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Dedicated to the Memory of Professor Zdzisław Kamont

ON RECONSTRUCTING AN UNKNOWN COORDINATE OF A NONLINEAR SYSTEM OF DIFFERENTIAL EQUATIONS

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Abstract. The paper discusses a method of auxiliary controlled models and the application of this method to solving problems of dynamical reconstruction of an unknown coordinate in a nonlinear system of differential equations. The solving algorithm, which is stable with respect to informational noises and computational errors, is presented.

Keywords: ordinary differential equations, inverse problems.

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1. INTRODUCTION

Problems of reconstructing unknown characteristics of dynamical systems through measurements of a part of the phase coordinates are embedded into the theory of inverse problems of dynamics. This theory is intensively developed at the present time. One of approaches to solving similar problems based on methods of the theory of positional control [1] was suggested in [2] and developed in [3–10]. In the present paper following the research in this field, an algorithm of dynamical reconstruction of an unmeasured coordinate of a second-order system is designed. This algorithm is dynamical and works in the real time mode. It is stable with respect to informational noises and computational errors.

Briefly, the essence of the problems under consideration can be formulated in the following way. There is a dynamical system Σ functioning on a finite time interval $T = [0, \vartheta]$, $\vartheta = \text{const} \in (0, +\infty)$. Its trajectory is

$$x(t) = x(t; x_0) \in \mathbb{R}^q, \quad t \in T.$$

On the interval T, a uniform net $\Delta = \{\tau_i\}_{i=0}^m$ with a step $\delta > 0$ is taken, $\tau_0 = 0$, $\tau_{i+1} = \tau_i + \delta$, $\tau_m = \vartheta$. An output

$$y(t) = Cx(t)$$

is measured at the moments τ_i (C is an $r\times q$-dimensional matrix). Results of inaccurate measurements are vectors $\xi_i \in \mathbb{R}^r$ satisfying the inequalities

$$\|\xi_i - y(\tau_i)\| \le h, \quad i \in [0 : m-1],$$

where h is some given accuracy; the symbol $\|x\|$ denotes the Euclidean norm of the vector x, the notation $i \in [0 : m-1]$ means that i takes all integer values from 0 up to $m-1$. It is required to design an algorithm that allows us to reconstruct an approximation to the solution $x(\cdot)$.

The algorithms suggested in the works cited above realize the reconstruction process in the mean-square metric. In this paper, a solving algorithm for reconstructing unmeasured coordinates in the uniform metric is presented. We consider a second-order system. For other dynamical algorithms for reconstructing unknown characteristics (in the L_2-metric) of the system considered in the paper, see [10].

2. PROBLEM STATEMENT. SOLUTION SCHEME

We consider an initial-value problem for a system of the form

$$\begin{aligned} \dot{x}_1(t) &= k(t)x_2(t) + x_1(t)(\lambda x_2(t) - \nu), \\ \dot{x}_2(t) &= -k(t)x_2(t) - (\lambda x_1(t) + \mu)x_2(t) + \gamma(t), \end{aligned} \tag{2.1}$$

$$t \in T, \quad x_1(0) = x_{10}, \quad x_2(0) = x_{20}.$$

This model describes the process of the diffusion of innovations [11]. We assume that the constants λ, μ and the continuous functions $k(t)$ and $\gamma(t)$ are known. The constant ν ($-\infty < a \le \nu \le b < +\infty$) is unknown as well. At discrete, frequent enough, time moments

$$\tau_i \in \Delta = \{\tau_i\}_{i=0}^m, \quad \tau_{i+1} = \tau_i + \delta, \quad \tau_0 = 0, \quad \tau_m = \vartheta,$$

the coordinate $x_2(\tau_i)$ is inaccurately measured. Results of measurements (numbers $\xi_i^h \in \mathbb{R}$) satisfy the inequalities

$$|x_2(\tau_i) - \xi_i^h| \le h, \tag{2.2}$$

where $h \in (0, 1)$ is the measurement accuracy. Here and below, $|x|$ denotes the modulus of the number x. It is required to design an algorithm for reconstructing the unknown coordinate $x_1(\cdot)$. This is the informal statement of the problem considered in the paper.

