JOURNAL OF SUSTAINABLE MINING

Volume 23 | Issue 1 Article 3

2024

Mathematical model for the management of the wave processes in three-winding transformers with consideration of the main magnetic flux in mining industry

Author(s) ORCID Identifier:
M. S. Seheda: 0000-0001-8459-5758
O. S. Beshta: 0000-0002-4648-0260
P. F. Gogolyuk: 0000-0003-0002-4638
Yu. V. Blyznak: 0000-0002-4914-2283
R. D. Dychkovskyi: 0000-0002-3143-8940
Adam Smoliński: 0000-0002-4901-7546

Recommended Citation

Seheda, M. S.; Beshta, O. S.; Gogolyuk, P. F.; Blyznak, Yu. V.; Dychkovskyi, R. D.; and Smoliński, A. (2024) "Mathematical model for the management of the wave processes in three-winding transformers with consideration of the main magnetic flux in mining industry," *Journal of Sustainable Mining*: Vol. 23 : Iss. 1 , Article 3.
Available at: https://doi.org/10.46873/2300-3960.1402

Mathematical model for the management of the wave processes in three-winding transformers with consideration of the main magnetic flux in mining industry

Abstract

The aim of the work is to study the wave processes in three-winding power transformers caused by impulse overvoltage, to create an improved mathematical model for reproducing the process of distribution and transmission of the impulse in the windings of a three-winding power transformer. A mathematical model has been developed for the study of internal overvoltage in the windings of three-winding power transformers, based on the proposed substitute circuit of an infinitesimal element, taking into account the longitudinal and transverse inductive connections between the turns of the winding, the electromagnetic connections between the windings and the flux splitting from the main magnetic flux of the magnetic wire, in the form of a system of differential equations in partial derivatives using a modified method of variable separation. The formation of initial and boundary conditions for this mathematical model is presented. The results of the study of the distribution of overvoltage along the windings of a three-winding power transformer as a function of distance and time during the action of a voltage pulse on them are presented, as well as the distribution of overvoltage at different points of the winding of high, medium and low voltage as a function of time. The study of the wave processes in the windings of a three-winding power transformer makes it possible to form new approaches to the coordination of the insulation in the windings of the transformer, replacing physical experiments. The choice of insulation for high and ultra-high-voltage power transformers remains a particularly difficult engineering task since it is necessary to know the maximum voltage values at different points of the winding. The mathematical model presented can be used to create more complex models that allow a more detailed study of the wave processes.

Keywords

managing the wave process; mathematical model; transformer; differential equations with partial derivatives; initial and boundary conditions

Authors

M. S. Seheda, O. S. Beshta, P. F. Gogolyuk, Yu. V. Blyznak, R. D. Dychkovskyi, and A. Smoliński

RESEARCH ARTICLE

Mathematical model for the management of the wave processes in three-winding transformers with consideration of the main magnetic flux in mining industry

M. S. Seheda [a], O. S. Beshta [b], P. F. Gogolyuk [a], Yu. V. Blyznak [a], R. D. Dychkovskyi [c], A. Smoliński [d,e,*]

[a] Lviv Polytechnic National University, 79013, L'vov, Stepana Bandera St., 12, Ukraine
[b] Dnipro University of Technology, 49005, Dnipro, av. Dmytra Yavornytskoho, Ukraine
[c] AGH University of Science and Technology, Faculty of Management, 30059, al. Mickiewicza 30, Krakow, Poland
[d] Central Mining Institute, 40166, Plac Gwarkow 1, Katowice, Poland
[e] Spółka Restrukturyzacji Kopalń S.A., 30059, Strzelców Bytomskich 207, Bytom, Poland

Abstract

The aim of the work is to study the wave processes in three-winding power transformers caused by impulse overvoltage, to create an improved mathematical model for reproducing the process of distribution and transmission of the impulse in the windings of a three-winding power transformer. A mathematical model has been developed for the study of internal overvoltage in the windings of three-winding power transformers, based on the proposed substitute circuit of an infinitesimal element, taking into account the longitudinal and transverse inductive connections between the turns of the winding, the electromagnetic connections between the windings and the flux splitting from the main magnetic flux of the magnetic wire, in the form of a system of differential equations in partial derivatives using a modified method of variable separation. The formation of initial and boundary conditions for this mathematical model is presented. The results of the study of the distribution of overvoltage along the windings of a three-winding power transformer as a function of distance and time during the action of a voltage pulse on them are presented, as well as the distribution of overvoltage at different points of the winding of high, medium and low voltage as a function of time. The study of the wave processes in the windings of a three-winding power transformer makes it possible to form new approaches to the coordination of the insulation in the windings of the transformer, replacing physical experiments. The choice of insulation for high and ultra-high-voltage power transformers remains a particularly difficult engineering task since it is necessary to know the maximum voltage values at different points of the winding. The mathematical model presented can be used to create more complex models that allow a more detailed study of the wave processes.

Keywords: Managing the wave process, Mathematical model, Transformer, Differential equations with partial derivatives, Initial and boundary conditions

1. Introduction

Managing wave processes in three-winding transformers requires a comprehensive understanding of their electrical behavior, modelling techniques and system interactions. Careful consideration of operating modes, impedance matching, transient phenomena and protection schemes is essential to ensure reliable and efficient operation within the power system.

A three-winding transformer has three separate sets of windings, each connected to a different electrical circuit. These windings interact with each other through the magnetic field they produce when current flows through them [1]. The primary winding produces the main magnetic flux, which induces voltages in the secondary and tertiary windings according to the turns ratio of the transformer and the magnetic coupling between the windings [2].

Received 14 September 2023; revised 23 September 2023; accepted 1 October 2023.
Available online 1 January 2024

* Corresponding author at: Central Mining Institute, 40166, Plac Gwarkow 1, Katowice, Poland.
E-mail address: smolin@gig.katowice.pl (A. Smoliński).

https://doi.org/10.46873/2300-3960.1402

The magnetic flux in the core of a transformer is responsible for transferring power from one winding to another. When an AC voltage is applied to the primary winding, it creates an alternating magnetic field that induces voltage in the secondary and tertiary windings. This process allows power to be transferred from the primary circuit to the secondary and tertiary circuits [3].

Consideration of the principal magnetic flux in a three-winding transformer is important in understanding how the windings interact and how power is transferred between them. When designing and analysing such transformers, factors such as the turns ratio of the windings, the core material and its magnetic properties, and the degree of coupling between the windings must be considered [4].

When analyzing the behavior of a three-winding transformer, we can use mathematical models and simulations that include the magnetic flux [5]. These models help to predict how changes in voltage, current and load on one winding can affect the others, taking into account the magnetic coupling and phase relationships involved [6]. The magnetic flux and its interaction with the windings are crucial for transformers, so this topic needs to be researched on the basis of modern science and technology [7]. Statistical data processing systems and the formation of multi-purpose theorems [8], including neural networks, can be involved in this process [9].

The study of wave processes in the windings of power transformers is a difficult engineering task, especially for high and very high-voltage transformers. This is due to the peculiarities of the technical implementation and design, namely the insulation of the winding of transformers of this voltage class. One way of solving this problem is to create mathematical models that adequately reproduce the processes that occur in transformers when they are subjected to impulse overvoltage, i.e. to create a mathematical model for the analysis of wave processes in transformer windings.

The need to take into account the electromagnetic connections between the windings and the windings of the three-winding transformer, taking into account the main magnetic flux, ensures the appropriate adequacy of the results of the mathematical modelling. The development of mathematical models for the study of wave processes in transformers taking into account these factors is relevant [10]. The main purpose of such research is to adequately reproduce the physical processes that occur during wave processes in transformers.

Linnik et al. [11] developed a model to study the electromagnetic processes in a power transformer, taking into account the magnetic connections and the saturation of the magnetic wire. The model is based on differential equations in complete derivatives of the electric and magnetic states of a single-phase two-winding transformer, and the algorithm for solving the system of differential equations using numerical simulation methods in Matlab 6.5 is presented. The results of the mathematical modelling of the electromagnetic processes in the no-load transformer are given.

High-frequency electromagnetic processes caused by the interaction of the electrical network and the power transformer are considered in [12]. The model of the transformer is created on the basis of the white-box modelling method, and the methodology for determining the impedance using the finite element method is given.

Mombello et al. [13] enabled the use of the software complex Alternative Transients Program (ATP) for modelling detailed equivalent transformer circuits. In creating these circuits, the following considerations were taken into account: reducing the size of the model, solving the magnetic circuit, avoiding numerical instability, limiting and optimizing the use of resistors, and creating versions of ATP capable of handling large models [14].

Three transformer models are given in [15]: a three-winding transformer, a tap-changer transformer (ULTC) and a phase-shift transformer (PST). To evaluate the effectiveness of the presented models, a test model of the IEEE 14-bus power supply system was implemented in the Modelica and PSAT software environments.

Frequency Response Analysis (FRA) is widely recognized as a reliable diagnostic tool for detecting damage, such as winding and core deformation due to short circuits. This paper [16] proposes a model for the study of high frequencies in a transformer, taking into account the winding structure, interturn capacitance and all mutual inductances [17].

High-frequency modeling in a three-winding transformer using frequency response sweep analysis is given in [18]. A mathematical model consisting of RLC sections and mutual inductance parameters is developed, and the SFRA curve is calculated and compared with that measured in practical experiments. This allows the parameters of the model to be adjusted and the accuracy of the estimation to be achieved, which in turn allows the parameters of the power transformer replacement circuits to be estimated effectively and simply on the basis of the SFRA data.

The study of partial discharges (PD) is also an issue for power transformers, namely for monitoring their condition. In order to analyze the transient process in the winding of the transformer, it is necessary to have a model that accurately reflects its behavior at high frequencies. The article [19] presents an improved mathematical model for the study of such processes. A generalized state space algorithm for the study of PD in the transformer winding has also been developed. The modelling of the layer and disc winding and the propagation of the PD pulse are presented, and the calculation is based on the finite element method.

The characteristics of resonant overvoltage for a transformer winding of different types are considered in [20,21], even without considering the mutual inductive connections between the turns of the winding. In [22,23], Laplace transforms and transfer functions are proposed to solve partial differential equations without considering the mutual inductive connections between windings.

Deatsonu et al. [24] present a study based on methods of numerical modelling of electric circuits of the SPICE family (Simulation Program with Integrated Circuit Emphasis), which analyses the effect of changing the winding parameters of the equivalent distributed circuit model of an electric transformer on the propagation of the overvoltage wave along the high-voltage winding of the transformer [25].

At present, methods of analyzing wave processes in transformer windings are aimed at developing mathematical models that take into account the main magnetic flux, the own and mutual windings and the mutual interwinding currents of the transformer winding dissipation. This approach requires the formation of mathematical models of the elements of the power system, taking into account all the parameters of the alternative circuit of the element, which makes it possible to study its internal transient processes [26,27].

Methods for the calculation of wave processes in two-turn transformers are given in works [28,29], on the basis of which it is possible to analyze the voltage distribution along the windings during the action of impulse overvoltage on them, which makes it possible to adjust their insulation capacity.

To study the wave processes in the windings of three-winding power transformers, develop a mathematical model based on the proposed substitute circuit of an infinitesimal element, taking into account the longitudinal and transverse inductive connections between the turns of the winding, the electromagnetic connections between the windings and the flux connection of the main magnetic flux of the magnetic wire in the form of a system of differential equations in partial derivatives. Form initial and boundary conditions and solve a system of partial differential equations using a modified method of separation of variables. To study the wave processes in the windings of three-winding power transformers as a function of distance and time and at different points of the winding of high, medium and low voltage as a function of time, caused by impulse overvoltage.

2. Materials and method

The formation of the mathematical model of the transformer winding with a grounded or insulated terminal is given in [30]. On the basis of the proposed model, the voltage distribution along the winding and the frequency characteristics for the effect of overvoltage of various forms on the winding were obtained. The mathematical model [31] of the three-winding transformer was developed on the basis of the proposed substitution scheme, which is shown in Figure 1.

The equations for the change in winding currents based on Kirchhoff's first law are given below:

$$-\frac{\partial i_1(x,t)}{\partial x}=g_{01}u_1(x,t)+(C_{012}+C_{013}+C_{01})\frac{\partial u_1(x,t)}{\partial t} - C_{012}\frac{\partial u_2(x,t)}{\partial t}-C_{013}\frac{\partial u_3(x,t)}{\partial t}-C_{M01}\frac{\partial^3 u_1(x,t)}{\partial x^2\partial t}; \quad (1)$$

$$-\frac{\partial i_2(x,t)}{\partial x}=g_{02}u_2(x,t)-C_{012}\frac{\partial u_1(x,t)}{\partial t} + (C_{012}+C_{023}+C_{02})\frac{\partial u_2(x,t)}{\partial t}-C_{023}\frac{\partial u_3(x,t)}{\partial t} - C_{M02}\frac{\partial^3 u_2(x,t)}{\partial x^2\partial t}; \quad (2)$$

$$-\frac{\partial i_3(x,t)}{\partial x}=g_{03}u_3(x,t)-C_{013}\frac{\partial u_1(x,t)}{\partial t}-C_{023}\frac{\partial u_2(x,t)}{\partial t} + (C_{013}+C_{023}+C_{03})\frac{\partial u_3(x,t)}{\partial t}-C_{M03}\frac{\partial^3 u_3(x,t)}{\partial x^2\partial t}. \quad (3)$$

where $i_1, i_2, i_3, u_1, u_2, u_3$ denote transformer winding currents and voltages, respectively; $g_1, g_2, g_3, C_{01}, C_{01}, C_{03}$ denote conductivity and self-capacitance of the transformer winding per unit of their length; $C_{012}, C_{013}, C_{023}$ denote mutual capacitances of the transformer winding per unit of their length; and $C_{M01}, C_{M02}, C_{M03}$ denote inter-turn capacitance winding along the axis per unit of its length.

In order to increase the adequacy and efficiency of the mathematical model of the winding wave processes, the second group of equations of the electromagnetic state of the transformer is improved by taking into account the main magnetic flux Φ_m of its

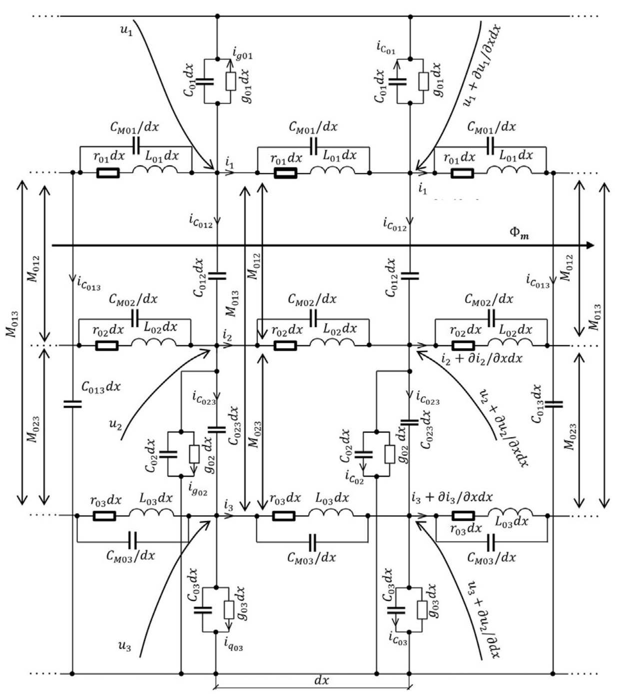

Fig. 1. Alternate circuit of a three-winding transformer per unit length along the winding axes.

own and mutual interwinding and mutual inter-winding fluxes of the winding dispersion.

