an extensive bibliography on this subject see [1–11] and references therein.

## 2. Nonlocal generalizations of the Fourier law

For materials with time nonlocality (with memory) the effect at a point $\mathbf{x}$ at time $t$ depends on the histories of causes at a point $\mathbf{x}$ at all past and present times. In the theory proposed by Gurtin and Pipkin [12] the law of heat conduction is given by general time–nonlocal dependence

$$\mathbf{q}(t) = -k \int_0^{\infty} K(u) \operatorname{grad} T(t-u) \, \mathrm{d}u. \tag{9}$$

Using substitution $\tau = t - u$ leads to the following equation

$$\mathbf{q}(t) = -k \int_{-\infty}^{t} K(t-\tau) \operatorname{grad} T(\tau) \, \mathrm{d}\tau. \tag{10}$$

Choosing 0 instead of $-\infty$ as a "starting point", we obtain

$$\mathbf{q}(t) = -k \int_0^{t} K(t-\tau) \operatorname{grad} T(\tau) \, \mathrm{d}\tau \tag{11}$$

and the heat conduction equation with memory [13]:

$$\frac{\partial T}{\partial t} = a \int_0^{t} K(t-\tau) \, \Delta T(\tau) \, \mathrm{d}\tau. \tag{12}$$

The time-nonlocal dependences between the heat flux vector and the temperature gradient with the "long-tale" power kernel $K(t-\tau)$ were considered in [5, 8, 9] (see also [14])

$$\mathbf{q}(t) = -\frac{k}{\Gamma(\alpha)} \frac{\partial}{\partial t} \int_0^{t} (t-\tau)^{\alpha-1} \operatorname{grad} T(\tau) \, \mathrm{d}\tau, \qquad 0 < \alpha \le 1; \tag{13}$$

$$\mathbf{q}(t) = -\frac{k}{\Gamma(\alpha-1)} \int_0^{t} (t-\tau)^{\alpha-2} \operatorname{grad} T(\tau) \, \mathrm{d}\tau, \qquad 1 < \alpha \le 2, \tag{14}$$

where $\Gamma(\alpha)$ is the gamma function. Equations (13) and (14) can be interpreted in terms of fractional integrals and derivatives

$$\mathbf{q}(t) = -k D_{RL}^{1-\alpha} \operatorname{grad} T(t), \qquad 0 < \alpha \le 1; \tag{15}$$

$$\mathbf{q}(t) = -kI^{\alpha-1}\mathrm{grad}\, T(t), \qquad 1 < \alpha \le 2, \tag{16}$$

where $I^{\alpha} f(t)$ and $D^{\alpha}_{RL} f(t)$ are the Riemann–Liouville fractional integral and derivative of the order $\alpha$, respectively [15–18]:

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, \mathrm{d}\tau, \quad \alpha > 0, \tag{17}$$

$$D^{\alpha}_{RL} f(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n} \left[ \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f(\tau)\, \mathrm{d}\tau \right], \quad n-1 < \alpha < n. \tag{18}$$

The constitutive equations (15) and (16) yield the time-fractional heat conduction equation

$$\frac{\partial^{\alpha} T}{\partial t^{\alpha}} = a \Delta T, \qquad 0 < \alpha \le 2, \tag{19}$$

with the Caputo fractional derivative of order $0 < \alpha \le 2$

$$\frac{\mathrm{d}^{\alpha} f(t)}{\mathrm{d}t^{\alpha}} = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \frac{\mathrm{d}^n f(\tau)}{\mathrm{d}\tau^n}\, \mathrm{d}\tau, \quad n-1 < \alpha < n. \tag{20}$$

## 3. Boundary conditions

The Dirichlet boundary condition (the boundary condition of the first kind) specifies the temperature over the surface of the body under consideration

$$T\big|_S = g_0(\mathbf{x}_S, t). \tag{21}$$

For fractional heat conduction equations, two types of Neumann boundary condition (the boundary condition of the second kind) can be considered: the mathematical condition with the prescribed boundary value of the normal derivative

$$\frac{\partial T}{\partial n}\Big|_S = G_0(\mathbf{x}_S, t) \tag{22}$$

and the physical condition with the prescribed boundary value of the heat flux

$$\begin{aligned} D^{1-\alpha}_{RL} \frac{\partial T}{\partial n}\bigg|_S &= G_0(\mathbf{x}_S, t), \quad 0 < \alpha \le 1, \\ I^{\alpha-1} \frac{\partial T}{\partial n}\bigg|_S &= G_0(\mathbf{x}_S, t), \quad 1 < \alpha \le 2. \end{aligned} \tag{23}$$

In the case of the classical heat conduction equation ($\alpha = 1$), these two types of boundary condition are identical, but for fractional heat conduction they are essentially different. Similarly, the mathematical Robin boundary condition (the boundary condition of the third kind) is a specification of a linear combination of the values of temperature and the values of its normal derivative at the boundary of the domain

$$\left( c_1 T + c_2 \frac{\partial T}{\partial n} \right) \Bigg|_S = H_0(\mathbf{x}_S, t) \tag{24}$$

with some nonzero constants $c_1$ and $c_2$, while the physical Robin boundary condition specifies a linear combination of the values of temperature and the values of the heat flux at the boundary of the domain. The condition of convective heat exchange between a body and the environment with the temperature $T_e$ leads to

$$\begin{aligned} \left( hT + k D_{RL}^{1-\alpha} \frac{\partial T}{\partial n} \right) \Bigg|_S &= hT_e(\mathbf{x}_S, t), \quad 0 < \alpha \leq 1, \\ \left( hT + k I^{\alpha-1} \frac{\partial T}{\partial n} \right) \Bigg|_S &= hT_e(\mathbf{x}_S, t), \quad 1 < \alpha \leq 2, \end{aligned} \tag{25}$$

where $h$ is the convective heat transfer coefficient.

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