The algorithm for solving the problem consists of the following. An auxiliary dynamical system M (a model) is introduced. This model functioning on the time interval T has an unknown input (control) $u^h(\cdot)$ and an output $w^h(\cdot)$. Then, the problem of reconstructing $x_1(\cdot)$ is replaced by the problem of forming a control $u^h(\cdot)$ in the model (by the feedback principle) in such a way that the deviation of $x_1(\cdot)$ from $u^h(\cdot)$ in the uniform metric is small if the measurement accuracy h is small enough. The process of synchronous feedback control of the systems Σ and M is organized on the interval T. This process is decomposed into $(m-1)$ identical steps. At the i-th step carried out on the time interval $\delta_i = [\tau_i, \tau_{i+1})$, the following actions are fulfilled. First, at the time moment τ_i, according to the chosen rule u^h, the control

$$u^h(t) = u_i^h \in U(\tau_i, \xi_i^h, w^h(\tau_i)), \quad t \in [\tau_i, \tau_{i+1}),$$

is calculated. Then (till the moment τ_{i+1}), the control $u^h = u^h(t)$, $\tau_i \le t < \tau_{i+1}$, is fed onto the input of the system M. The value $w^h(\tau_{i+1})$ is the result of the work of the algorithm at the i-th step. Thus, the complexity of solving the problem is reduced to the appropriate choice of the model M and the function $u^h(\cdot)$.

So, the procedure for solving the reconstruction problem is, in essence, equivalent to the procedure for solving the following two problems:

a) the problem of choosing the model M;

b) the problem of choosing the rule u^h for forming the control in the model.

Let us proceed to the rigorous statement of the problem in question. Fix a family of partitions of the interval T:

$$\Delta_h = \{\tau_{i,h}\}_{h=0}^{m_h}, \quad \tau_{i+1,h} = \tau_{i,h} + \delta(h), \quad \tau_{0,h} = 0, \quad \tau_{m_h,h} = \vartheta. \tag{2.3}$$

Problem 2.1. It is required to specify differential equations of the model M:

$$\dot{w}^h(t) = f_1(\xi_i^h, w^h(\tau_i), u_i^h), \quad t \in \delta_{h,i} = [\tau_i, \tau_{i+1}), \tag{2.4}$$

$$\tau_i = \tau_{i,h}, \quad i \in [0 : m_h - 1], \quad w^h(t_0) = w_0^h, \quad w^h(t) \in \mathbb{R},$$

and the rule for forming controls u_i^h at the moments τ_i as some mapping

$$U : \{\tau_i, \xi_i^h, w^h(\tau_i)\} \to u_i^h \in \mathbb{R} \tag{2.5}$$

such that the convergence

$$\sup_{t\in T} |u^h(t) - x_1(t)| \to 0 \tag{2.6}$$

takes place as h tends to 0. Here, $u^h(t) = x_{10}$ for $t \in [0, \zeta(h))$, $u^h(t) = u_i^h$ for $t \in \delta_{h,i}$, $\tau_{i+1,h} > \zeta(h)$, $\zeta(h) \to 0$ as $h \to 0$.

3. SOLVING METHOD

Let us proceed to the description of the solving method of the problem under consideration. We introduce the following notation:

$$f(t, x_2) = -k(t)x_2 - \mu x_2 + \gamma(t), \quad \tilde{u}(t) = -x_1(t).$$

In this case, the second equation of system (2.1) can be rewritten in the form

$$\dot{x}_2(t) = f(t, x_2(t)) + \lambda x_2(t)\tilde{u}(t).$$

Let a number $M_* > 0$ be such that

$$|x_2(t)| \le M_* \quad \text{for} \quad t \in [0, \vartheta], \tag{3.1}$$

$$\|f(t, x_2(t)) - f(\tau_i, \xi_i^h)\| \le M_*(\delta + h + \omega(\delta)) \quad \text{for} \quad t \in \delta_i = [\tau_i, \tau_{i+1}), \tag{3.2}$$

where $\tau_i = \tau_{i,h}$, $\omega(\delta)$ is the modulo of continuity of the function $t \to \gamma(t)$, $t \in T$, i.e.,

$$\omega(\delta) = \sup\{|\gamma(t) - \gamma(t - \delta)| : t \in [\delta, \vartheta], 0 < \delta < \vartheta\}.$$

Inequality (3.2) is a consequence of (2.2) and (3.1).