The equation for the voltage drop per unit length of the winding is written on the basis of Kirchhoff's second law:

$$-\frac{\partial u_1(x,t)}{\partial x} = r_{01} i_1(x,t) + L_{01}\frac{\partial i_1(x,t)}{\partial t} + M_{012}\frac{\partial i_2(x,t)}{\partial t} + M_{013}\frac{\partial i_3(x,t)}{\partial t} + \frac{\partial \psi_{\sigma 1}(x,t)}{\partial t}; \tag{4}$$

$$-\frac{\partial u_2(x,t)}{\partial x} = r_{02} i_2(x,t) + L_{02}\frac{\partial i_2(x,t)}{\partial t} + M_{012}\frac{\partial i_1(x,t)}{\partial t} + M_{023}\frac{\partial i_3(x,t)}{\partial t} + \frac{\partial \psi_{\sigma 2}(x,t)}{\partial t}; \tag{5}$$

$$-\frac{\partial u_3(x,t)}{\partial x} = r_{03} i_2(x,t) + L_{03}\frac{\partial i_3(x,t)}{\partial t} + M_{013}\frac{\partial i_1(x,t)}{\partial t} + M_{023}\frac{\partial i_2(x,t)}{\partial t} + \frac{\partial \psi_{\sigma 3}(x,t)}{\partial t}, \tag{6}$$

where $L_{01} = L_{\mu 0} + L_{\sigma 01}$; $L_{02} = \frac{L_{\mu 0}}{k_{12}^2} + L_{\sigma 02}$; $L_{03} = \frac{L_{\mu 0}}{k_{12}^2} + L_{\sigma 03}$; $M_{012} = \frac{L_{\mu 0}}{k_{12}} + M_{\sigma 012}$; $M_{013} = \frac{L_{\mu 0}}{k_{13}} + M_{\sigma 013}$; $M_{230} = \frac{L_{\mu 0}}{k_{12}k_{13}} + M_{\sigma 230}$; $L_{\sigma 01}, L_{\sigma 02}, L_{\sigma 03}, M_{\sigma 012}, M_{\sigma 013}, M_{\sigma 023}$ – own and mutual inductances of dissipation of winding elements; $\psi_{\sigma 1}$, $\psi_{\sigma 2}$, $\psi_{\sigma 3}$ – flux leakage of length elements from their own and mutual inter-turn inductances of winding dissipation; $L_{\mu 0}$ – inductance of the magnetic system of the transformer; k_{12}, k_{13} – transformation coefficients between transformer windings.

Expressions for finding derivatives of flow leakage are as follows:

$$\frac{\partial \psi_{\sigma 1}}{\partial t} = \int_0^1 \left(M_{\sigma 1}(x,s)\frac{\partial i_1}{\partial t} + M_{\sigma 12}(x,s)\frac{\partial i_2}{\partial t} + M_{\sigma 13}(x,s)\frac{\partial i_3}{\partial t} \right) ds; \tag{7}$$

$$\frac{\partial \psi_{\sigma 2}}{\partial t} = \int_0^1 \left(M_{\sigma 2}(x,s)\frac{\partial i_2}{\partial t} + M_{\sigma 12}(x,s)\frac{\partial i_1}{\partial t} + M_{\sigma 23}(x,s)\frac{\partial i_3}{\partial t} \right) ds; \tag{8}$$

$$\frac{\partial \psi_{\sigma 3}}{\partial t} = \int_0^1 \left(M_{\sigma 3}(x,s)\frac{\partial i_3}{\partial t} + M_{\sigma 13}(x,s)\frac{\partial i_1}{\partial t} + M_{\sigma 23}(x,s)\frac{\partial i_2}{\partial t} \right) ds; \tag{9}$$

where $M_{\sigma 1}(x,s), M_{\sigma 2}(x,s), M_{\sigma 3}(x,s)$, $M_{\sigma 12}(x,s), M_{\sigma 13}(x,s)$, $M_{\sigma 23}(x,s)$– own and mutual inter-turn dissipation inductances of the primary, secondary and tertiary winding, respectively; l – winding length, x – current longitude coordinate, s – the current coordinate, by which the distance from the location x to the coordinate of any other location of the winding axis is determined.

Equations (1)–(3) are distinguished by t:

$$-\frac{\partial^2 i_1(x,t)}{\partial x \partial t} = g_{01}\frac{\partial u_1(x,t)}{\partial t} - (C_{012} + C_{013} + C_{01})\frac{\partial^2 u_1(x,t)}{\partial t^2} - C_{012}\frac{\partial^2 u_2(x,t)}{\partial t^2} - C_{013}\frac{\partial^2 u_3(x,t)}{\partial t^2} - C_{M01}\frac{\partial^4 u_1(x,t)}{\partial x^2 \partial t^2}; \tag{10}$$

$$-\frac{\partial^2 i_2(x,t)}{\partial x \partial t} = g_{02}\frac{\partial u_2(x,t)}{\partial t} - C_{012}\frac{\partial^2 u_1(x,t)}{\partial t^2} + (C_{012} + C_{023} + C_{01})\frac{\partial^2 u_2(x,t)}{\partial t^2} - C_{023}\frac{\partial^2 u_3(x,t)}{\partial t^2} - C_{M02}\frac{\partial^4 u_2(x,t)}{\partial x^2 \partial t^2}; \tag{11}$$

$$-\frac{\partial^2 i_3(x,t)}{\partial x \partial t} = g_{03}\frac{\partial u_3(x,t)}{\partial t} - C_{013}\frac{\partial^2 u_1(x,t)}{\partial t^2} - C_{023}\frac{\partial^2 u_2(x,t)}{\partial t^2} + (C_{013} + C_{023} + C_{03})\frac{\partial^2 u_3(x,t)}{\partial t^2} - C_{M03}\frac{\partial^4 u_3(x,t)}{\partial x^2 \partial t^2}. \tag{12}$$

Differentiating equations (4)–(6), we obtain:

$$-\frac{\partial^2 i_1(x,t)}{\partial x^2}=r_{01}\frac{\partial i_1(x,t)}{\partial x}+L_{01}\frac{\partial^2 i_1(x,t)}{\partial t\partial x}+M_{012}\frac{\partial^2 i_2(x,t)}{\partial t\partial x}+M_{013}\frac{\partial^2 i_3(x,t)}{\partial t\partial x}+\frac{\partial^2 \psi\sigma_1(x,t)}{\partial t\partial x}; \tag{13}$$

$$-\frac{\partial^2 u_2(x,t)}{\partial x}=r_{02}\frac{\partial i_2(x,t)}{\partial x}+L_{02}\frac{\partial^2 i_2(x,t)}{\partial t\partial x}+M_{012}\frac{\partial^2 i_1(x,t)}{\partial t\partial x}+M_{023}\frac{\partial^2 i_3(x,t)}{\partial t\partial x}+\frac{\partial^2 \psi\sigma_2(x,t)}{\partial t\partial x}; \tag{14}$$

$$-\frac{\partial^2 u_3(x,t)}{\partial x^2}=r_{30}\frac{\partial i_2(x,t)}{\partial x}+L_{30}\frac{\partial^2 i_3(x,t)}{\partial t\partial x}+M_{130}\frac{\partial^2 i_1(x,t)}{\partial t\partial x}+M_{230}\frac{\partial^2 i_2(x,t)}{\partial t\partial x}+\frac{\partial^2 \psi\sigma_3(x,t)}{\partial t\partial x}. \tag{15}$$

Substituting equations (1)–(3) and (10)–(12) into (13)–(15), we obtain:

$$-\frac{\partial^2 u_1(x,t)}{\partial x^2}=r_{01}\Bigg(-g_{01}u_1(x,t)-(C_{012}+C_{013}+C_{01})\frac{\partial u_1(x,t)}{\partial t}+C_{012}\frac{\partial u_2(x,t)}{\partial t}+C_{013}\frac{\partial u_3(x,t)}{\partial t}+C_{M01}\frac{\partial^3 u_1(x,t)}{\partial x^2\partial t}\Bigg)$$
$$+L_{01}\Bigg(-g_{01}\frac{\partial u_1(x,t)}{\partial t}-(C_{012}+C_{013}+C_{01})\frac{\partial^2 u_1(x,t)}{\partial t^2}+C_{012}\frac{\partial^2 u_2(x,t)}{\partial t^2}+C_{013}\frac{\partial^2 u_3(x,t)}{\partial t^2}+C_{M01}\frac{\partial^4 u_1(x,t)}{\partial x^2\partial t^2}\Bigg)+$$
$$+M_{012}\Bigg(-g_{02}\frac{\partial u_2(x,t)}{\partial t}+C_{012}\frac{\partial^2 u_1(x,t)}{\partial t^2}-(C_{012}+C_{023}+)\frac{\partial^2 u_2(x,t)}{\partial t^2}+C_{023}\frac{\partial^2 u_3(x,t)}{\partial t^2}\Bigg)+C_{M02}\frac{\partial^4 u_2(x,t)}{\partial x^2 t^2}$$
$$+M_{013}\Bigg(-g_{03}\frac{\partial u_3(x,t)}{\partial t}+C_{013}\frac{\partial^2 u_1(x,t)}{\partial t^2}+C_{023}\frac{\partial^2 u_2(x,t)}{\partial t^2}\Bigg)-(C_{013}+C_{023}+C_{03})\frac{\partial^2 u_3(x,t)}{\partial t^2}+C_{M03}\frac{\partial^4 u_3(x,t)}{\partial x^2\partial t^2}+\frac{\partial^2 \psi_{\sigma 1}(x,t)}{\partial t}; \tag{16}$$

$$-\frac{\partial^2 u_2(x,t)}{\partial x}=r_{02}\Bigg(-g_{02}u_2(x,t)+C_{012}\frac{\partial u_1(x,t)}{\partial t}-(C_{012}+C_{023}+C_{02})\frac{\partial u_2(x,t)}{\partial t}+C_{023}\frac{\partial u_3(x,t)}{\partial t}+C_{M02}\frac{\partial^3 u_2(x,t)}{\partial x^2\partial t}\Bigg)$$
$$+L_{02}\Bigg(-g_{02}\frac{\partial u_2(x,t)}{\partial t}+C_{012}\frac{\partial^2 u_1(x,t)}{\partial t^2}-(C_{012}+C_{023}+C_{02})\frac{\partial^2 u_2(x,t)}{\partial t^2}+C_{023}\frac{\partial^2 u_3(x,t)}{\partial t^2}+C_{M02}\frac{\partial^4 u_2(x,t)}{\partial x^2\partial t^2}\Bigg)$$
$$+M_{012}\Bigg(-g_{01}\frac{\partial u_1(x,t)}{\partial t}-(C_{012}+C_{013}+C_{01})\frac{\partial^2 u_1(x,t)}{\partial t^2}+C_{012}\frac{\partial^2 u_2(x,t)}{\partial t^2}+C_{013}\frac{\partial^2 u_3(x,t)}{\partial t^2}+C_{M01}\frac{\partial^4 u_1(x,t)}{\partial x^2\partial t^2}\Bigg)$$
$$+M_{023}\Bigg(-g_{03}\frac{\partial u_3(x,t)}{\partial t}+C_{013}\frac{\partial^2 u_1(x,t)}{\partial t^2}+C_{023}\frac{\partial^2 u_2(x,t)}{\partial t^2}-(C_{013}+C_{023}+C_{03})\frac{\partial^2 u_3(x,t)}{\partial t^2}+C_{M03}\frac{\partial^4 u_3(x,t)}{\partial x^2\partial t^2}\Bigg)$$
$$+\frac{\partial^2 \psi_{\sigma 2}(x,t)}{\partial t\partial x}; \tag{17}$$

$$-\frac{\partial^2 u_3(x,t)}{\partial x^2}=r_{03}\Bigg(-g_{03}u_3(x,t)+C_{013}\frac{\partial u_1(x,t)}{\partial t}+C_{023}\frac{\partial u_2(x,t)}{\partial t}-(C_{013}+C_{023}+C_{03})\frac{\partial u_3(x,t)}{\partial t}+C_{M03}\frac{\partial^3 u_3(x,t)}{\partial x^2\partial t}\Bigg)$$
$$+L_{03}\Bigg(-g_{03}\frac{\partial u_3(x,t)}{\partial t}+C_{013}\frac{\partial^2 u_1(x,t)}{\partial t^2}+C_{023}\frac{\partial^2 u_2(x,t)}{\partial t^2}-(C_{013}+C_{023}+C_{03})\frac{\partial^2 u_3(x,t)}{\partial t^2}+C_{M03}\frac{\partial^4 u_3(x,t)}{\partial x^2\partial t^2}\Bigg)$$
$$+M_{013}\Bigg(-g_{01}\frac{\partial u_1(x,t)}{\partial t}-(C_{012}+C_{013}+C_{01})\frac{\partial^2 u_1(x,t)}{\partial t^2}+C_{012}\frac{\partial^2 u_2(x,t)}{\partial t^2}+C_{013}\frac{\partial^2 u_3(x,t)}{\partial t^2}+C_{M01}\frac{\partial^4 u_1(x,t)}{\partial x^2\partial t^2}\Bigg)$$
$$+M_{023}\Bigg(-g_{02}\frac{\partial u_2(x,t)}{\partial t}+C_{012}\frac{\partial^2 u_1(x,t)}{\partial t^2}-(C_{012}+C_{023})\frac{\partial^2 u_2(x,t)}{\partial t^2}+C_{023}\frac{\partial^2 u_3(x,t)}{\partial t^2}+C_{M02}\frac{\partial^4 u_2(x,t)}{\partial x^2\partial t^2}\Bigg)$$
$$+\frac{\partial^2 \psi_{\sigma 3}(x,t)}{\partial t\partial x}. \tag{18}$$

