CIRCLE ORTHOGONALLY INTERSECTING TWO NON - COPLANAR CIRCLES

Andrzej BIELIŃSKI[1/], Cecylia ŁAPIŃSKA[2/]

[1/] Warsaw University of Technology
20 Nowowiejska st., 00-653 Warsaw, Poland
email: andrzej.bielinski@is.pw.edu.pl
[2/] Warsaw University of Technology
20 Nowowiejska st., 00-653 Warsaw, Poland
email: cecylia.lapinska@is.pw.edu.pl

Abstract: The problem of finding a circle orthogonally intersecting two non-coplanar circles is discussed. It is shown that if the required circle exists, it can be obtained as the intersection line of two adequately chosen spheres. Conditions which must be satisfied by such two spheres are given. Using the properties of these spheres, the problem of constructing a circle orthogonally intersecting two circles in chain position can be reduced to finding a straight line intersecting four suitably determined straight lines.

Keywords: power of a point, cross ratio, pole, polarity, polar (plane) of a point with respect to a sphere, conjugates with respect to a sphere, Steiner's construction.

1 Introduction

In this paper we are dealing with circles in the real space.

If we say that two circles intersect, we understand that they meet at two distinct points. The straight line perpendicular to the plane of a circle and passing through its center will be called *axis of the circle.*

The problem of finding a circle intersecting two given circles, which is not difficult for the coplanar circles (see for ex. [4] §9), becomes complicated in the case of circles not lying in the same plane.

Our idea of how to resolve this problem is based on the two following statements, well known and not difficult to prove.

(I) There exists exactly one sphere passing through two non-coplanar circles which meet at two distinct points (because their axes intersect).

(II) Two circles lying in a sphere are orthogonally intersecting at two distinct points if and only if their planes are conjugate with respect to this sphere (Fig.1 – $P_1 \in \alpha_2$, $P_2 \in \alpha_1$).

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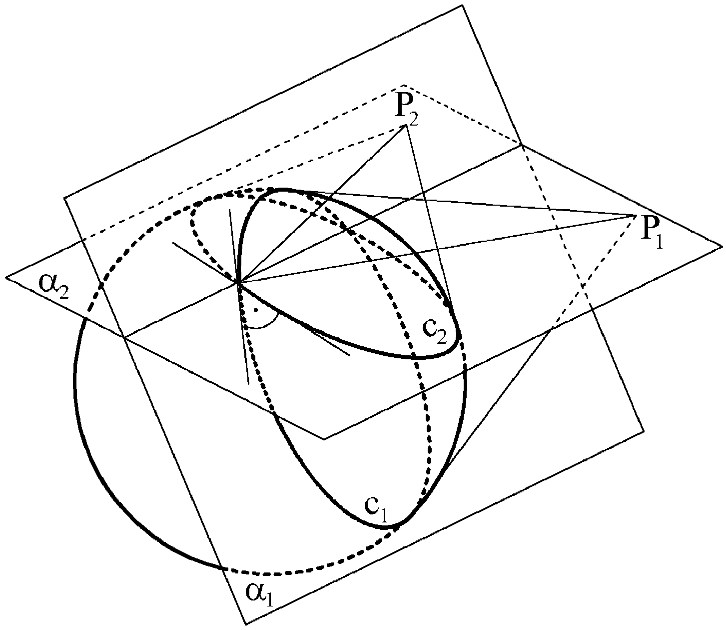

Fig.1

In fact, it follows from the above that if the required circle exists it may be obtained as the intersection line of two appropriate spheres. Each of these spheres contains one of the given circles and the inquired circle – evidently they are not fully determined for now.

However, we will examine such spheres in order to obtain some conditions describing them in a way that enables to make a construction.

2 Analysis

Let's now take into consideration two given circles c_1 and c_2 with axes l_1 and l_2 and lying in the planes α_1 and α_2 respectively.

Denote by S_1 and S_2 spheres containing the circles c_1 and c_2 respectively.

In Fig.2 a simplified representation of the spatial situation is given – some elements taken into our consideration in the following analysis are omitted for better clarity.