Fix a family Δ_h of partitions of the interval T of form (2.3) and some auxiliary function $\alpha(h) : (0, 1) \to (0, 1)$ and choose a linear system M (a model) described by the following equation:

$$\dot{w}^h(t) = f(\tau_i, \xi_i^h) - \lambda \xi_i^h u_i^h \quad \text{for a.a.} \quad t \in \delta_i = [\tau_i, \tau_{i+1}), \tag{3.3}$$

$i \in [0 : m - 1]$, $\tau_i = \tau_{i,h}$, $m = m_h$, with the initial condition

$$w^h(0) = x_2(0).$$

Let

$$\begin{aligned} u_i^h \in U(\tau_i, \xi_i^h, w^h(\tau_i)) &= -\arg\min\{2\lambda\xi_i^h(w^h(\tau_i) - \xi_i^h)u + \alpha u^2 : u \in \mathbb{R}\} = \\ &= \frac{1}{\alpha}\lambda\xi_i^h[w^h(\tau_i) - \xi_i^h] \quad \text{for} \quad t \in \delta_i, \quad \alpha = \alpha(h). \end{aligned} \tag{3.4}$$

Thus, in the equation of the model (see (2.4)), we have

$$f_1(\xi_i^h, w^h(\tau_i), u_i^h) = f(\tau_i, \xi_i^h) - \lambda \xi_i^h u_i^h.$$

The model control u_i^h is determined by the feedback principle (see (2.5) and (3.4)). Hence, equation (3.3) takes the form

$$\dot{w}^h(t) = f(\tau_i, \xi_i^h) - \frac{1}{\alpha}(\lambda\xi_i^h)^2[w^h(\tau_i) - \xi_i^h] \quad \text{for a.a.} \quad t \in \delta_i. \tag{3.5}$$

Let us describe the algorithm for reconstructing the unmeasured coordinate $x_1(\cdot)$ in the real time mode. Before the algorithm starts, we fix some accuracy $h \in (0, 1)$ and a partition Δ_h. The work of the algorithm is decomposed into $m - 1$ identical steps. At the i-th step carried out on the time interval $\delta_i = [\tau_i, \tau_{i+1})$, $\tau_i = \tau_{i,h}$, the following sequence of actions is fulfilled. First, at the moment τ_i, the control u_i^h is calculated by formula (3.4). After that, this constant control is fed onto the input of model (3.3) on the time interval $[\tau_i, \tau_{i+1})$. As a result, under the action of this control, the model passes from the state $w^h(\tau_i)$ into the state $w^h(\tau_{i+1}) = w^h(\tau_{i+1}; \tau_i, w^h(\tau_i), u_i^h)$. The procedure stops at the moment ϑ.

Hereinafter, we assume that the following condition is fulfilled.

Condition 3.1. $\min_{s \in T}(\lambda x_2(s))^2 \ge b > 0$.

By the symbol $\Xi(x(\cdot), h)$, we denote the set of all piecewise constant functions $\xi(\cdot) : T \to \mathbb{R}$, $\xi(t) = \xi_i^h$ for $t \in [\tau_i, \tau_{i+1})$, $i \in [0 : m_h - 1]$, where the numbers ξ_i^h satisfy inequalities (2.2).

The following lemma is valid.

Lemma 3.2. *Let the conditions*

$$\alpha(h) \to 0, \quad \delta(h) \to 0, \quad \delta(h)\alpha^{-1}(h) \to 0, \quad h\alpha^{-1}(h) \to 0 \quad \text{as} \quad h \to 0 \tag{3.6}$$

be fulfilled. Then the inequalities

$$\int_{\tau_i}^{\tau_{i+1}} |\dot{w}^h(s)|\, ds \le C\delta \tag{3.7}$$

are fulfilled uniformly with respect to any $h \in (0, 1)$, $\xi^h(\cdot) \in \Xi(x(\cdot), h)$, *and* $i \in [0 : m_h - 1]$. *Here* $C = \text{const} > 0$, $\delta = \delta(h)$, $\tau_i = \tau_{i,h}$.