Based on physical considerations, the values of the derivatives of the flux linkages in equations (16)–(18) are neglected due to their smallness. Having grouped equation (16), we introduce the notation:

$$a_{11} = r_{01}g_{01}; b_{11} = r_{01}(C_{01} + C_{012} + C_{013}) + L_{01}g_{01};$$

$$c_{11} = -r_{01}C_{012} + M_{012}g_{02}; d_{11} = -r_{01}C_{013} + M_{013}g_{03};$$

$$e_{11} = L_{01}(C_{01} + C_{012} + C_{013}) - M_{012}C_{012} - M_{013}C_{013};$$

$$f_{11} = -L_{01}C_{012} + M_{012}(C_{02} + C_{012} + C_{023}) - M_{013}C_{023};$$

$$g_{11} = -L_{01}C_{013} - M_{012}C_{023} + M_{013}(C_{03} + C_{013} + C_{023});$$

$$h_{11} = -r_{01}C_{M01}; i_{11} = -L_{01}C_{M01}; j_{11} = -M_{012}C_{M02}; k_{11} = -M_{013}C_{M03}.$$

Equation (16) takes the form:

$$\frac{\partial^2 u_1(x,t)}{\partial x^2} = a_{11}u_1(x,t) + b_{11}\frac{\partial u_1(x,t)}{\partial t} + c_{11}\frac{\partial u_2(x,t)}{\partial t} + d_{11}\frac{\partial u_3(x,t)}{\partial t} + e_{11}\frac{\partial^2 u_1(x,t)}{\partial t^2} + f_{11}\frac{\partial^2 u_2(x,t)}{\partial t^2} + g_{11}\frac{\partial^2 u_3(x,t)}{\partial t^2} + h_{11}\frac{\partial^3 u_1(x,t)}{\partial^2 xt^2} + i_{11}\frac{\partial^4 u_2(x,t)}{\partial x^2 \partial t^2} + j_{11}\frac{\partial^4 u_2(x,t)}{\partial x^2 \partial t^2} + k_{11}\frac{\partial^4 u_3(x,t)}{\partial x^2 \partial t^2} \tag{19}$$

Having grouped equation (17), we introduce the notation:

$$a_{21} = r_{02}g_{02}; b_{21} = -r_{02}C_{01} + M_{012}g_{01}; c_{21} = r_{02}(C_{02} + C_{012} + C_{023}) + L_{02}g_{02}; d_{21} = -r_{02}C_{023} + M_{023}g_{02};$$

$$e_{11} = -L_{02}C_{012} + M_{012}(C_{01} + C_{012} + C_{013}) - M_{023}C_{013};$$

$$f_{21} = L_{02}(C_{02} + C_{012} + C_{023}) - M_{012}C_{012} - M_{023}C_{023}; g_{21} = -L_{02}C_{023} - M_{012}C_{013} + M_{023}(C_{03} + C_{013} + C_{023});$$

$$h_{21} = -r_{02}C_{M02}; i_{21} = -L_{02}C_{M02}; j_{11} = -M_{012}C_{M01}; k_{21} = -M_{012}C_{M03}.$$

Equation (17) takes the form:

$$\frac{\partial^2 u_2(x,t)}{\partial x^2} = a_{21}u_2(x,t) + b_{21}\frac{\partial u_1(x,t)}{\partial t} + c_{21}\frac{\partial u_2(x,t)}{\partial t} + d_{21}\frac{\partial u_3(x,t)}{\partial t} + e_{21}\frac{\partial^2 u_1(x,t)}{\partial t^2} + f_{21}\frac{\partial^2 u_2(x,t)}{\partial t^2} + g_{21}\frac{\partial^2 u_3(x,t)}{\partial t^2} + h_{21} = \frac{\partial^3 u_2(x,t)}{\partial^2 xt^2} + i_{21}\frac{\partial^4 u_2(x,t)}{\partial x^2 xt^2} + j_{21}\frac{\partial^4 u_1(x,t)}{\partial x^2 xt^2} + k_{21}\frac{\partial^4 u_3(x,t)}{\partial x^2 xt^2} \tag{20}$$

Having grouped equation (18), we introduce the notation:

$$a_{31} = +r_{03}g_{03}; b_{31} = -r_{03}C_{013} + M_{013}g_{01}; c_{31} = -r_{03}C_{023} + M_{023}g_{02}; d_{31} = r_{03}(C_{03} + C_{013} + C_{023}) + L_{03}g_{03};$$

$$e_{31} = -L_{03}C_{013} + M_{013}(C_{01} + C_{012} + C_{013}) - M_{023}C_{013};$$

$$f_{31} = -L_{03}C_{023} - M_{013}C_{012} + M_{023}(C_{02} + C_{012} + C_{023});$$

$$g_{31} = +L_{03}(C_{03} + C_{013} + C_{023}) - M_{013}C_{013} - M_{023}C_{023};$$

$$h_{31} = -r_{03}C_{M03}; i_{31} = -L_{03}C_{M03}; j_{31} = -M_{013}C_{M01}; k_{31} = -M_{013}C_{M02}$$

Equation (18) takes the form:

$$\frac{\partial^2 u_3(x,t)}{\partial x^2} = a_{31}u_3(x,t) + b_{31}\frac{\partial u_1(x,t)}{\partial t} + C_{31}\frac{\partial u_2(x,t)}{\partial t} + d_{31}\frac{\partial u_3(x,t)}{\partial t} + e_{31}\frac{\partial^2 u_1(x,t)}{\partial t^2} + f_{31}\frac{\partial^2 u_2(x,t)}{\partial t^2} + g_{31}\frac{\partial^2 u_3(x,t)}{\partial t^2} + h_{31}\frac{\partial^3 u_3(x,t)}{\partial^2 xt^2} + i_{31}\frac{\partial^4 u_3(x,t)}{\partial x^2 \partial t^2} + j_{31}\frac{\partial^4 u_1(x,t)}{\partial x^2 \partial t^2} + k_{31}\frac{\partial^4 u_2(x,t)}{\partial x^2 \partial t^2} \tag{21}$$

The initial conditions have the form:

$$u_1(x,t)|_{t=0} = u_{10}(x) = U_{m1} - k_1 x; \left.\frac{\partial u_1(x,t)}{\partial}\right|_{t=0} = u_{11}(x) = 0; \tag{22}$$

$$u_2(x,t)|_{t=0} = u_{20}(x) = U_{m2} - k_2 x; \left.\frac{\partial u_2(x,t)}{\partial}\right|_{t=0} = u_{21}(x) = 0; \tag{23}$$

$$u_3(x,t)|_{t=0} = u_{30}(x) = U_{m3} - k_3 x; \left.\frac{\partial u_3(x,t)}{\partial}\right|_{t=0} = u_{31}(x) = 0; x \in (0;l) \tag{24}$$

where U_{m1}, U_{m2}, U_{m3} denote amplitude values of the steady-state voltage of the primary, secondary, and tertiary windings, respectively; and k_1, k_2, k_3 denote coefficients of the rate of change of voltages along the winding for the instant of time $t = 0$, respectively.

Boundary conditions:

$$u_{ilmn}(x,t)|_{x=10} = f_{10x}(t_l) = e_{1l}(t); u(x,t)|_{=} = f(t) = 0; \quad (25)$$

$$u_2(x,t)|_{x=0} = f_{20}(t) = 0; u_2(x,t)|_{x=1} = f_{21}(t) = 0; \quad (26)$$

$$u_3(x,t)|_{x=0} = f_{30}(t) = 0; u_3(x,t)|_{x=1} = f_{31}(t) = 0; t>0. \quad (27)$$

Consistency of conditions:

$$\begin{aligned}
&u_{10}(x)|_{t=0} = u_1(x,t)|_{t=0} = f_{10}(t)|_{t=0}; u_{10}(x)|_{x=l} = u_1(x,t)|_{t=0} = f_{1l}(t)|_{t=0};\\
&u_{20}(x)|_{t=0} = u_2(x,t)|_{t=0} = f_{20}(t)|_{t=0}; u_{20}(x)|_{x=l} = u_2(x,t)|_{t=0} = f_{2l}(t)|_{t=0};\\
&u_{30}(x)|_{t=0} = u_3(x,t)|_{t=0} = f_{30}(t)|_{t=0}; u_{30}(x)|_{x=l} = u_3(x,t)|_{t=0} = f_{3l}(t)|_{t=0};\\
&f_{10}(0) = u_{10}(0); f_{1l}(0) = u_{1l}(l); f_{20}(0) = u_{20}(0); f_{2l}(0) = u_{2l}(l);\\
&f_{30}(0) = u_{30}(0); f_{3l}(0) = u_{3l}(l);\\
&\left.\frac{\partial f_{10}(t)}{\partial t}\right|_{t=0} = u_{10}(0)|_{t=0}; \left.\frac{\partial f_{1l}(t)}{\partial t}\right|_{t=0} = u_{11}(t); \left.\frac{\partial f_{20}(t)}{\partial t}\right|_{t=0} = u_{20}(0)|_{t=0}; \left.\frac{\partial f_{2l}(t)}{\partial t}\right|_{t=0} = u_{21}(t);\\
&\left.\frac{\partial f_{30}(t)}{\partial t}\right|_{t=0} = u_{30}(0)|_{t=0}; \left.\frac{\partial f_{3l}(t)}{\partial t}\right|_{t=0} = u_{31}(t);
\end{aligned} \quad (28)$$

In equation (19) [30], let's make the following substitution:

$$u_1(x,t) = V_1(x,t) + A_1(t) + xB_1(t) \quad (29)$$

We are looking for a function $A_1(t)$,$B_1(t)$ so that replacing $V_1(x,t)$ gives consistent conditions (28), i.e. $V_1(x,t)|_{x=0} = 0$, $V_1(x,t)|_{x=1} = 0$.

Then we write (29) for $x = 0$:

$$u_1(x,t)|_{x=0} = V_1(x,t)|_{x=0} + A_1(t) = f_{10}(t). \quad (30)$$

From equation (30) we get:

$$A_{ilmn}(t) = f_{10}(t) = e(t). \quad (31)$$

For $x = l$:

$$u_1(x,t)|_{x=1} = V_1(x,t)|_{x=1} + A_1(t) + lB_1(t) = f_{11}(t). \quad (32)$$

From equation (32) we get:

$$B_{ilmn}(t) = \frac{1}{l}(f_{1l}(t) - A_1(t)) = -\frac{1}{l}e(t). \quad (33)$$

There are conditions instead of initial conditions (22):

$$u_1(x,t)|_{t=0} = V_1(x,t)|_{t=0} + A_1(t)|_{t=0} + xB_1(t)|_{t=0} \equiv u_{10}(x);$$

$$\left.\frac{\partial u_1(x,t)}{\partial t}\right|_{t=0} = \left.\frac{\partial V_1(x,t)}{\partial t}\right|_{t=0} + \left.\frac{dA_1(t)}{dt}\right|_{t=0} + x\left.\frac{dB_1(t)}{dt}\right|_{t=0} \equiv u_{11}(x). \quad (34)$$

From equation (34) we get:

$$V_1(x,t)|_{t=0} = u_{10}(x) - A_1(t)|_{t=0} - xB_1(t)|_{t=0} \equiv V_{10}(x);$$

$$\left.\frac{\partial V_1(x,t)}{\partial t}\right|_{t=0} = u_{11}(x) - \left.\frac{dA_1(t)}{dt}\right|_{t=0} - x\left.\frac{dB_1(t)}{dt}\right|_{t=0} = -\left.\frac{dA_1(t)}{dt}\right|_{t=0} - x\left.\frac{dB_1(t)}{dt}\right|_{t=0} \equiv V_{11}(x). \quad (35)$$

Let's make the following substitution in equation (20):

$$u_2(x,t) = V_2(x,t) + A_2(t) + xB_2(t). \quad (36)$$

We are looking for a function $A_2(t)$,$B_2(t)$ so that expression (36) satisfies $V_2(x,t)$ for conditions (28), namely $V_2(x,t)|_{x=0} = 0$, $V_2(x,t)|_{x=1} = 0$.

Then equation (36) for $x = 0$ will take the form:

$$u_2(x,t)|_{x=0} = V_2(x,t)|_{t=0} + A_2(t) = f_{20}(t). \quad (37)$$

From equation (37) we get:

$$A_2(t) = f_{20}(t) = 0. \quad (38)$$

For $x = l$:

$$u_2(x,t)|_{x=1} = V_2(x,t)|_{x=1} + A_2(t) + lB_2(t) = f_{21}(t); \quad (39)$$

From equation (39) we get:

$$B_2(t)=\frac{1}{l}f_{21}(t). \tag{40}$$

Conditions arise instead of initial conditions expressed in equation (23):

$$u_2(x,t)|_{t=0}=V_2(x,t)|_{t=0}+A_2(t)|_{t=0}+xB_2(t)|_{t=0}\equiv u_{20}(x);$$

$$\left.\frac{\partial u_2(x,t)}{\partial t}\right|_{t=0}=\left.\frac{\partial V_2(x,t)}{\partial t}\right|_{t=0}+\left.\frac{dA_2(t)}{dt}\right|_{t=0}+x\left.\frac{dB_2(t)}{dt}\right|_{t=0}\equiv u_{21}(x). \tag{41}$$

From equation (41) we get:

$$V_2(x,t)|_{t=0}=u_{20}(x)-A_2(t)|_{t=0}-xB_2(t)|_{t=0}\equiv V_{20}(x).$$

$$\left.\frac{\partial V_2(x,t)}{\partial t}\right|_{t=0}=u_{2l}(x)-\left.\frac{dA_2(t)}{dt}\right|_{t=0}-x\left.\frac{dB_2(t)}{dt}\right|_{t=0}=-\left.\frac{dA_2(t)}{dt}\right|_{t=0}-x\left.\frac{dB_2(t)}{dt}\right|_{t=0}\equiv V_{21}(x). \tag{42}$$

Let's make the following substitution in equation (21):

$$u_3(x,t)=V_3(x,t)+A_3(t)+xB_3(t) \tag{43}$$

We are looking for a function $A_3(t)$, $B_3(t)$ so that the replacement of (43) gives for $V_3(x,t)$ consistent conditions (28), i.e. $V_3(x,t)|_{x=0}=0$, $V_3(x,t)|_{x=1}=0$.
Then we write (43) for $x=0$ so

$$u_3(x,t)|_{x=0}=V_3(x,t)|_{x=0}+A_3(t)=f_{30}(t). \tag{44}$$

From (44) we get

$$A_3(t)=f_{30}(t)=0 \tag{45}$$

For $x=l$.

$$u_3(x,t)|_{x=1}=V_3(x,t)|_{x=1}+A_3(t)+lB_3(t)=f_{31}(t). \tag{46}$$

From (46) we get

$$B_3(t)=\frac{1}{l}f_{31}(t) \tag{47}$$

Instead of the initial conditions (24), conditions arise

$$u_3(x,t)|_{t=0}=V_3(x,t)|_{t=0}+A_3(t)|_{t=0}+xB_3(t)|_{t=0}\equiv u_{30}(x);$$

$$\left.\frac{\partial u_3(x,t)}{\partial t}\right|_{t=0}=\left.\frac{\partial V_3(x,t)}{\partial t}\right|_{t=0}+\left.\frac{dA_3(t)}{dt}\right|_{t=0}+x\left.\frac{dB_3(t)}{dt}\right|_{t=0}\equiv u_{31}(x). \tag{48}$$