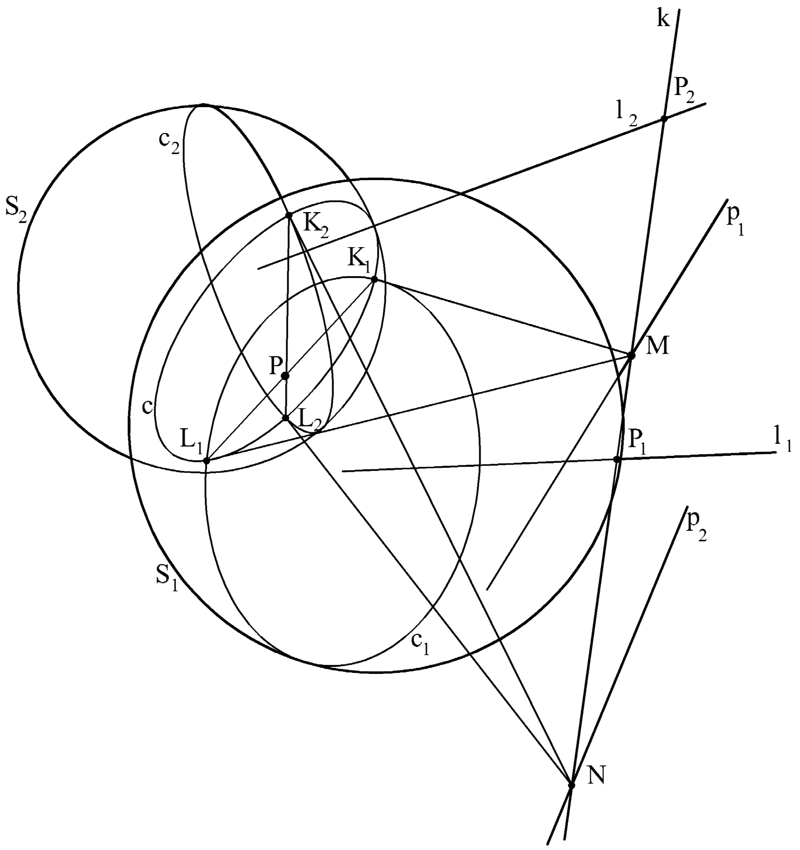

Fig.*2*

We will determine under what conditions the spheres S_1 and S_2 intersect at a circle *c* which meets both c_1 and c_2 at right angles.

Let α denote the plane of this circle *c*.

By the property (II), if the circle *c* intersects perpendicularly the circle c_1 then the planes α and α_1 are conjugate with respect to the sphere S_1 (i.e. the pole of α is lying in α_1 and reciprocally the pole of α_1 is lying in α). By the same argument the planes α and α_2 are conjugate with respect to the sphere S_2. Hence the points P_1 and P_2, P_1 being the pole of the plane α_1 with respect to the sphere S_1 and P_2 the pole of the plane α_2 with respect to the sphere S_2 , both lie in the plane α.

Denote by *P* the common point of the three planes α, α_1 and α_2 . Observe that powers of the point P with respect to the three circles *c*, c_1 and c_2 are equal.

We will first study some relationships on the sphere S_1.

Remark that if a straight line passing through the point *P* pierces the sphere S_1 at points *X* and *Y* then it pierces the polar plane of *P* with respect to S_1 at point *Z* which is the harmonic conjugate of *P* with respect to *X, Y* and conversely: the harmonic conjugate of *P* with respect to *X, Y* belongs to the polar of P with respect to S_1.

Designate the polar plane of *P* with respect to S_1 as β_1, the line PP_1 as *q* and the line PP_2 as *r*. Because P_1 and P_2 lie in α, the lines *q* and *r* intersect the sphere S_1 at points lying on the circle *c* – the line *q* at points Q_1 and Q_2, the line *r* at points R_1 and R_2 respectively. Thus we obtain:

$$(Q_1 Q_2, PP_1) = -1 \quad \text{and} \quad (R_1 R_2, PP_1) = -1$$

Hence conformably to the previous remark the points P_1 and P_2 belong to the plane β_1, which is the polar of the point P with respect to the sphere S_1.

Let the circle c intersect the circle c_1 at points K_1 and L_1 and the circle c_2 at points K_2 and L_2. So the pole N of the line K_1L_1 with respect to the circle c_1 lies on the line p_1 in α_1 which is the polar of the point P with respect to the circle c_1. Similarly the pole M of the line K_2L_2 with respect to the circle c_2 lies on the line p_2 in α_2 which is the polar of the point P with respect to the circle c_2.

Let now the straight line PN pierce the sphere S_1 at points A_1 and A_2 and the line PM pierce this sphere at points B_1 and B_2. It is clear that

$$(A_1A_2, PN) = -1 \qquad \text{and} \qquad (B_1B_2, PM) = -1$$

Thus we can conclude that the points N and M together with the points P_1 and P_2 lie in the plane β_1 which is the polar of the point P with respect to the sphere S_1.