Proof. Taking into account (2.5), we conclude that the following equalities

$$\frac{d}{dt}[w^h(t) - x_2(t)] = f(\tau_i, \xi_i^h) - \frac{1}{\alpha}(\lambda\xi_i^h)^2[w^h(\tau_i) - \xi_i^h] - f(t, x_2(t)) - \lambda x_2(t)\tilde{u}(t) =$$
$$= -\frac{1}{\alpha}(\lambda x_2(s))^2[w^h(t) - x_2(t)] + \Psi_h^{(1)}(t) \quad \text{for a. a.} \quad t \in \delta_i$$

are fulfilled. In addition,

$$w^h(0) = x_2(0),$$

where

$$\Psi_h^{(1)}(s) = \Psi_h(s) + \Psi_h^{(4)}(s),$$
$$\Psi_h(s) = \Psi_h^{(2)}(s) + \Psi_h^{(3)}(s),$$

$$\Psi_h^{(2)}(s) = -\frac{1}{\alpha}(\lambda x_2(s))^2[x_2(s) - \xi_i^h] + [f(\tau_i, \xi_i^h) - f(s, x_2(s))], \quad \Psi_h^{(3)}(s) = -\lambda x_2(s)\tilde{u}(s),$$

$$\Psi_h^{(4)}(s) = \frac{1}{\alpha}(\lambda x_2(s))^2[w^h(s) - w^h(\tau_i)] - \frac{1}{\alpha}(\lambda\xi_i^h)^2[(\xi_i^h)^2 - x_2^2(s)][w^h(\tau_i) - \xi_i^h], \quad s \in \delta_i.$$

Here, in view of (2.2), (3.2), (3.3), and (3.7), the family of the functions $\Psi_h(\cdot)$ is bounded:

$$|\Psi_h(s)| \le M^{(1)} \quad \text{for a.a.} \quad t \in T \tag{3.8}$$

uniformly with respect to $h \in (0, 1)$. Further, we have

$$w^h(t) - x_2(t) = \int_0^t e^{-\int_s^t \frac{1}{\alpha}(\lambda x_2(s))^2 ds} \Psi_h^{(1)}(s)\, ds, \quad t \in T. \tag{3.9}$$

Introduce the notation:

$$\mu(t) = \max_{0 \le \tau \le t} |w^h(\tau) - x_2(\tau)|, \quad f_h(t) = f(\tau_i, \xi_i^h) \quad \text{for} \quad t \in \delta_i.$$

In virtue of the inequality

$$\sup\{|f_h(t)| : h \in (0,1), t \in T\} \le K_0, \tag{3.10}$$

the estimates

$$\begin{aligned}\frac{\lambda^2}{\alpha}\int\limits_{\tau_i}^{\tau_{i+1}} x_2^2(s)|\dot w^h(s)|\,ds \le \frac{K_2}{\alpha}\int\limits_{\tau_i}^{\tau_{i+1}} |f_h(s) - \frac{1}{\alpha}(\lambda\xi_i^h)^2[w^h(\tau_i) - \xi_i^h]|\,ds \le \\ \le K_3\frac{\delta}{\alpha} + K_4\frac{\delta}{\alpha^2}(\mu(\tau_i)+h), \quad \mu(\tau_i) \le \mu(\tau_{i+1})\end{aligned} \tag{3.11}$$

are true. Note that

$$|\Psi_h^{(1)}(t)| \le |\Psi_h(t)| + K_5\frac{h+\delta}{\alpha}(\mu(\tau_i)+h) + \frac{1}{\alpha}(\lambda x_2(t))^2\int\limits_{\tau_i}^{\tau_{i+1}} |\dot w^h(s)|\,ds \quad \text{for} \quad t \in \delta_i. \tag{3.12}$$

Thus, taking into account (3.9)–(3.12), we obtain

$$\begin{aligned}\mu(t) \le K_6\Big(\frac{\delta}{\alpha} + \frac{h}{\alpha}\mu(\tau_i) + \frac{h^2}{\alpha} + \frac{\delta}{\alpha^2}\mu(\tau_i) + \frac{\delta h}{\alpha^2}\Big)\int\limits_0^t e^{-\int\limits_s^t \frac{1}{\alpha}(\lambda x_2(v))^2 dv}\,ds + \\ + \int\limits_0^t e^{-\int\limits_s^t \frac{1}{\alpha}(\lambda x_2(v))^2\,dv}|\Psi_h(s)|\,ds, \quad t \in \delta_i.\end{aligned} \tag{3.13}$$

