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Stability analysis for discrete-time fractional-order LTI state-space systems. Part II: New stability criterion for FD-based systems

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Abstract. This paper presents a series of new results on the asymptotic stability of discrete-time fractional difference (FD) state space systems and their finite-memory approximations called finite FD (FFD) and normalized FFD (NFFD) systems. In Part I of the paper, new necessary and sufficient stability conditions have been given in a unified form for FD, FFD and NFFD-based systems. Part II offers a new, simple, ultimate stability criterion for FD-based systems. This gives rise to the introduction of new definitions of the so-called f-poles and f-zeros for FD-based systems, which are used in the closed-loop stability analysis for FD-based systems and, approximately, for FFD/NFFD-based ones.

Key words: Grünwald-Letnikov fractional difference, discrete-time fractional-order systems, stability criterion, nonminimum phase system, closed-loop stability.

1. Introduction

Up-to-date complete, ultimate, simple stability criteria for FD-based discrete-time LTI state space systems have been achieved for a special case of positive systems only [1–5]. General stability conditions of Refs. [6, 7] are not computationally simple on the one hand and are not referred to the ultimate stability criterion form known for fractional-order continuous-time systems on the other.

The celebrated stability criterion for continuous-time fractional-order LTI state space systems [8, 9] is reduced to a simple argument condition for eigenvalues of a state matrix. That is why in the discrete-time fractional-order case of Part I [10] we have been concerned with the stability conditions involving eigenvalues of the state matrix A_f on the one hand and the polar representation of complex numbers in new coordinate transformations $w(z) = z(1 - z^{-1})$ or $v(z) = w(z) + 1$ on the other. Such an approach has been applied for the first time ever, yielding a series of new stability results related with the roots of some simple characteristic equations and culminating with a unified, polar-form framework for stability analysis of FD/FFD/NFFD-based state space systems. Those stability testing results, essentially outperforming previous contributions [6, 7, 11, 12], in terms of conceptual simplicity and lower computational burden, have provided a firm basis for the introduction in Part II of a new stability criterion for FD-based systems. The criterion triggers the redefinition of poles and zeros for the FD-based systems, new definition of minimum/nonminimum phase FD-based systems and, consequently, provides means for the analysis of the closed-loop stability of FD-based systems under some control laws.

Part II of the paper is organized as follows. Having introduced the stability problem in Sec. 1, the FD/FFD/NFFD-based LTI state space system descriptions and properties are recalled in Sec. 2. The new stability analysis results of Part I [10] of the paper are briefly reminded in Sec. 3. The main result of Part II and of the whole paper, that is an original stability criterion for the FD-based systems is announced in Sec. 4, followed by an associated discussion on surjection, injection and bijection of the transformations and on the relationship to the continuous-time fractional-order derivative case. In Sec. 5 it is shown how the stability criterion for the FD-based systems contributes to the introduction of new types of poles and zeros for the systems, that is f-poles and f-zeros as well as new definitions of minimum/nonminimum phase property for the FD-based systems. Section 6 presents simple examples of control-related implications of f-poles and f-zeros. In Sec. 7, numerical stability testing procedures are proposed for FFD/NFFD-systems, with some approximate relationship with f-poles and f-zeros being indicated.

Conclusions of Sec. 8 summarize the achievements of Part II of the paper.

2. FD/FFD/NFFD-based LTI state space systems

Consider a discrete-time state space LTI system described by the (constant-order) fractional-order model

$$\Delta^{\alpha} x(t+1) = A_f x(t) + Bu(t), \tag{1}$$

$$y(t) = Cx(t) + Du(t), \tag{2}$$

where the fractional order $\alpha \in (0, 2)$, $x(t) \in \Re^n$, $u(t) \in \Re^{n_u}$ and $y(t) \in \Re^{n_y}$ are the state, input and output vectors, respectively, $A_f \in \Re^{n \times n}$, $B \in \Re^{n \times n_u}$, $C \in \Re^{n_y \times n}$ and $D \in \Re^{n_y \times n_u}$. Without loss of generality we will assume in the sequel that the initial vector x_0 is zero, especially that we will operate on finite-memory FD approximations that do not trace back to x_0.