Reiterating the same reasoning as above but in relation to the sphere S_2 we obtain the following conclusion analogous to the previous:

The points N, M, P_1 and P_2 lie in the plane β_2 which is the polar of the point P with respect to the sphere S_2.

Consequently the points N, M, P_1 and P_2 lie on the edge k of the polar planes β_1 and β_2, and then they are collinear.

The poles of planes α_1 and α_2 with respect to the spheres including the circles c_1 and c_2 lie on the axes l_1 and l_2 of these circles respectively.

The straight line k, which intersect the polars p_1 and p_2 of the point P with respect to the circles c_1 and c_2 at points N and M respectively, also intersect the lines l_1 and l_2 at the points at points P_1 and P_2.

3 Conclusion

It follows from the previous considerations that the required spheres S_1 and S_2, which intersect at the circle c orthogonally intersecting two given circles c_1 and c_2, are completely determined if the poles P_1 and P_2 of planes α_1 and α_2 with respect to these spheres respectively are known. Because the axes l_1 and l_2 are given together with the circles c_1 and c_2 then the required points P_1 and P_2 lying on k and also on l_1 and l_2 respectively are determined if the line k is determined.

The line k can be defined as the one intersecting the four lines l_1, l_2, p_1, p_2.

The lines p_1, p_2 being defined as the polars of the point P with respect to the circles c_1 and c_2 respectively are evidently determined by the point P, which is defined by the fact that its powers with respect to c_1 and c_2 are equal.

The point P is uniquely determined in the following case.

We will say that two circles are in *chain position* if they are not coplanar and if the edge of their planes intersects them in two separating pairs of points. Such circles are represented in Fig.3.

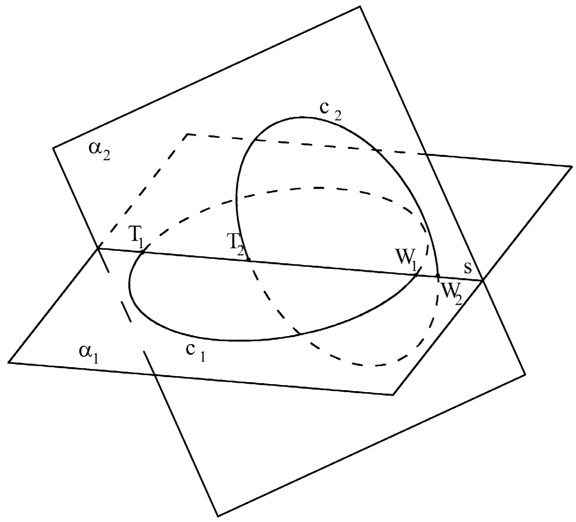

Fig.3

Denote these circles by c_1 and c_2 and their planes by α_1 and α_2 correspondingly. The edge s of planes α_1 and α_2 intersects the circle c_1 at the points T_1 and W_1 and the circle c_2 at the points T_2 and W_2 respectively. The pair of the points T_1, W_1 separates the pair T_2 , W_2 so the cross ratio (T_1W_1, T_2W_2) is negative. The four lines l_1, l_2, p_1, p_2 are evidently known.

Then in this case the problem of finding a circle orthogonally intersecting these circles can be reduced to constructing a straight line intersecting four given lines. The construction (together with a discussion of the existence) of that line based on the Steiner's construction is described by the authors in [1].

References

[1]. Bieliński A., Łapińska C.: *Straight line intersecting four given lines.,* The Journal BIULETYN of Polish Society for Geometry and Engineering Graphics,Vol.18 (2008).
[2]. Grochowski B.: *Geometria wykreślna z perspektywą stosowaną*. Państwowe Wydawnictwo Naukowe, Warszawa 1988.
[3]. Plamitzer A.: *Elementy geometrii rzutowej*. Lwów 1927.
[4]. Silvester J.R.: *Geometry Ancient and Modern*. Oxford Univ. Press Inc, New York 2001.

OKRĄG PRZECINAJĄCY ORTOGONALNIE DWA OKRĘGI NIEWSPÓŁPŁASZCZYZNOWE

W pracy pokazano, że okrąg przecinający dwa dane okręgi nieleżące w jednej płaszczyźnie może być uzyskany jako część wspólna dwóch odpowiednio dobranych sfer. Sformułowano warunki opisujące te sfery, z których wynikają warunki dla istnienia poszukiwanego okręgu. Pokazano, że w niektórych sytuacjach konstrukcja tego okręgu sprowadza się do wyznaczenia prostej przecinającej cztery odpowiednio dobrane proste. Ta prosta może być otrzymana przez zastosowanie konstrukcji Steinera.