Using inequality (3.8), we derive

$$\int\limits_0^t e^{-\frac{1}{\alpha}(\lambda x_2^2(s))^2(t-s)}|\Psi_h(s)|\,ds \le K_7\int\limits_0^t e^{-\int\limits_s^t \frac{1}{\alpha}(\lambda x_2(v))^2\,dv}\,ds. \tag{3.14}$$

Then, in virtue of condition 3.1, the following relations

$$\int\limits_0^t e^{-\int\limits_s^t \frac{1}{\alpha}(\lambda x_2(v))^2\,dv}\,ds \le \int\limits_0^t e^{-\frac{b}{\alpha}(t-s)}\,ds = \frac{\alpha}{b}e^{-\frac{b}{\alpha}(t-s)}\Big|_0^t = \frac{\alpha}{b}(1-e^{-\frac{b}{\alpha}t}) \le K_8\alpha \tag{3.15}$$

hold. Thus, from (3.14) and (3.15) we obtain

$$\int\limits_0^t e^{-\int\limits_s^t \frac{1}{\alpha}(\lambda x_2(v))^2\,dv}|\Psi_h(s)|\,ds \le K_9\alpha. \tag{3.16}$$

In turn, from (3.13) we derive (assuming $t = \tau_i$ and taking into account (3.8), (3.16))

$$\mu(\tau_i) \le K_6K_8\Big(\delta + h^2 + \frac{\delta}{\alpha}\mu(\tau_i) + h\mu(\tau_i) + \frac{\delta h}{\alpha}\Big) + K_9\alpha.$$

In this case,

$$\Big(1 - K_6 K_8 \Big(h + \frac{\delta}{\alpha}\Big)\Big)\mu(\tau_i) \le K_{10}\Big(\alpha + h + \delta + \frac{\delta h}{\alpha}\Big).$$

Therefore, for sufficiently small h (for example, such that $1 - K_6 K_8 (h + \frac{\delta}{\alpha}) \le \frac{1}{2}$), we have

$$\mu(\tau_i) \le K_{11}\Big(\alpha + \delta + h + \frac{\delta h}{\alpha}\Big) \le K_{12}(\alpha + h + \delta) \tag{3.17}$$

(see (3.6)). From (3.11), in virtue of Condition 3.1, we deduce that

$$\int\limits_{\tau_i}^{\tau_{i+1}} |\dot{w}^h(s)|\, ds \le K_{13}\Big\{\delta + \frac{\delta}{\alpha}(\mu(\tau_i) + h)\Big\}.$$

Using again (3.6), we have

$$\delta + \frac{\delta}{\alpha}(\mu(\tau_i) + h) \le \delta + K_{14}\frac{\delta}{\alpha}\Big(\alpha + \delta + h + \frac{\delta h}{\alpha}\Big) \le K_{15}\delta.$$

Consequently,

$$\int\limits_{\tau_i}^{\tau_{i+1}} |\dot{w}^h(s)|\, ds \le K_{16}\delta.$$

Inequality (3.7) is established. The lemma is proved. □

Lemma 3.3. *Let conditions* (3.6) *be fulfilled and* $\delta^\beta(h)\alpha^{-1}(h) \to +\infty$ (*for some* $\beta \in (0,1)$) *as* $h \to 0$. *Let*

$$u^h(t) = \begin{cases} x_{10}, & t \in [0, \zeta(h)), \\ -v^h(t), & t \in [\zeta(h), \vartheta], \end{cases}$$

where

$$v^h(t) = v_i^h = -u_i^h \quad \textit{for} \quad t \in [\tau_i, \tau_{i+1}), \quad \zeta(h) = \delta^\beta(h).$$

Then the inequality

$$\sup_{t\in T} |u^h(t) - x_1(t)| \le C(\alpha(h) + (h + \delta(h))\alpha^{-1}(h) + \omega(\delta(h)) + \alpha(h)\delta^{-\beta}(h)),$$

is true. Here, the constant C *does not depend on* $h \in (0,1)$, $t \in T$, $\xi^h(\cdot) \in \Xi(x(\cdot), h)$.