From (48) we get

$$V_3(x,t)|_{t=0}=u_{30}(x)-A_3(t)|_{t=0}-xB_3(t)|_{t=0}\equiv V_{30}(x);$$

$$\left.\frac{\partial V_3(x,t)}{\partial t}\right|_{t=0}=u_{31}(x)-\left.\frac{dA_3(t)}{dt}\right|_{t=0}-x\left.\frac{dB_3(t)}{dt}\right|_{t=0}=-\left.\frac{dA_3(t)}{dt}\right|_{t=0}-x\left.\frac{dB_3(t)}{dt}\right|_{t=0}\equiv V_{31}(x). \tag{49}$$

We obtain the equation for the variable $V_1(x,t)$ by substituting (29) (36) and (43) into (19), i.e.

$$\begin{aligned}\frac{\partial^2 V_1(x,t)}{\partial x^2}=&a_{11}(V_1(x,t)+A_1(t)+xB_1(t))+b_{11}\left(\frac{\partial V_1(x,t)}{dt}+\frac{dA_1(t)}{dt}+x\frac{dB_1(t)}{dt}\right)+c_{11}\left(\frac{\partial V_2(x,t)}{dt}+\frac{dA_2(t)}{dt}+x\frac{dB_2(t)}{dt}\right)+d_{11}\left(\frac{\partial V_3(x,t)}{dt}+\frac{dA_3(t)}{dt}+x\frac{dB_3(t)}{dt}\right)\\&+e_{11}\left(\frac{\partial^2 V_1(x,t)}{dt^2}+\frac{d^2A_1(t)}{dt^2}+x\frac{d^2B_2(t)}{dt^2}\right)+f_{11}\left(\frac{\partial^2 V_2(x,t)}{dt^2}+\frac{d^2A_2(t)}{dt^2}+x\frac{d^2B_2(t)}{dt^2}\right)\\&+g_{11}\left(\frac{\partial^2 V_3(x,t)}{dt^2}+\frac{d^2A_3(t)}{dt^2}+x\frac{d^2B_3(t)}{dt^2}\right)+h_{11}\frac{\partial^3 V_1(x,t)}{\partial x^2\partial t}+i_{11}\frac{\partial^4 V_1(x,t)}{\partial x^2\partial t^2}+j_{11}\frac{\partial^4 V_2(x,t)}{\partial x^2\partial t^2}+k_{11}\frac{\partial^4 V_3(x,t)}{\partial x^2\partial t^2}\end{aligned} \tag{50}$$

We obtain the equation for the variable $V_2(x,t)$ by substituting (29) (36) and (43) into (20), i.e.

$$\begin{aligned}\frac{\partial^2 V_2(x,t)}{\partial x^2}=&a_{21}(V_2(x,t)+A_2(t)+xB_2(t))+b_{21}\left(\frac{\partial V_1(x,t)}{dt}+\frac{dA_1(t)}{dt}+x\frac{dB_1(t)}{dt}\right)+c_{21}\left(\frac{\partial V_2(x,t)}{dt}+\frac{dA_2(t)}{dt}+x\frac{dB_2(t)}{dt}\right)+d_{21}\left(\frac{\partial V_3(x,t)}{dt}+\frac{dA_3(t)}{dt}+x\frac{dB_3(t)}{dt}\right)\\&+e_{21}\left(\frac{\partial^2 V_1(x,t)}{dt^2}+\frac{d^2A_1(t)}{dt^2}+x\frac{d^2B_1(t)}{dt^2}\right)+f_{21}\left(\frac{\partial^2 V_2(x,t)}{dt^2}+\frac{d^2A_2(t)}{dt^2}+x\frac{d^2B_2(t)}{dt^2}\right)\\&+g_{21}\left(\frac{\partial^2 V_3(x,t)}{dt^2}+\frac{d^2A_3(t)}{dt^2}+x\frac{d^2B_3(t)}{dt^2}\right)+h_{21}\frac{\partial^3 V_2(x,t)}{\partial x^2\partial t}+i_{21}\frac{\partial^4 V_2(x,t)}{\partial x^2\partial t^2}+j_{21}\frac{\partial^4 V_1(x,t)}{\partial x^2\partial t^2}+k_{21}\frac{\partial^4 V_3(x,t)}{\partial x^2\partial t^2}\end{aligned} \tag{51}$$

We obtain the equation for the variable $V_3(x,t)$ by substituting (29) (36) and (43) into (21), i.e.

$$\frac{\partial^2 V_3(x,t)}{\partial x^2}=a_{31}(V_3(x,t)+A_3(t)+xB_3(t))+b_{31}\left(\frac{\partial V_1(x,t)}{dt}+\frac{dA_1(t)}{dt}+x\frac{dB_1(t)}{dt}\right)+c_{31}\left(\frac{\partial V_2(x,t)}{dt}+\frac{dA_2(t)}{dt}+x\frac{dB_2(t)}{dt}\right)+d_{31}\left(\frac{\partial V_3(x,t)}{dt}+\frac{dA_3(t)}{dt}+x\frac{dB_3(t)}{dt}\right)+e_{31}\left(\frac{\partial^2 V_1(x,t)}{dt^2}+\frac{d^2A_1(t)}{dt^2}+x\frac{d^2B_1(t)}{dt^2}\right)+f_{31}\left(\frac{\partial^2 V_2(x,t)}{dt^2}+\frac{d^2A_2(t)}{dt^2}+x\frac{d^2B_2(t)}{dt^2}\right)+g_{31}\left(\frac{\partial^2 V_3(x,t)}{dt^2}+\frac{d^2A_3(t)}{dt^2}+x\frac{d^2B_3(t)}{dt^2}\right)+h_{31}\frac{\partial^3 V_3(x,t)}{\partial x^2\partial t}+i_{21}\frac{\partial^4 V_3(x,t)}{\partial x^2\partial t^2}+j_{31}\frac{\partial^4 V_1(x,t)}{\partial x^2\partial t^2}+k_{31}\frac{\partial^4 V_2(x,t)}{\partial x^2\partial t^2} \tag{52}$$

In equations (50)–(52), we leave the known parts on the right, i.e.:

$$F_1(x,t)=a_{11}(A_1(t)+xB_1(t))+b_{11}\left(\frac{dA_1(t)}{dt}+x\frac{dB_1(t)}{dt}\right)+c_{11}\left(\frac{dA_2(t)}{dt}+x\frac{dB_2(t)}{dt}\right)+d_{11}\left(\frac{dA_3(t)}{dt}+x\frac{dB_3(t)}{dt}\right)+e_{11}\left(\frac{d^2A_1(t)}{dt^2}+x\frac{d^2B_1(t)}{dt^2}\right)+f_{11}\left(\frac{d^2A_2(t)}{dt}+x\frac{d^2B_2(t)}{dt}\right)+g_{11}\left(\frac{d^2A_3(t)}{dt}+x\frac{d^2B_3(t)}{dt}\right); \tag{53}$$

$$F_2(x,t)=a_{21}(A_2(t)+xB_2(t))+b_{21}\left(\frac{dA_1(t)}{dt}+x\frac{dB_1(t)}{dt}\right)+c_{21}\left(\frac{dA_2(t)}{dt}+x\frac{dB_2(t)}{dt}\right)+d_{21}\left(\frac{dA_3(t)}{dt}+x\frac{dB_3(t)}{dt}\right)+e_{21}\left(\frac{d^2A_1(t)}{dt^2}+x\frac{d^2B_1(t)}{dt^2}\right)+f_{21}\left(\frac{d^2A_2(t)}{dt}+x\frac{d^2B_2(t)}{dt}\right)+g_{21}\left(\frac{d^2A_3(t)}{dt^2}+x\frac{d^2B_3(t)}{dt^2}\right); \tag{54}$$

$$F_3(x,t)=a_{31}(A_3(t)+xB_3(t))+b_{31}\left(\frac{dA_1(t)}{dt}+x\frac{dB_1(t)}{dt}\right)+c_{31}\left(\frac{dA_2(t)}{dt}+x\frac{dB_2(t)}{dt}\right)+d_{31}\left(\frac{dA_3(t)}{dt}+x\frac{dB_3(t)}{dt}\right)+e_{31}\left(\frac{d^2A_1(t)}{dt^2}+x\frac{d^2B_1(t)}{dt^2}\right)+f_{31}\left(\frac{d^2A_2(t)}{dt^2}+x\frac{d^2B_2(t)}{dt^2}\right)+g_{31}\left(\frac{d^2A_3(t)}{dt^2}+x\frac{d^2B_3(t)}{dt^2}\right); \tag{55}$$

Let's write the equation for the variables $V_1(x,t)$, $V_2(x,t)$ and $V_3(x,t)$ as follows

$$-\frac{\partial^2 V_1(x,t)}{\partial x^2}+a_{11}V_1(x,t)+b_{11}\frac{\partial V_1(x,t)}{\partial t}+c_{11}\frac{\partial V_2(x,t)}{\partial t}+d_{11}\frac{\partial V_3(x,t)}{\partial t}+e_{11}\frac{\partial^2 V_1(x,t)}{\partial t^2}+f_{11}\frac{\partial^2 V_2(x,t)}{\partial t^2}+g_{11}\frac{\partial^2 V_3(x,t)}{\partial t^2}+h_{11}\frac{\partial^3 V_1(x,t)}{\partial x^2\partial t}+i_{11}\frac{\partial^4 V_1(x,t)}{\partial x^2\partial t^2}+i_{11}\frac{\partial^4 V_2(x,t)}{\partial x^2\partial t^2}+k_{11}\frac{\partial^4 V_3(x,t)}{\partial x^2\partial t^2}=F_1(x,t); \tag{56}$$

$$-\frac{\partial^2 V_2(x,t)}{\partial x^2}+a_{21}V_2(x,t)+b_{21}\frac{\partial V_1(x,t)}{\partial t}+c_{21}\frac{\partial V_2(x,t)}{\partial t}+d_{21}\frac{\partial V_3(x,t)}{\partial t}+e_{21}\frac{\partial^2 V_1(x,t)}{\partial t^2}+f_{21}\frac{\partial^2 V_2(x,t)}{\partial t^2}+g_{21}\frac{\partial^2 V_3(x,t)}{\partial t^2}+h_{21}\frac{\partial^3 V_2(x,t)}{\partial^2 x\partial t}+i_{21}\frac{\partial^4 V_2(x,t)}{\partial x^2\partial t^2}+j_{21}\frac{\partial^4 V_1(x,t)}{\partial x^2\partial t^2}+k_{11}\frac{\partial^4 V_3(x,t)}{\partial x^2\partial t^2}=F_2(x,t); \tag{57}$$

$$-\frac{\partial^2 V_3(x,t)}{\partial x^2}+a_{31}V_3(x,t)+b_{31}\frac{\partial V_1(x,t)}{\partial t}+c_{31}\frac{\partial V_2(x,t)}{\partial t}+d_{31}\frac{\partial V_3(x,t)}{\partial t}+e_{31}\frac{\partial^2 V_1(x,t)}{\partial t^2}+f_{31}\frac{\partial^2 V_2(x,t)}{\partial t^2}+g_{31}\frac{\partial^2 V_3(x,t)}{\partial t^2}+h_{31}\frac{\partial^3 V_3(x,t)}{\partial^2 x\partial t}+i_{31}\frac{\partial^4 V_3(x,t)}{\partial x^2\partial t^2}+j_{31}\frac{\partial^4 V_1(x,t)}{\partial x^2\partial t^2}+k_{31}\frac{\partial^4 V_2(x,t)}{\partial x^2\partial t^2}=F_3(x,t) \tag{58}$$

Considering $V_1(x,t)|_{x=0}=0$, $V_1(x,t)|_{x=1}=0$, $V_2(x,t)|_{x=0}=0$ $V_2(x,t)|_{x=1}=0$ i $V_3(x,t)|_{x=0}=0$, $V_3(x,t)|_{x=1}=0$ we are looking for solutions for $V_1(x,t)$, $V_2(x,t)$ and $V_3(x,t)$ as follows:

$$V_1(x,t)=\sum_{k=1}^{m} C_k(t)sin\left(\frac{\pi kx}{l}\right),0<x<1; \quad (59)$$

$$V_2(x,t)=\sum_{k=1}^{m} C_k(t)sin\left(\frac{\pi kx}{l}\right),0<x<1; \quad (60)$$

$$V_3(x,t)=\sum_{k=1}^{m} C_k(t)sin\left(\frac{\pi kx}{l}\right),0<x<1. \quad (61)$$

Let's find the derivatives of equations (59)–(61):

$$\frac{\partial V_1(x,t)}{\partial t}=\sum_{k=1}^{m}\frac{dC_k(t)}{dt}\sin\left(\frac{\pi kx}{l}\right); \quad (62)$$

$$\frac{\partial V_2(x,t)}{\partial t}=\sum_{k=1}^{m}\frac{dD_k(t)}{dt}\sin\left(\frac{\pi kx}{l}\right); \quad (63)$$

$$\frac{\partial V_1(x,t)}{\partial t}=\sum_{k=1}^{m}\frac{dH_k(t)}{dt}\sin\left(\frac{\pi kx}{l}\right). \quad (64)$$

For $t=0$ from equations (59)–(64) we get:

$$V_1(x,t)|_{t=0}=V_{10}(x)=\sum_{k=1}^{m}C_k(t)|_{t=0}\sin(\pi kx/l);$$

$$\partial V_1(x,t)/\partial t|_{t=0}=V_{11}(x)=\sum_{k=1}^{m}dC(t)/dt|_{t=0}\sin(\pi kx/l); \quad (65)$$

$$V_2(x,t)|_{t=0}=V_{20}(x)=\sum_{k=1}^{m}D_k(t)|_{t=0}\sin(\pi kx/l);$$

$$\partial V_2(x,t)/\partial t|_{t=0}=V_{21}(x)=\sum_{k=1}^{m}dD(t)/dt|_{t=0}\sin(\pi kx/l); \quad (66)$$

$$V_3(x,t)|_{t=0}=V_{30}(x)=\sum_{k=1}^{m}H_k(t)|_{t=0}\sin(\pi kx/l);$$

$$\partial V_3(x,t)/\partial t|_{t=0}=V_{31}(x)=\sum_{k=1}^{m}dH(t)/dt|_{t=0}\sin(\pi kx/l). \quad (67)$$