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Note that $A_f = A - I$, with $A \in \Re^{n \times n}$ representing a discrete-time state space system in a 'regular' form (with $\alpha = 1$) and $I \in \Re^{n \times n}$ is the identity matrix. Accounting for the Grünwald-Letnikov fractional difference (FD), Eq. (1) can be presented in the following form [10, 12]

$$x(t+1) = \left(A_f + \alpha I\right)x(t) - \sum_{j=2}^{t+1} P_j(\alpha)x(t-j+1) + Bu(t), \tag{3}$$

where q^{-1} is the backward shift operator and

$$P_j(\alpha) = (-1)^j C_j(\alpha), \tag{4}$$

with

$$C_j(\alpha) = \binom{\alpha}{j} = \begin{cases} 1 & j = 0 \\ \dfrac{\alpha(\alpha-1)...(\alpha-j+1)}{j!} & j > 0 \end{cases}. \tag{5}$$

Using the Definition 1 and Definition 2 of Part I of the paper [10] we can jointly present FD/FFD/NFFD-based discrete-time state equations as

$$\begin{aligned} x(t+1) &= \left(A_f + \frac{\alpha}{N} I\right) x(t) \\ &- \frac{1}{N}\sum_{j=2}^{J} P_j(\alpha)x(t-j+1) + Bu(t), \end{aligned} \tag{6}$$

with $N = N(\overline{J}) = \sum_{j=1}^{\overline{J}} P_j(\alpha)$ defined as in Definition 2 of Part I [10] and J redefined as $J = \min(t+1, \overline{J})$.

Note that Eq. (6) can be considered the most general fractional-difference state equation including the NFFD one, FFD one (for $N = 1$) and FD one (for $\overline{J} \to \infty$ implying $N \to 1$).

Remark 1. Possible accounting for the sampling period T (when transferring from a continuous-time derivative to the discrete-time difference) results in the substitutions $A_f \to A_f T^\alpha$ and $B \to BT^\alpha$ in Eqs. (3) and (6) [12].

Remark 2. The steady-state accuracy analysis has shown [13] that steady-state error-free output modeling (with respect to FD) can be obtained for the NFFD-based system only.

3. New results of Part I [10]

Most important results of Part I [10] are now briefly recalled. The results are related with two important mappings

$$w = z(1 - z^{-1})^\alpha \tag{7}$$

and

$$v = z(1 - z^{-1})^\alpha + 1 \tag{8}$$

rewritten respectively in the polar form for $|z| = 1$

$$w = e^{i\varphi}(1 - e^{-i\varphi})^\alpha \tag{9}$$

and

$$v = e^{i\varphi}(1 - e^{-i\varphi})^\alpha + 1 \tag{10}$$

with $\varphi = \arg(z)$.

Theorem 1. [10] The FD-based discrete-time state equation (3) with $\alpha \in (0, 2)$ is asymptotically stable if and only if all the roots of the characteristic equation

$$\det\left[z(1 - z^{-1})^\alpha I - A_f\right] = 0 \tag{11}$$

are strictly inside the unit circle.

Theorem 2. [10] Consider the FD-stability solid $\mathcal{S} = \left\{e^{i\varphi}(1 - e^{-i\varphi})^\alpha; 0 < \alpha < 2; 0 \le \varphi \le 2\pi\right\}$ related with the coordinate transformation (7), with $\alpha = \text{par}$ and $\varphi = \arg(z)$. The FD-based discrete-time state equation (3) with $\alpha \in (0, 2)$ is asymptotically stable if and only if all the roots of the characteristic equation

$$\det[wI - A_f] = 0 \tag{12}$$

that is all eigenvalues of A_f, are strictly inside the FD stability solid.