Proof. It is easily seen that the equality

$$\begin{aligned} \frac{1}{\alpha}(\lambda x_2(t))^2[w^h(t) - x_2(t)] &= \int\limits_0^t \frac{d}{ds}\Big(e^{-\frac{1}{\alpha}\int_s^t (\lambda x_2(v))^2 dv}\Big)\Psi_h^{(1)}(s)\, ds = \\ = -\lambda \int\limits_0^t \frac{d}{ds}\Big(e^{-\frac{1}{\alpha}\int_s^t (\lambda x_2(v))^2 dv}\Big)& x_2(s)\tilde{u}(s)\, ds + \sum_{j=1}^4 \int\limits_0^t \frac{d}{ds}\Big(e^{-\frac{1}{\alpha}\int_s^t (\lambda x_2(v))^2 dv}\Big)\gamma_\delta^{(j)}(s)\, ds, \end{aligned} \tag{3.18}$$

is valid. Here,

$$\gamma_\delta^{(1)}(s) = \frac{1}{\alpha}(\lambda x_2(s))^2[w^h(s) - w^h(\tau_i)],$$

$$\gamma_\delta^{(2)}(s) = -\frac{1}{\alpha}(\lambda x_2(s))^2[x_2(s) - \xi_i^h],$$

$$\gamma_\delta^{(3)}(s) = f(\tau_i, \xi_i^h) - f(s, x_2(s)),$$

$$\gamma_\delta^{(4)}(s) = -\frac{1}{\alpha}\lambda^2[(\xi_i^h)^2 - x_2^2(s)][w^h(\tau_i) - \xi_i^h] \quad \text{for a.a.} \quad s \in \delta_i.$$

Due to (3.7), we conclude that

$$|\gamma_\delta^{(1)}(s)| \le C_1\frac{\delta}{\alpha}, \quad s \in T. \tag{3.19}$$

In turn, using (2.2) and (3.1), we have

$$|\gamma_\delta^{(2)}(s)| \le C_2\frac{\delta + h}{\alpha}, \quad s \in T. \tag{3.20}$$

In addition (see (2.2) and (3.3)), the estimates

$$|\gamma_\delta^{(3)}(s)| \le M_*(\delta + h + \omega(\delta)), \quad s \in T, \tag{3.21}$$

$$|\gamma_\delta^{(4)}(s)| \le c_3\frac{\delta + h}{\alpha}, \quad s \in T, \tag{3.22}$$

are true. In this case, taking into account (3.15) and (3.16), from (3.19)–(3.22) we deduce that

$$\Big|\sum_{j=1}^{4}\int_0^t \frac{d}{ds}\Big(e^{-\frac{1}{\alpha}\int_s^t(\lambda x_2(v))^2 dv}\Big)\gamma_\delta^{(j)}(s)\,ds\Big| \le \tag{3.23}$$

$$\le \varrho(h, \alpha, \delta) = C_4\Big(\delta + h + \omega(\delta) + \frac{\delta + h}{\alpha}\Big).$$

Integrating the first term on the right-hand side of equality (3.18) by parts, we get

$$-\int_0^t \frac{d}{ds}\Big(e^{-\frac{1}{\alpha}\int_s^t(\lambda x_2(v))^2 dv}\Big)\lambda x_2(s)\tilde{u}(s)\,ds = \tag{3.24}$$

$$= \lambda e^{-\frac{1}{\alpha}\int_0^t(\lambda x_2(v))^2 dv} x_2(0)\tilde{u}(0) - \lambda x_2(t)\tilde{u}(t) + \int_0^t e^{-\frac{1}{\alpha}\int_s^t(\lambda x_2(v))^2 dv}\frac{d}{ds}(\lambda x_2(s)\tilde{u}(s))\,ds.$$

Then, by virtue of (2.2), (3.2), and (3.7) (see Lemma 3.2), we have for $t \in \delta_i$

$$\Big|\frac{1}{\alpha}(\lambda x_2(t))^2[w^h(t) - x_2(t)] - \frac{1}{\alpha}(\lambda\xi_i^h)^2[w^h(\tau_i) - \xi_i^h]\Big| \le \tag{3.25}$$

$$\leq \frac{C_5}{\alpha}\Big\{ \int\limits_{\tau_i}^{\tau_{i+1}} |\dot{w}^h(s)|\,ds + h + \delta \Big\} \leq C_6 \frac{h+\delta}{\alpha}.$$

In view of (3.16), we derive the inequality

$$\Big| \int\limits_0^t e^{-\frac{1}{\alpha}\int\limits_s^t (\lambda x_2(v))^2 dv} \lambda \frac{d}{ds}(x_2(s)\tilde{u}(s))\,ds \Big| \leq C_7 \alpha.$$