Let us expand the initial conditions equations (65) and (66) into Fourier series:

$$V_{10}(x)=\sum_{k=1}^{m}\alpha_{k1}\sin\left(\frac{\pi kx}{l}\right);V_{11}(x)=\sum_{k=1}^{m}\beta_{k1}\sin(\pi kx/l); \quad (68)$$

$$V_{20}(x)=\sum_{k=1}^{m}\alpha_{k2}\sin\left(\frac{\pi kx}{l}\right);V_{21}(x)=\sum_{k=1}^{m}\beta_{k2}\sin(\pi kx/l); \quad (69)$$

$$V_{30}(x)=\sum_{k=1}^{m}\alpha_{k3}\sin\left(\frac{\pi kx}{l}\right);V_{31}(x)=\sum_{k=1}^{m}\beta_{k3}\sin(\pi kx/l). \quad (70)$$

Equation (68)–(70) show that:

$$\alpha_{k1}=1/l\int_{x=0}^{1}V_{10}(x)\sin(\pi kx/l)dx;\quad \beta_{k1}=1/l\int_{x=0}^{1}V_{11}(x)\sin(\pi kx/l)dx; \quad (71)$$

$$\alpha_{k2}=1/l\int_{x=0}^{1}V_{20}(x)\sin(\pi kx/l)dx;\quad \beta_{k2}=1/l\int_{x=0}^{1}V_{21}(x)\sin(\pi kx/l)dx; \quad (72)$$

$$\alpha_{k3}=1\setminus/l\int_{x=0}^{1}V_{30}(x)\sin(\pi kx/l)dx;\quad \beta_{k3}=1/l\int_{x=0}^{1}V_{31}(x)\sin(\pi kx/l)dx. \quad (73)$$

Then from equations (65)–(67) we get:

$$C_k(t)|_{t=0}=\alpha_{k1}\text{and } dC(t)/dt|_{t=0}=\beta_{k1}; \quad (74)$$

$$D_k(t)|_{t=0}=\alpha_{k2}\text{ and } dD(t)/dt|_{t=0}=\beta_{k2}; \quad (75)$$

$$H_k(t)|_{t=0}=\alpha_{k3}\text{ and } dH(t)/dt|_{t=0}=\beta_{k3}. \quad (76)$$

The differential equations for $C_k(t)$, $D_k(t)$ and $H_k(t)$ are found by expanding $F_1(x,t)$ $F_2(x,t)$ and $F_3(x,t)$ into Fourier series, i.e.:

$$F_1(x,t)=\sum_{k=1}^{m}\gamma_k(t)\sin\left(\frac{\pi kx}{l}\right); \quad (77)$$

$$F_2(x,t)=\sum_{k=1}^{m}\mu_k(t)\sin\left(\frac{\pi kx}{l}\right); \quad (78)$$

$$F_3(x,t)=\sum_{k=1}^{m}\sigma_k(t)\sin\left(\frac{\pi kx}{l}\right). \quad (79)$$

Using equations (59)–(61), let's transform the left part of equation (56) as follows:

$$e_{11}\frac{\partial^2 V_1(x,t)}{\partial t^2}+f_{11}\frac{\partial^2 V_2(x,t)}{\partial t^2}+g_{11}\frac{\partial^2 V_3(x,t)}{\partial t^2}+i_{11}\frac{\partial^4 V_1(x,t)}{\partial x^2\partial t^2}+j_{11}\frac{\partial^4 V_2(x,t)}{\partial x^2\partial t^2}+k_{11}\frac{\partial^4 V_3(x,t)}{\partial x^2\partial t^2}=\sum_{k=1}^{m}\left(\left(e_{11}+i_{11}\left(\frac{\pi k}{l}\right)^2\right)\frac{d^2C_k(t)}{dt^2}+\left(f_{11}+j_{11}\left(\frac{\pi k}{l}\right)^2\right)\frac{d^2D_k(t)}{dt^2}+\left(g_{11}+k_{11}\left(\frac{\pi k}{l}\right)^2\right)\frac{d^2H_k(t)}{dt^2}\right)\sin\left(\frac{\pi kx}{l}\right);$$

$$b_{11}\frac{\partial V_1(x,t)}{\partial t}+c_{11}\frac{\partial V_2(x,t)}{\partial t}+d_{11}\frac{\partial V_3(x,t)}{\partial t}+h_{11}\frac{\partial^3 V_1(x,t)}{\partial x^2\partial t}=\sum_{k=1}^{m}\left(\left(b_{11}+h_{11}\left(\frac{\pi k}{l}\right)^2\right)\frac{dC_k(t)}{dt}+c_{11}\frac{dD_k(t)}{dt}+d_{11}\frac{dH_k(t)}{dt}\sin\left(\frac{\pi kx}{l}\right);\right.$$

$$-\frac{\partial^2 V_1(x,t)}{\partial x^2}+a_{11}V_1(x,t)=\sum_{k=1}^{m}\left(-\left(\frac{\pi k}{l}\right)^2+a_{11}\right)C_k(t)\sin\left(\frac{\pi kx}{l}\right). \tag{80}$$

For equation (80), we introduce the notation:

$$a_1=e_{11}+i_{11}\left(\frac{\pi k}{l}\right)^2;\ b_1=f_{11}+j_{11}\left(\frac{\pi k}{l}\right)^2;\ c_1=g_{11}+k_{11}\left(\frac{\pi k}{l}\right)^2;\ d_1=b_{11}+h_{11}\left(\frac{\pi k}{l}\right)^2;\ e_1=c_{11};f_1=d_{11};g_1=-\left(\frac{\pi k}{l}\right)^2+a_{11}. \tag{81}$$

Equation (56), taking into account equations (80) and (81), is written as follows:

$$a_1\frac{dC^2(t_k)}{dt^2}+b_1\frac{dD^2(t_k)}{dt^2}+c_1\frac{dH^2(t_k)}{dt^2}+d_1\frac{dC(t_k)}{dt}+e_1\frac{dD(t_k)}{dt}+f_1\frac{dH(t_k)}{dt}+g_1C(t_k)=\gamma(t_k),k=1\ldots m. \tag{82}$$

Using equations (59)–(61), let's transform the left part of equation (57) as follows:

$$e_{21}\frac{\partial^2 V_1(x,t)}{\partial t^2}+f_{21}\frac{\partial^2 V_2(x,t)}{\partial t^2}+g_{21}\frac{\partial^2 V_3(x,t)}{\partial t^2}+i_{21}\frac{\partial^4 V_2(x,t)}{\partial x^2\partial t^2}+j_{11}\frac{\partial^4 V_1(x,t)}{\partial x^2\partial t^2}+k_{11}\frac{\partial^4 V_3(x,t)}{\partial x^2\partial t^2}=\sum_{k=1}^{m}\left(\left(e_{21}+j_{21}\left(\frac{\pi k}{l}\right)^2\right)\frac{d^2C_k(t)}{dt^2}+\left(f_{21}+i_{21}\left(\frac{\pi k}{l}\right)^2\right)\frac{d^2D_k(t)}{dt^2}+\left(g_{21}+k_{21}\left(\frac{\pi k}{l}\right)^2\right)\frac{d^2H_k(t)}{dt^2}\right)\sin\left(\frac{\pi kx}{l}\right);$$

$$b_{21}\frac{\partial V_1(x,t)}{\partial t}+c_{21}\frac{\partial V_2(x,t)}{\partial t}+d_{21}\frac{\partial V_3(x,t)}{\partial t}+h_{21}\frac{\partial^3 V_2(x,t)}{\partial x^2\partial t}=\sum_{k=1}^{m}\left(b_{21}\frac{dC_k(t)}{dt}+\left(c_{21}+h_{21}\left(\frac{\pi k}{l}\right)^2\right)\frac{dD_k(t)}{dt}+d_{21}\frac{dH_k(t)}{dt}\sin\left(\frac{\pi kx}{l}\right);\right.$$

$$-\frac{\partial^2 V_2(x,t)}{\partial x^2}+a_{21}V_2(x,t)=\sum_{k=1}^{m}\left(-\left(\frac{\pi k}{l}\right)^2+a_{21}\right)D_k(t)\sin\left(\frac{\pi kx}{l}\right). \tag{83}$$

For (83), we introduce the notation:

$$a_2=e_{21}+j_{21}\left(\frac{\pi k}{l}\right)^2;\ b_2=f_{21}+i_{21}\left(\frac{\pi k}{l}\right)^2;\ c_2=g_{21}+k_{21}\left(\frac{\pi k}{l}\right)^2;\ d_2=b_{21};\ e_2=c_{21}+h_{21}\left(\frac{\pi k}{l}\right)^2;\ f_2=d_{21};g_2=-\left(\frac{\pi k}{l}\right)^2+a_{21}. \tag{84}$$

Equation (57), taking into account equations (83) and (84), is written as follows:

$$a_2\frac{dC^2(t_k)}{dt^2}+b_2\frac{dD^2(t_k)}{dt^2}+c_2\frac{dH^2(t_k)}{dt^2}+d_2\frac{dC(t_k)}{dt}+e_2\frac{dD(t_k)}{dt}+f_2\frac{dH(t_k)}{dt}+g_2D_k(t)=\mu_k(t),k=1\ldots m. \tag{85}$$

Using equations (59)–(61), let's transform the left part of equation (58) as follows:

$$e_{31}\frac{\partial^2 V_1(x,t)}{\partial t^2}+f_{31}\frac{\partial^2 V_2(x,t)}{\partial t^2}+g_{31}\frac{\partial^2 V_3(x,t)}{\partial t^2}+i_{31}\frac{\partial^4 V_2(x,t)}{\partial x^2\partial t^2}+j_{31}\frac{\partial^4 V_1(x,t)}{\partial x^2\partial t^2}+k_{31}\frac{\partial^4 V_3(x,t)}{\partial x^2\partial t^2}=\sum_{k=1}^{m}\left(\left(\left(e_{31}+j_{31}\left(\frac{\pi k}{l}\right)^2\right)\frac{d^2C_k(t)}{dt^2}+\left(f_{31}+i_{21}\left(\frac{\pi k}{l}\right)^2\right)\frac{d^2D_k(t)}{dt^2}+\left(g_{31}+k_{31}\left(\frac{\pi k}{l}\right)^2\right)\frac{d^2H_k(t)}{dt^2}\right)\sin\left(\frac{\pi kx}{l}\right);$$

$$b_{31}\frac{\partial V_1(x,t)}{\partial t}+c_{31}\frac{\partial V_2(x,t)}{\partial t}+d_{31}\frac{\partial V_3(x,t)}{\partial t}+h_{31}\frac{\partial^3 V_3(x,t)}{\partial x^2\partial t}=\sum_{k=1}^{m}\left(b_{31}\frac{dC_k(t)}{dt}+c_{31}\frac{dD_k(t)}{dt}+\left(d_{31}+h_{31}\left(\frac{\pi k}{l}\right)^2\frac{dH_k(t)}{dt}\right)\sin\left(\frac{\pi kx}{l}\right);\right.$$

$$-\frac{\partial^2 V_3(x,t)}{\partial x^2}+a_{31}V_3(x,t)=\sum_{k=1}^{m}\left(-\left(\frac{\pi k}{l}\right)^2+a_{31}\right)H_k(t)\sin\left(\frac{\pi kx}{l}\right). \tag{86}$$

For equation (86), we introduce the notation:

$$a_3=e_{31}+j_{31}\left(\frac{\pi k}{l}\right)^2;\ b_3=f_{31}+i_{31}\left(\frac{\pi k}{l}\right)^2;\ c_3=g_{31}+k_{31}\left(\frac{\pi k}{l}\right)^2;\ d_3=b_{31};$$

$$e_3=c_{31};f_3=d_{31}+h_{31}\left(\frac{\pi k}{l}\right)^2;g_3=-\left(\frac{\pi k}{l}\right)^2+a_{31}. \tag{87}$$

Equation (58), taking into account equations (86) and (87), is written as follows:

$$a_3\frac{dC^2(t_k)}{dt^2}+b_3\frac{dD^2(t_k)}{dt^2}+c_3\frac{dH^2(t_k)}{dt^2}+d_3\frac{dC(t_k)}{dt}+e_3\frac{dD(t_k)}{dt}+f_3\frac{dH(t_k)}{dt}+g_3H_k(t)=\sigma_k(t),k=1\dots m. \tag{88}$$

Consider equations (82), (85) and (88) as homogeneous with constant coefficients and zero initial conditions:

$$a_1\frac{d^2e_1(t)}{dt^2}+b_1\frac{d^2e_2(t)}{dt^2}+c_1\frac{d^2e_3(t)}{dt^2}+d_1\frac{de_1(t)}{dt}+e_1\frac{de_2(t)}{dt}+f_1\frac{de_3(t)}{dt}+g_1e_1(t)=0; \tag{89}$$

$$a_2\frac{d^2e_1(t)}{dt^2}+b_2\frac{d^2e_2(t)}{dt^2}+c_2\frac{d^2e_3(t)}{dt^2}+d_2\frac{de_1(t)}{dt}+e_2\frac{de_2(t)}{dt}+f_2\frac{de_3(t)}{dt}+g_2e_2(t)=0; \tag{90}$$

$$a_3\frac{d^2e_1(t)}{dt^2}+b_3\frac{d^2e_2(t)}{dt^2}+c_3\frac{d^2e_3(t)}{dt^2}+d_3\frac{de_1(t)}{dt}+e_3\frac{de_2(t)}{dt}+f_3\frac{de_3(t)}{dt}+g_3e_3(t)=0. \tag{91}$$

Equations (89)–(91) are written in the operator form:

$$\left(a_1\lambda^2+d_1\lambda+g_1\right)e_1(\lambda)+\left(b_1\lambda^2+e_1\lambda\right)e_2(\lambda)+\left(c_1\lambda^2+f_1\lambda\right)e_3(\lambda)=0; \tag{92}$$

$$\left(a_2\lambda^2+d_2\lambda+g_1\right)e_1(\lambda)+\left(b_2\lambda^2+e_2\lambda+g_2\right)e_2(\lambda)+\left(c_2\lambda^2+f_2\lambda\right)e_3(\lambda)=0; \tag{93}$$

$$\left(a_3\lambda^2+d_3\lambda\right)e_1(\lambda)+\left(b_3\lambda^2+e_3\lambda\right)e_2(\lambda)+\left(c_3\lambda^2+f_3\lambda+g_3\right)e_3(\lambda)=0. \tag{94}$$

The characteristic equation of the system of equations (92)–(94) has the form:

$$s_6\lambda^6+s_5\lambda^5+s_4\lambda^4+s_3\lambda^3+s_2\lambda^2+s_1\lambda+s_0=0, \tag{95}$$

where:

$$s_6=a_1b_2c_3+b_1c_2a_3+c_1b_3a_2-c_1a_3b_2-a_2b_1c_3-c_2b_3a_1;$$

$$s_5=a_1b_2f_3+a_1e_2c_3+d_1b_2c_3+b_1c_2d+f_2b_1a_3+e_1c_2a_3+c_1b_3d_2+c_1e_3a_2+f_1b_3a_2-c_1a_3e_2-c_1d_3b_2-f_1a_3b_2-a_2b_1f_3-a_2e_1c_3-d_2b_1c_3-c_2b_3d_1-c_2e_3a_1-f_2b_3a_1;$$

$$s_4=a_1b_2f_3+d_1b_2f_3+a_1g_2c_3+d_1e_2c_3+g_1b_2c_3+f_2b_1d_3+e_1c_2d_3+e_1f_2a_3+c_1e_3d_2+f_1b_3d_2+f_1e_3a_2-c_1a_3g_2-c_1d_3e_2-f_1a_3e_2-f_1d_3b_2-a_2b_1g_3-a_2e_1f_3-d_2b_1f_3-d_2e_1c_3-c_2b_3g_1-c_2e_3d_1-f_2b_3d_1-f_2e_3a_1;$$

$$s_3=a_1e_2g_3+d_1b_2g_3+a_1g_2f_3+d_1e_2f_3+g_1b_2f_3+d_1g_2c_3+g_1e_2c_3+e_1f_2d_3+f_1e_3d_2-c_1d_3g_2-f_1a_3g_2--f_1d_3e_2-a_2e_1g_3-d_2b_1g_3-d_2e_1f_3-c_2e_3g_1-f_2b_3g_1-f_2e_3d_1;$$

$$s_2 = a_1 g_2 g_3 + d_1 e_2 g_3 + g_1 b_2 g_3 + d_1 g_2 f_3 + g_1 e_2 f_3 + g_1 g_2 c_3 - f_1 d_3 g_2 - d_2 e_1 g_3 - f_2 e_3 g_1;$$

$$s_1 = d_1 g_2 g_3 + g_1 e_2 g_3 + g_1 g_2 f_3;$$

$$s_0 = g_1 g_2 g_3.$$

Consider the case where the characteristic equation (95) has complex conjugate roots:

$\lambda_{1,2} = \delta_1 \pm j\theta_1, \lambda_{3,4} = \delta_2 \pm j\theta_2$ та $\lambda_{5,6} = \delta_3 \pm j\theta_3$

The homogeneous equations (89)–(91) have the following solution:

$e_1(t) = e^{\delta_1 t}\cos(\theta_1 t), e_2(t) = e^{\delta_1 t}\sin(\theta_1 t), e_3(t) = e^{\delta_2 t}\cos(\theta_2 t),$
$e_4(t) = e^{\delta_2 t}\sin(\theta_2 t),$

$e_5(t) = e^{\delta_3 t}\cos(\theta_3 t),\ e_6(t) = e^{\delta_3 t}\sin(\theta_3 t),$ respectively.

3. Results and discussions

The ultimate solution to these problems is based on the use of variation of arbitrary constants. This typically involves solving ordinary differential equations (ODEs) or systems of ODEs. The method of variation of arbitrary constants is used to find solutions to inhomogeneous linear differential equations. To do this, it is necessary to determine whether the differential equation is linear and homogeneous or inhomogeneous. The variation of arbitrary constants method is mainly used for inhomogeneous linear ODEs. It is then necessary to solve the corresponding homogeneous equation to find the general solution. The general solution of the homogeneous equation will involve arbitrary constants, and the result will be the function of the independent variable or a combination of functions. For the inhomogeneous equation, we assume a particular solution based on the form of the inhomogeneous term. This solution will involve constants, but they won't be arbitrary constants. Once we've determined the values of the arbitrary constants using the initial and boundary conditions, we'll substitute them into the general solution along with the solution. This gives you the final solution that satisfies both the inhomogeneous equation and any given initial and boundary conditions.

Using the method of variation of arbitrary constants, we look for solutions to equations (82), (85) and (88) in the form [30]:

$$C_k(t) = B_1(t)e_1(t) + B_2(t)e_2(t) + B_3(t)e_3(t) + B_4(t)e_4(t) + B_5(t)e_5(t) + B_6(t)e_6(t); \quad (96)$$

$$D_k(t) = B_1(t)e_1(t) + B_2(t)e_2(t) + B_3(t)e_3(t) + B_4(t)e_4(t) + B_5(t)e_3(t) + B_6(t)e_4(t); \quad (97)$$

$$H_k(t) = B_1(t)e_1(t) + B_2(t)e_2(t) + B_3(t)e_3(t) + B_4(t)e_4(t) + B_5(t)e_3(t) + B_6(t)e_4(t). \quad (98)$$

We look for functions $B_1(t)$, $B_2(t)$, $B_3(t)$, $B_4(t)$, $B_5(t)$, and $B_6(t)$ from the system of equations:

$$dB_1(t)/dte_1(t)/dt + dB_2(t)/dte_2(t)dt + dB_3(t)/dte_3(t)/dt + dB_4(t)/dte_4(t)/dt + dB_5(t)/dte_5(t)/dt + dB_6(t)/dte_6(t) = 0;$$

$$dB_1(t)/dtde_1(t)/dt + dB_2(t)/dtde_2(t)dt + dB_3(t)/dtde_3(t)/dt + dB_4(t)/dtde_4(t)/dt + dB_5(t)/dtde_5(t)/dt + dB_6(t)/dtde_6(t)/dt = 1/a_1\gamma_k(t); \quad (99)$$

$$\begin{aligned} &dB_1(t)/dte_1(t)/dt + dB_2(t)/dte_2(t)dt + dB_3(t)/dte_3(t)/dt + dB_4(t)/dte_4(t)/dt + dB_5(t)/dte_5(t)/dt + dB_6(t)/dte_6(t) = 0; \\ &dB_1(t)/dtde_1(t)/dt + dB_2(t)/dtde_2(t)dt + dB_3(t)/dtde_3(t)/dt + dB_4(t)/dtde_4(t)/dt + dB_5(t)/dtde_5(t)/dt + dB_6(t)/dtde_6(t)/dt = 1/a_2\mu_k(t); \end{aligned} \quad (100)$$

$$\begin{aligned} &dB_1(t)/dte_1(t)/dt + dB_2(t)/dte_2(t)dt + dB_3(t)/dte_3(t)/dt + dB_4(t)/dte_4(t)/dt + dB_5(t)/dte_5(t)/dt + dB_6(t)/dte_6(t) = 0; \\ &dB_1(t)/dtde_1(t)/dt + dB_2(t)/dtde_2(t)dt + dB_3(t)/dtde_3(t)/dt + dB_4(t)/dtde_4(t)/dt + dB_5(t)/dtde_5(t)/dt + dB_6(t)/dtde_6(t)/dt = 1/a_3\sigma_k(t); \end{aligned} \quad (101)$$

We write the determinant of the matrix of coefficients of the system of equations (99)–(101) as follows:

$$\Delta = \begin{vmatrix} e_1(t) & e_2(t) & e_3(t) & e_4(t) & e_5(t) & e_6(t) \\ \frac{de_1(t)}{dt} & \frac{de_2(t)}{dt} & \frac{de_3(t)}{dt} & \frac{de_4(t)}{dt} & \frac{de_5(t)}{dt} & \frac{de_6(t)}{dt} \\ e_1(t) & e_2(t) & e_3(t) & e_4(t) & e_5(t) & e_6(t) \\ \frac{de_1(t)}{dt} & \frac{de_2(t)}{dt} & \frac{de_3(t)}{dt} & \frac{de_4(t)}{dt} & \frac{de_5(t)}{dt} & \frac{de_6(t)}{dt} \\ e_1(t) & e_2(t) & e_3(t) & e_4(t) & e_5(t) & e_6(t) \\ \frac{de_1(t)}{dt} & \frac{de_2(t)}{dt} & \frac{de_3(t)}{dt} & \frac{de_4(t)}{dt} & \frac{de_5(t)}{dt} & \frac{de_6(t)}{dt} \end{vmatrix} \quad (102)$$

We also find Δ_1, Δ_2, Δ_3, Δ_4, Δ_5 and Δ_6, accordingly, replacing the columns in equation (102) with the right part of equations (99)–(101), respectively:

$$dB_1(t)/dt = \Delta_1/\Delta; dB_2(t)/dt = \Delta_2/\Delta; dB_3(t)/dt = \Delta_3/\Delta;$$
$$dB_4(t)/dt = \Delta_4/\Delta; dB_5(t)/dt = \Delta_5/\Delta; dB_6(t)/dt = \Delta_6/\Delta. \quad (103)$$

To find the initial conditions $B_1(t)|_{t=0}$,$B_2(t)|_{t=0}$, $B_3(t)|_{t=0}$,$B_4(t)|_{t=0}$,$B_5(t)|_{t=0}$ and $B_6(t)|_{t=0}$, we differentiate equations (96)–(98) and, using equations (99)–(101), we obtain:

$$\begin{aligned} dC(t_k)/dt &= dB_1(t)/dte_1(t) + dB_2(t)/dte_2(t) \\ &+ dB_3(t)/dte_3(t) + dB_4(t)/dte_4(t) + dB_5(t)/dte_5(t) \\ &+ dB_6(t)/dte_6(t) + B_1(t)de_1(t)/dt + B_2(t)de_2(t)/dt \\ &+ B_3(t)de_3(t)/dt + B_4(t)de_4(t)/dt + B_5(t)de_5(t)/dt \\ &+ B_6(t)de_6(t)/dt = B_1(t)de_1(t)/dt + B_2(t)de_2(t)/dt \\ &+ B_3(t)de_3(t)/dt + B_4(t)de_4(t)/dt + B_5(t)de_5(t)/dt \\ &+ B_6(t)de_6(t)/dt; \end{aligned} \quad (104)$$

$$\begin{aligned} dD_k(t)/dt &= dB_1(t)/dte_1(t) + dB_2(t)/dte_2(t) \\ &+ dB_3(t)/dte_3(t) + dB_4(t)/dte_4(t) + dB_5(t)/dte_5(t) \\ &+ dB_6(t)/dte_6(t) + B_1(t)de_1(t)/dt + B_2(t)de_2(t)/dt \\ &+ B_3(t)de_3(t)/dt + B_4(t)de_4(t)/dt + B_5(t)de_5(t)/dt \\ &+ B_6(t)de_6(t)/dt = B_1(t)de_1(t)/dt + B_2(t)de_2(t)/dt \\ &+ B_3(t)de_3(t)/dt + B_4(t)de_4(t)/dt + B_5(t)de_5(t)/dt \end{aligned}$$

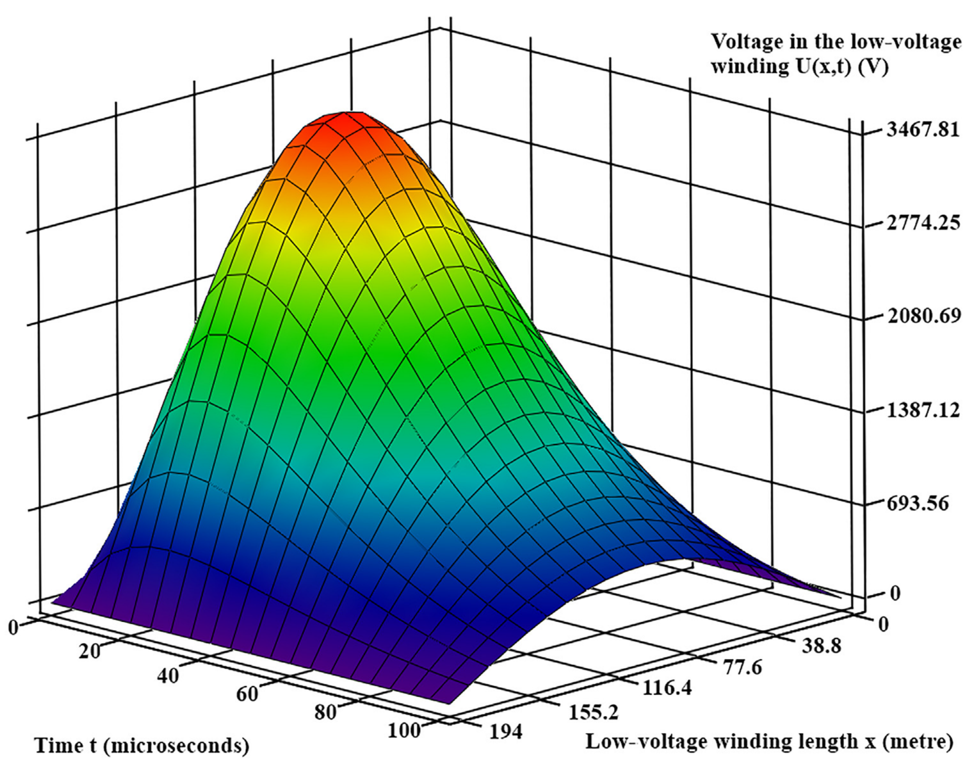

Fig. 2. Voltage distribution as a function of distance and time in the primary winding of a three-winding transformer.

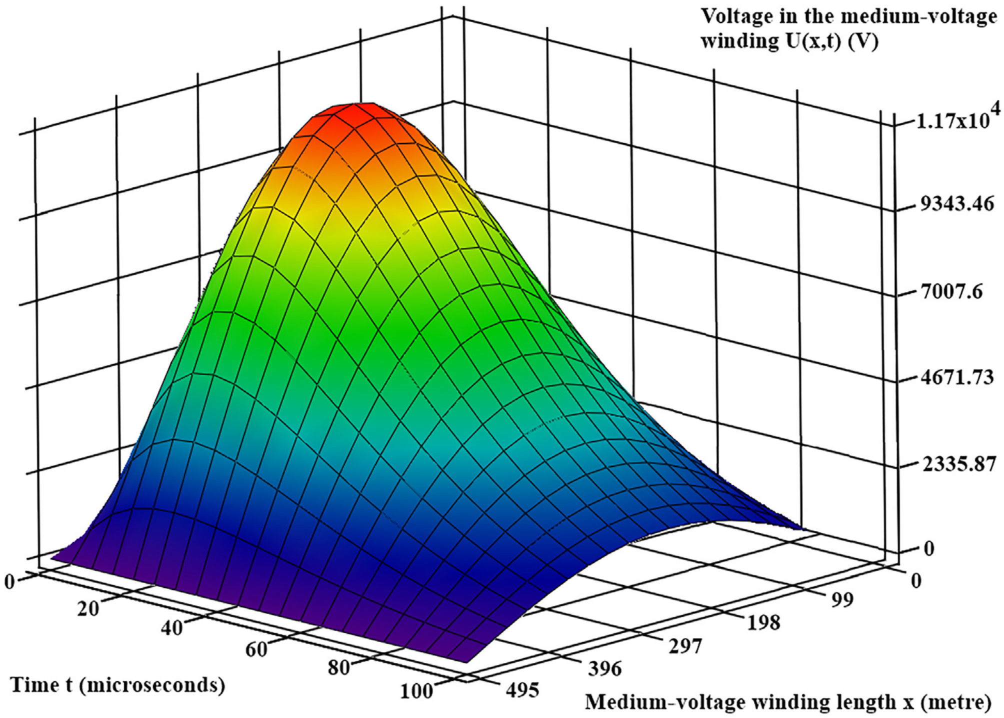

Fig. 3. Voltage distribution as a function of distance and time in the secondary winding of a three-winding transformer.