Combining together the last inequality with (3.18), (3.23)–(3.25), and (3.6), we obtain for $t \in \delta_i$

$$\begin{aligned} &\Big| -\frac{1}{\alpha}\lambda^2[w^h(\tau_i) - x_2(\tau_i)] - \lambda x_2(t)\tilde{u}(t) \Big| \leq \\ &\leq \varrho(h,\delta,\alpha) + C_6 \frac{h+\delta}{\alpha} + C_7\alpha \leq C_8\Big(\alpha + \omega(\delta) + \frac{\delta + h}{\alpha}\Big). \end{aligned} \tag{3.26}$$

Note that for $t = \delta^\beta$ $(\beta \in (0,1))$ the inequality

$$\Big| \lambda e^{-\frac{1}{\alpha}\int\limits_0^t (\lambda x_2(v))^2 dv} x_2(0)\tilde{u}(0) \Big| \leq \frac{\alpha |\lambda x_1(0) x_2(0)|}{b\delta^\beta} \tag{3.27}$$

takes place. The lemma follows from (3.26) since, in virtue of (3.27), the following inequalities

$$|u^h(t) - \tilde{u}(t)| \leq C_9 \delta^\beta \quad \text{for} \quad t \in [0, \delta^\beta],$$

$$|u^h(t) - \tilde{u}(t)| \leq C_8\Big(\alpha + \omega(\delta) + \frac{\delta + h}{\alpha}\Big) + C_{10}\alpha(h)\delta^{-\beta}(h) \quad \text{for } t \in (\delta^\beta, \vartheta]$$

are valid. The lemma is proved. □

The next statement follows from Lemma 3.3.

Theorem 3.4. *Let the conditions of Lemma 3.3 be fulfilled. Then convergence of* (2.6) *takes place.*

We consider the problem of reconstructing the coordinate $x_1(\cdot)$ through the measurements of the coordinate $x_2(\cdot)$. Similarly, we can solve the "inverse" problem, namely, the problem of reconstructing the coordinate $x_2(\cdot)$ through the measurements of the coordinate $x_1(\cdot)$.

Actually, let the values λ, ν and μ, as well as the function $k(t)$ be known. The function $\gamma(t)$ is unknown. It is only known that this function is continuous and satisfies the condition $-\infty < a \leq \gamma(t) \leq b < +\infty$. Assume that the part of the current state, namely, the coordinate $x_1(\tau_i)$ is measured at the time moments τ_i. The measurement results $\xi_{1i}^h \in \mathbb{R}$ satisfy the inequalities

$$|x_1(\tau_i) - \xi_{1i}^h| \leq h.$$

It is required to design an algorithm for reconstructing the coordinate $x_2(\cdot)$.

Let the following condition be fulfilled.

Condition 3.5. a) The function $k(\cdot)$ is differentiable and

$$\operatorname{vrai}\sup_{t\in T}|\dot{k}(t)| \le c = \text{const} \ > \ 0.$$

b) $\min_{s\in T}(k(s)+\lambda x_1(s))^2 > 0$.

As a model M, we take a linear system described by the following scalar equation

$$\dot{w}_1^h(t) = f_*(\xi_{1i}^h) + (k(\tau_i) + \lambda\xi_{1i}^h)v_1^h(t) \quad \text{for a.a.} \quad t \in \delta_i = [\tau_i, \tau_{i+1}),$$

$i \in [0 : m-1]$, $\tau_i = \tau_{i,h}$, $m = m_h$, with the initial state

$$w_1^h(0) = x_1(0).$$

Here, $f_*(x_1) = -\nu x_1$. The control $v_1^h(\cdot)$ in the model is calculated by the rule

$$v_1^h(t) = v_{1i}^h = \arg\min\{2(k(\tau_i) + \lambda\xi_{1i}^h)(w_1^h(\tau_i) - \xi_{1i}^h)v + \alpha v^2 : v \in \mathbb{R}\} =$$
$$= -\frac{1}{\alpha}(k(\tau_i) + \lambda\xi_{1i}^h)[w_1^h(\tau_i) - \xi_{1i}^h], \quad t \in \delta_i.$$

Similarly to Lemma 3.3, the next lemma can be proved.