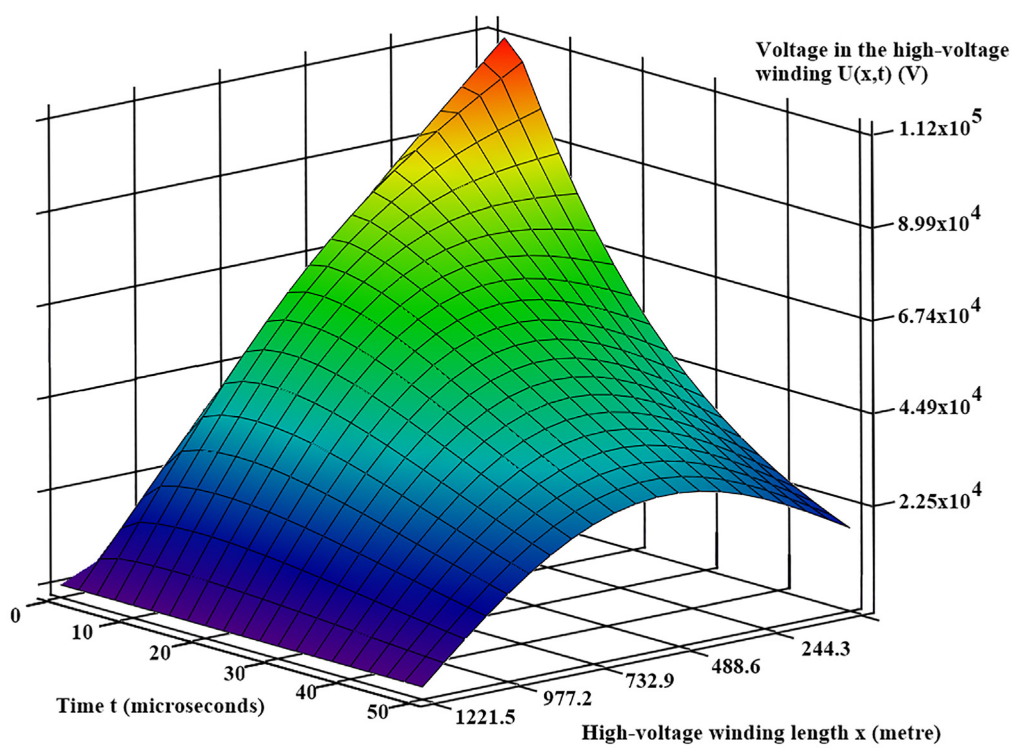

Fig. 4. Voltage distribution depending on distance and time in the tertiary winding of a three-winding transformer.

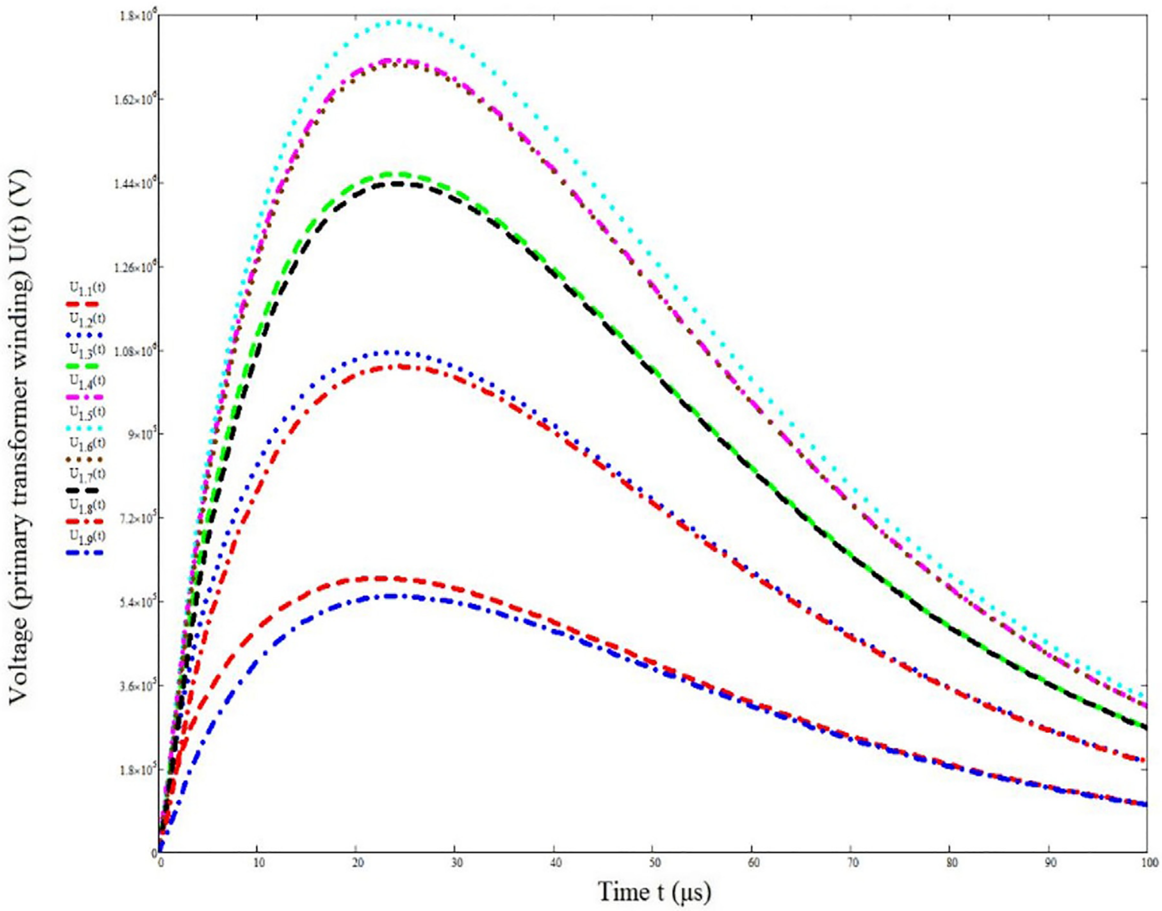

Fig. 5. Voltage distribution at different points of the primary winding of a three-winding power transformer depending on time.

$$+B_6(t)de_6(t)/dt; \tag{105}$$

$$\begin{aligned} dH_k(t)/dt &= dB_1(t)/dte_1(t) + dB_2(t)/dte_2(t) \\ &+ dB_3(t)/dte_3(t) + dB_4(t)/dte_4(t) + dB_5(t)/dte_5(t) \\ &+ dB_6(t)/dte_6(t) + B_1(t)de_1(t)/dt + B_2(t)de_2(t)/dt \\ &+ B_3(t)de_3(t)/dt + B_4(t)de_4(t)/dt + B_5(t)de_5(t)/dt \\ &+ B_6(t)de_6(t)/dt = B_1(t)de_1(t)/dt + B_2(t)de_2(t)/dt \\ &+ B_3(t)de_3(t)/dt + B_4(t)de_4(t)/dt + B_5(t)de_5(t)/dt \\ &+ B_6(t)de_6(t)/dt. \end{aligned} \tag{106}$$

From equations (104)–(106), as well as equations (96)–(98), we obtain for $t = 0$:

$$\begin{aligned} C_k(t)|_{t=0} &= B_1(t)|_{t=0}\, e_1(t)|_{t=0} + B_2(t)|_{t=0}\, e_2(t)|_{t=0} \\ &+ B_3(t)|_{t=0}\, e_3(t)|_{t=0} + B_4(t)|_{t=0}\, e_4(t)|_{t=0} \\ &+ B_5(t)|_{t=0}\, e_5(t)|_{t=0} + B_6(t)|_{t=0}\, e_6(t)|_{t=0}; \end{aligned}$$

$$\begin{aligned} dC_k(t)/dt|_{t=0} &= B_1(t)|_{t=0}\, de_1(t)/dt|_{t=0} \\ &+ B_2(t)|_{t=0}\, de_2(t)/dt|_{t=0} + B_3(t)|_{t=0}\, de_3(t)/dt|_{t=0} \\ &+ B_4(t)|_{t=0} de_4(t)/dt|_{t=0} + B_5(t)|_{t=0} de_5(t)/dt|_{t=0} \\ &+ B_6(t)|_{t=0} de_6(t)/dt|_{t=0}; \end{aligned} \tag{107}$$

$$\begin{aligned} D_k(t)|_{t=0} &= B_1(t)|_{t=0}\, e_1(t)|_{t=0} + B_2(t)|_{t=0}\, e_2(t)|_{t=0} \\ &+ B_3(t)|_{t=0}\, e_3(t)|_{t=0} + B_4(t)|_{t=0}\, e_4(t)|_{t=0} \\ &+ B_5(t)|_{t=0}\, e_5(t)|_{t=0} + B_6(t)|_{t=0}\, e_6(t)|_{t=0}; \end{aligned}$$

$$\begin{aligned} dD_k(t)/dt|_{t=0} &= B_1(t)|_{t=0}\, de_1(t)/dt|_{t=0} \\ &+ B_2(t)|_{t=0}\, de_2(t)/dt|_{t=0} + B_3(t)|_{t=0}\, de_3(t)/dt|_{t=0} \\ &+ B_4(t)|_{t=0} de_4(t)/dt|_{t=0} + B_5(t)|_{t=0} de_5(t)/dt|_{t=0} \\ &+ B_6(t)|_{t=0} de_6(t)/dt|_{t=0}; \end{aligned} \tag{108}$$

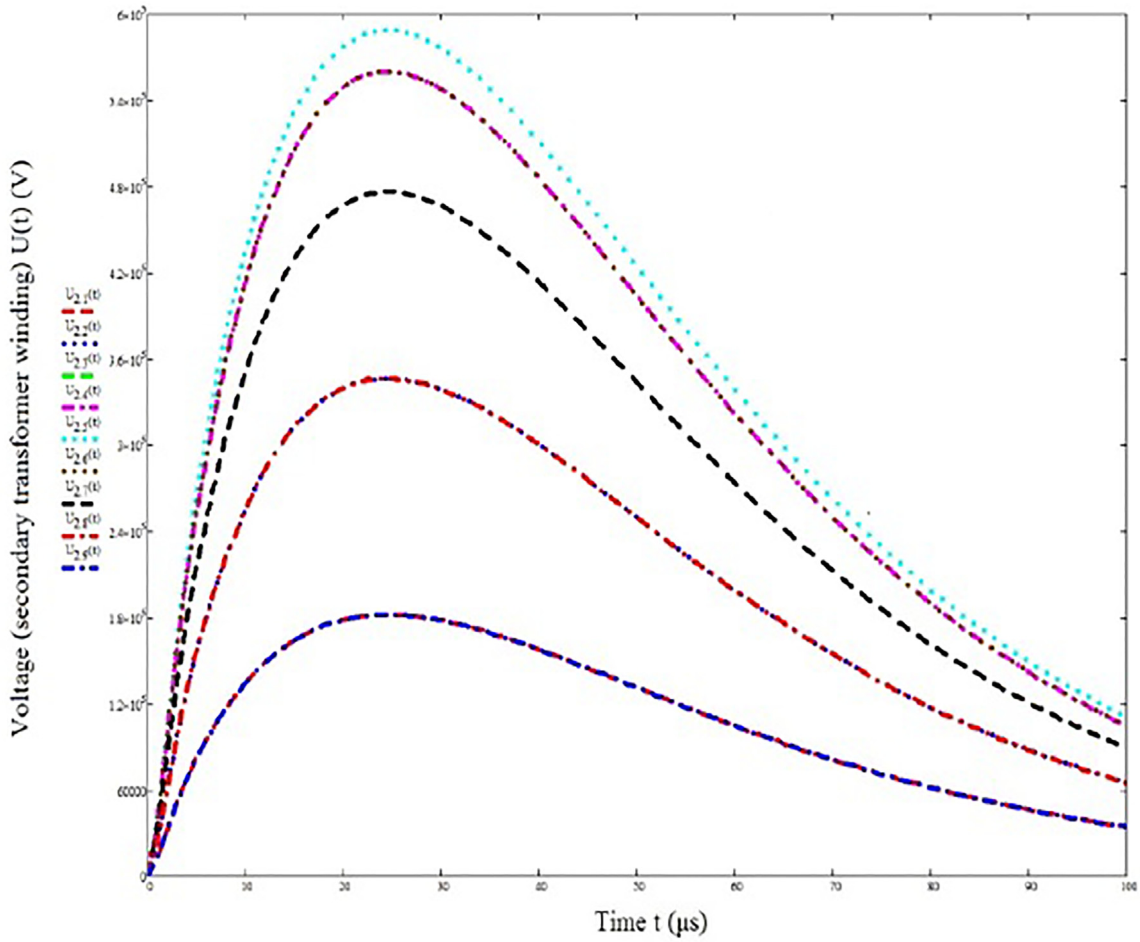

Fig. 6. Voltage distribution at different points of the secondary winding of a three-winding power transformer depending on time.

$$H_k(t)|_{t=0} = B_1(t)|_{t=0}\, e_1(t)|_{t=0} + B_2(t)|_{t=0}\, e_2(t)|_{t=0} + B_3(t)|_{t=0}\, e_3(t)|_{t=0} + B_4(t)|_{t=0}\, e_4(t)|_{t=0} + B_5(t)|_{t=0}\, e_5(t)|_{t=0} + B_6(t)|_{t=0}\, e_6(t)|_{t=0};$$

$$dH_k(t)/dt|_{t=0} = B_1(t)|_{t=0}\, de_1(t)/dt|_{t=0} + B_2(t)|_{t=0}\, de_2(t)/dt|_{t=0} + B_3(t)|_{t=0}\, de_3(t)/dt|_{t=0} + B_4(t)|_{t=0} de_4(t)/dt|_{t=0} + B_5(t)|_{t=0} de_5(t)/dt|_{t=0} + B_6(t)|_{t=0} de_6(t)/dt|_{t=0}. \quad (109)$$

Considering equations (74)–(76), we reformulate equations (107)–(109) as follows:

$$B_1(t) = \int_0^1 \frac{dB_1(t)}{dt} + B_1(t)|_{t=0};\; B_2(t) = \int_0^1 \frac{dB_2(t)}{dt} + B_2(t)|_{t=0};\; B_3(t) = \int_0^1 dB_3(t)\,/\,dt + B_3(t)|_{t=0};\; B_4(t) = \int_0^1 dB_4(t)\,/\,dt + B_5(t)|_{t=0};\; B_5(t) = \int_0^1 dB_5(t)\,/\,dt + B_5(t)|_{t=0};\; B_6(t) = \int_0^1 dB_6(t)\,/\,dt + B_6(t)|_{t=0}. \quad (110)$$

According to equations (96)–(98) we find $C_k(t)$, $D_k(t)$ and $H_k(t)$; according to equations (59)–(61) and $V_1(x,t), V_2(x,t)$ and $V_3(x,\ t)$; according to

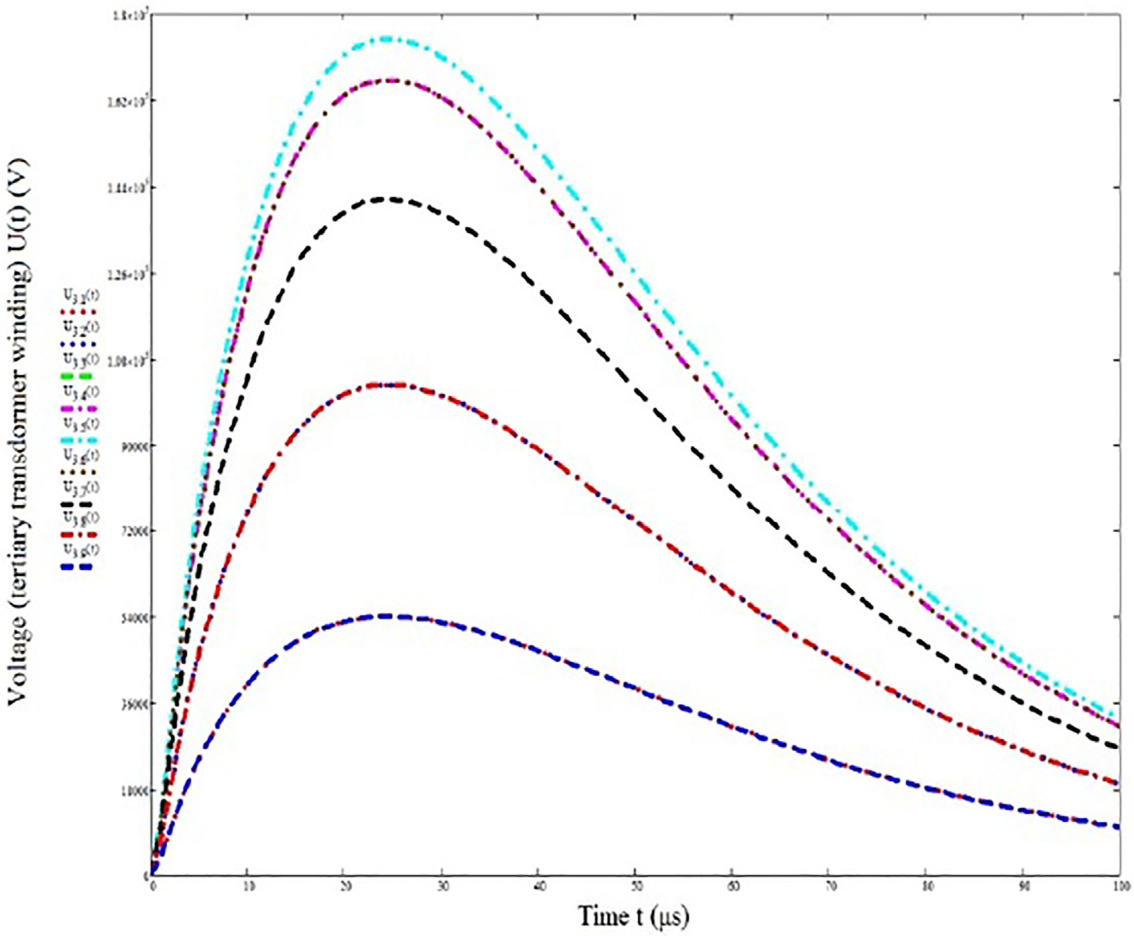

Fig. 7. Voltage distribution at different points of the tertiary winding of a three-winding power transformer depending on time.

equations (29), (36) and (43) we determine $u_1(x,t), u_2(x,t)$ and $u_3(x,t)$.

Figures 2–4 show the distribution of voltages in the primary, secondary and tertiary windings of a three-winding transformer as a function of distance and time, respectively.

Figures 5–7 show the distribution of voltages at various points in the primary, secondary and tertiary windings of a three-winding power transformer as a function of time.

The results obtained are the final solution and they do indeed satisfy the original inhomogeneous differential equation and the conditions given. It's important to note that the variation of arbitrary constants method is useful for linear inhomogeneous equations; it could apply to all types of differential equations. However, non-linear equations or equations with more complex forms may require other solution techniques.

4. Conclusions

Three-winding transformers, also known as tertiary transformers, are electrical devices used to transfer power between three separate circuits. In power systems, wave processes refer to the propagation of electrical signals and disturbances, such as voltage and current, through transmission lines, transformers and other components. These processes are characterized by various wave phenomena such as reflection, transmission and interaction. Wave processes can have significant effects on the operation and performance of the transformer. These effects can include voltage and current oscillations, transient overvoltage and other disturbances that can affect the stability and reliability of the power system. Understanding and analyzing these wave processes is essential for the effective design and operation of power systems.

A mathematical model has been developed for the study of wave processes in three-winding power transformers, taking into account the main magnetic flux, own and mutual interwinding fluxes and mutual interwinding fluxes of the winding dispersion, the formation of initial and boundary conditions is given. A modified method of variable separation is proposed to solve a system of differential equations with partial derivatives. The proposed mathematical model allows modeling of voltage distribution in transformer windings, on the basis of which it is possible to develop means of protection against overvoltage and coordinate their isolation.

Ethical statement

The authors state that the study was accompanied according to ethical standards.

Funding body

None.

Data availability

Data will be made available on request.

Conflicts of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgment

M. S. Seheda – Development and implementation of the model, preparing results, writing original draft, O. S. Beshta – Conceptualization, scoping the study, writing, preparing the final manuscript, P.F. Gogolyuk – Conceptualization, scoping the study, validate modeling results, Yu. V. Blyznak – Conceptualization, scoping the study, validate modeling results. R. O. Dychkovskyi – modelling and results analysis, A. Smolinski – Conceptualization, modelling and results analysis supervision.

References

[1] Li F, Zhang Y, Xu J, Zhu M, Zhang C. Simulation research on sweep frequency impedance characteristics of transformer winding based on three-phase three-winding lumped parameter model. 2020 IEEE International Conference on High Voltage Engineering and Application (ICHVE) 2020;45(32): 346–65. https://doi.org/10.1109/ichve49031.2020.9280077.

[2] Kolb A, Pazynich Y, Mirek A, Petinova O. Influence of voltage reserve on the parameters of parallel power active compensators in mining. E3S Web of Conferences 2020;201: 01024. https://doi.org/10.1051/e3sconf/202020101024.

[3] Savukov I, Karaulanov T. Multi-flux-transformer MRI detection with an atomic magnetometer. J Magn Reson 2014; 249:49–52. https://doi.org/10.1016/j.jmr.2014.10.009.

[4] Kang Y-C, Lee B-E, Zheng T-Y, Kim Y-H, Crossley PA. Protection, faulted phase and winding identification for the three-winding transformer using the increments of flux linkages. IET Gener, Transm Distrib 2010;4(9):1060. https://doi.org/10.1049/iet-gtd.2010.0094.

[5] Polyanska A, Pazynich Y, Sabyrova M, Verbovska L. Directions and prospects of the development of educational services in conditions of energy transformation: the aspect of the coal industry. Polityka Energetyczna – Energy Policy Journal 2023; 26(2):195–216. https://doi.org/10.33223/epj/162054.

[6] Lüdtke N, Logothetis NK, Panzeri S. Testing methodologies for the nonlinear analysis of causal relationships in neurovascular coupling. Magn Reson Imag 2010;28(8):1113–9. https://doi.org/10.1016/j.mri.2010.03.028.

[7] Polyanska A, Savchuk S, Dudek M, Sala D, Pazynich Y, Cicho D. Impact of digital maturity on sustainable development effects in energy sector in the condition of Industry 4.0. Naukovyi Visnyk Natsionalnoho Hirnychoho Universytetu 2022;6:97–103. https://doi.org/10.33271/nvngu/2022-6/097.

[8] Vladyko O, Maltsev D, Sala D, Cichoń D, Buketov V, Dychkovskyi R. Simulation of leaching processes of polymetallic ores using the similarity theorem. Rudarsko-Geolosko-Naftni Zb 2022;37(5):169–80. https://doi.org/10.17794/rgn.2022.5.14.

[9] Dyczko A. Real-time forecasting of key coking coal quality parameters using neural networks and artificial intelligence. Rudarsko-Geolosko-Naftni Zb 2023;64:105–17. https://doi.org/10.17794/rgn.2023.3.9.

[10] Nikolsky V, Dychkovskyi R, Cabana EC, Howaniec N, Jura B, Widera K, et al. The hydrodynamics of Translational–Rotational motion of incompressible gas flow within the working space of a vortex heat generator. Energies 2022;15(4):1431. https://doi.org/10.3390/en15041431.

[11] Linnik KS, Neyman LA. Numerical simulation of electromagnetic processes in power transformers taking into account the nonlinearities of Magnetic Bonds. J Phys Conf 2021;2032(1):012091. https://doi.org/10.1088/1742-6596/2032/1/012091.

[12] Mombello Enrique E, Guillermo A, Florez Díaz. An improved high frequency white-box lossy transformer model for the calculation of power systems electromagnetic transients. Elec Power Syst Res 2021;190. https://doi.org/10.1016/j.epsr.2020.106838. Article106838,ISSN 0378-7796.

[13] Mombello EE, Venerdini GG, Díaz Flórez GA. Optimized high-frequency white-box transformer model for implementation in ATP-EMTP. Elec Power Syst Res 2022;213. https://doi.org/10.1016/j.epsr.2022.108709. Article108709, ISSN 0378-7796.

[14] Dychkovskyi RO. Forming the bilayer artificially shell of georeactor in underground coal gasification. Naukovyi Visnyk Natsionalnoho Hirnychoho Universytetu 2015;5:37–42.

[15] Murad Mohammed Ahsan Adib, Francisco José Orts Gómez, Vanfretti Luigi. Equation-based modeling of three-winding and regulating transformers using Modelica. 2015 IEEE Eindhoven PowerTech; 2015. p. 1–6. https://doi.org/10.1109/PTC.2015.7232503.

[16] Zhao X, Yao C, Abu-Siada A, Liao R. High frequency electric circuit modeling for transformer frequency response analysis studies. Int J Electr Power Energy Syst 2019;111:351–68. https://doi.org/10.1016/j.ijepes.2019.04.010.

[17] Pivnyak GG, Beshta OO. A complex source of electrical energy for three-phase current based on a stand-alone voltage inverter. Naukovyi Visnyk Natsionalnoho Hirnychoho Universytetu 2020;1:89–93. https://doi.org/10.33271/nvngu/2020-1/089.

[18] Yoon Yeunggurl, Son Yongju, Cho Jintae, Jang SuHyeong, Kim Young-Geun, Choi Sungyun. High-frequency modeling

of a three-winding power transformer using sweep frequency response analysis. Energies 2021;14(13):4009. https://doi.org/10.3390/en14134009.
[19] Baravati Peyman Rezaei, Moazzami Majid, Seyed Mohammad Hassan Hosseini, Mirzaei Hassan Reza, Fani Bahador. Achieving the exact equivalent circuit of a large-scale transformer winding using an improved detailed model for partial discharge study. Int J Electr Power Energy Syst 2022; 134:107451. https://doi.org/10.1016/J.IJEPES.2021.107451.
[20] Mikulović JČ, Šekara TB. The numerical method of inverse Laplace transform for calculation of overvoltages in power transformers and test results. Serbian J Elect Eng June 2014; 11(No. 2):243−56.
[21] Bontidean SG, Badic M, Iordache M, Galan N. Simulations and experimental tests on the distribution of overvoltage within transformer windings. U.P.B. Sci. Bull. Series C 2015; 77(3).
[22] de Azevedo AC, Rezende I, Delaiba AC, Oliveira C, de Carvalho BC, de S Bronzeado H. Investigation of transformer electromagnetic forces caused by external faults using FEM. In: IEEE PES transmission and distribution conference and exposition. − Venezuela, Latin America; 2006. p. 1−6.
[23] Bontidean SG, Badic M, Iordache M, Galan N. Simulations and experimental tests on the distribution of overvoltage within transformer windings. U.P.B. Sci. Bull. Series C 2015; 77(3).
[24] Deaconu D, Chirilă A-I, Năvrăpescu V, Ghiţă C, Răchiţeanu A, Viişoreanu A-M. The influence of parameters of a power transformer winding equivalent distributed circuit model on atmospheric overvoltage wave internal propagation along the windings. In: 2020 international conference and exposition on electrical and power engineering (EPE); 2020. p. 507−12. https://doi.org/10.1109/EPE50722.2020.9305584.
[25] Sala D, Bieda B. Application of uncertainty analysis based on Monte Carlo (MC) simulation for life cycle inventory (LCI). Inzynieria Mineralna 2019;2(2−44):263−8. https://doi.org/10.29227/im-2019-02-80.
[26] Beshta A, Beshta A, Balakhontsev A, Khudolii S. Performances of asynchronous motor within variable frequency drive with additional power source plugged via combined converter. In: 2019 IEEE 6th international conference on energy smart systems (ESS); 2019. https://doi.org/10.1109/ess.2019.8764192.
[27] CIGRE brochure 577A, electrical transient interaction between transformers and the power system. Part 1: expertise . Joint working group A2/C4.39. − CIGRE. April 2014. p. 176. http://www.http//xmlopez.webs.uvigo.es/Html/Info/2014_Electrical_Transients_Part1_Expertise.pdf.
[28] Seheda MS, Cheremnykh YV, Gogolyuk PF, Mazur TA, Blyznak YV. Mathematical model of wave processes in two-winding transformers//*Tekhnichna Elektrodynamika*. 2020. p. 5−14. https://doi.org/10.15407/techned2020.06.005. No 6.
[29] Seheda MS, Cheremnykh YV, Gogolyuk PF, Blyznak YV. Mathematical modeling of wave processes in two-winding transformers taking into account the main magnetic flux. *Scientific Bulletin of National Mining University*; 2021. p. 80−6. https://doi.org/10.33271/nvngu/2021-5/080. № 5 (185).
[30] Seheda MS, Cheremnykh YV, Mazur TA. Mathematical modeling of free voltage oscillations on transformer windings into accout winds mutual induction under surge overvoltages. *Scientific Bulletin of National Mining University*; 2013. p. 68−76. 1 (133).
[31] Seheda Mykhailo. Petro Gogolyuk and Yurii Blyznak Mathematical model of analysis of wave electromagnetic transients in three-winding transformers. In: 2022 IEEE 8th international conference on energy smart systems (ESS); 2022. p. 269−72. https://doi.org/10.1109/ESS57819.2022.9969302. Kyiv, Ukraine.