Lemma 3.6. *Let the conditions of Lemma 3.3 and the following relation*

$$u_h(t) = \begin{cases} x_2(0), & t \in [0, \delta^\beta(h)), \\ -v_1^h(t), & t \in [\delta^\beta(h), \vartheta] \end{cases}$$

be fulfilled. Then the inequality

$$\sup_{t\in T}|u_h(t) - x_2(t)| \le$$

$$\le C(\alpha(h) + (h + \delta(h))\alpha^{-1}(h) + \omega(\delta(h)) + \alpha(h)\delta^{-\beta}(h))$$

is valid.

The latest lemma implies the following theorem.

Theorem 3.7. *Let the conditions of Theorem 3.4 be fulfilled. Then the convergence*

$$\sup_{t\in T}|u_h(t) - x_2(t)| \to 0 \quad as \quad h \to 0$$

takes place.

4. EXAMPLE

The algorithm was tested with the use of a model example. System (2.1) was considered on the time interval $T = [0, 2]$, $x_1, x_2 \in \mathbb{R}$. The initial state was of the form

$x_1(0) = 1$, $x_2(0) = 2$. We took the model with the initial state $w(0) = 2$ and the controls according to (3.3) and (3.4). We used the following parameters:

$$\vartheta = 2, \qquad \nu = 1, \qquad \mu = 1, \qquad \lambda = 3, \qquad \gamma(t) = \sin t + 1.5, \qquad k(t) = 0.5.$$

The computation results are shown in Figures 1 and 2. The dashed line represents the coordinate $x_1(\cdot)$; the solid line, the control $u^h(\cdot)$.

The results of numerical experiments show that the uniform convergence of $u^h(\cdot)$ to $x_1(\cdot)$ takes place under decreasing the parameters h, $\alpha = \alpha(h)$, and $\delta = \delta(h)$ or one of them. In order not to lengthen the paper, we presented only two figures corresponding to different values of the partition step δ.

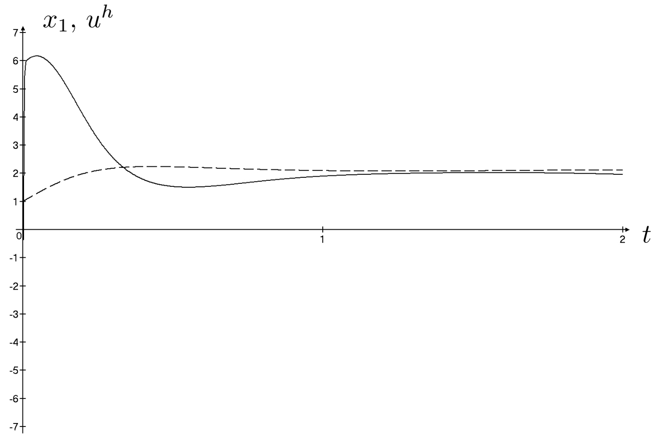

Fig. 1. $h = 0.001$, $\alpha = 0.1$, $\delta = 0.1$

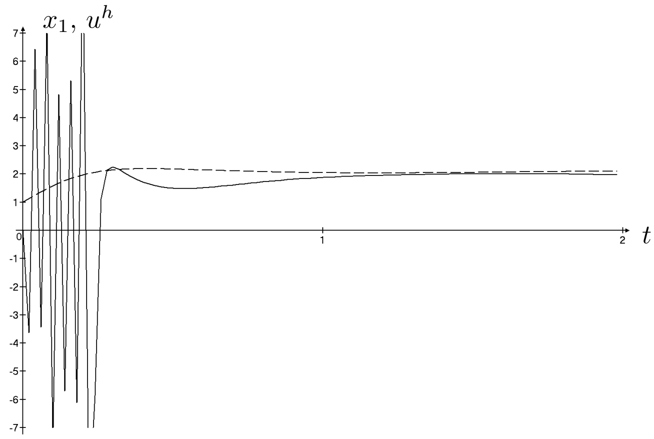

Fig. 2. $h = 0.001$, $\alpha = 0.1$, $\delta = 0.001$

5. CONCLUSION

In this paper, the algorithm for reconstructing characteristics of a system of nonlinear differential equations of the second order was designed. The algorithm is based on the theory of stable dynamical inversion. It is stable with respect to informational noises and computational errors. The performance of the algorithm is illustrated by a model example.

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