Theorem 3. [10] The NFFD/FFD/FD-based discrete-time state equation (6) with $\alpha \in (0, 2)$ is asymptotically stable if and only if all the roots of the characteristic equation

$$\det[wI - A_f] = 0 \tag{13}$$

that is all eigenvalues of A_f, are strictly inside the stability solid

$$\mathcal{S} = \left\{e^{i\varphi}\Psi(\varphi); 0 < \alpha < 2; 0 \le \varphi \le 2\pi\right\} \tag{14}$$

with $\varphi = \arg(z)$ and

i) $\Psi(\varphi) = 1 + \dfrac{1}{N}\sum\limits_{j=1}^{\overline{J}} P_j(\alpha)e^{-ij\varphi}$ for NFFD,
ii) $\Psi(\varphi)$ as above with $N = 1$ for FFD,
iii) $\Psi(\varphi) = (1 - e^{-i\varphi})^\alpha$ for FD.

Remark 3. Theorem 3 is the most important result to constitute a basis for the presentation of a new stability criterion for the FD-based systems.

4. New stability criterion for FD-based systems

4.1. FD-based system re-revisited – main result. The question arising from Theorem 3 of Part I of the paper [10] is why scanning the whole range of the argument φ from 0 to 2π instead of seeking for some features of, say, v at the *reference* argument φ_i^v related to the specific eigenvalue $\lambda_i^v = \lambda_i$, $i = 1, \ldots, n$, where n is the number of eigenvalues of A. The idea behind that concept is illustrated in Fig. 1, from which we can immediately infer the stability condition $|\lambda_i^v| < |v_i|$, $i = 1, \ldots, n$, where $|v_i|$ is the modulus of v corresponding to the *i*-th reference argument φ_i^v. Well, the problem reduces to the calculation of φ_i^v and $|v_i|$, $i = 1, \ldots, n$, and this is a difficult issue which is not likely to be solved in a simple analytical way. We will provide solution to the problem while using the transformation (7) instead of (8).

Here we offer a new, capital result which is a culmination of the whole FD-based system stability involvement.

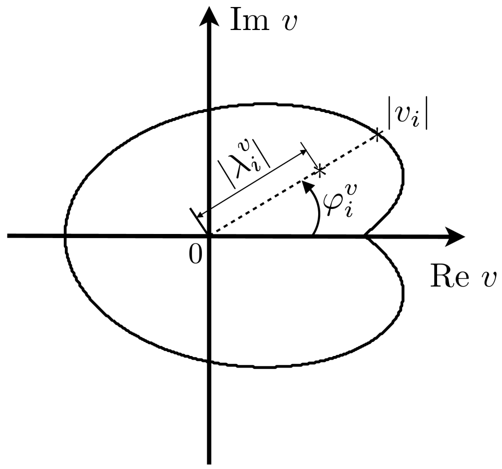

Fig. 1. Rationale for new stability condition

Theorem 4. The FD-based discrete-time state equation (3) with $\alpha \in (0,2)$ is asymptotically stable if and only if

$$\varphi_i^f \in \left[\alpha\frac{\pi}{2}, 2\pi - \alpha\frac{\pi}{2}\right] \quad \wedge \quad |\lambda_i^f| < |w_i| \qquad i = 1,\dots,n, \tag{15}$$

where $|\lambda_i^f|$ and φ_i^f are the modulus and argument, respectively, of the *i*-th eigenvalue λ_i^f of the matrix A_f and

$$|w_i| = \left(2\left|\sin\frac{\varphi_i^f - \alpha\frac{\pi}{2}}{2-\alpha}\right|\right)^\alpha \qquad i = 1,\dots,n. \tag{16}$$

Proof. See Appendix A.

Remark 4. Theorem 4 is the ultimate analytical stability criterion and nothing simpler, nothing more compact and nothing more general can be obtained for the FD-based state equation (3).

Remark 5. Note that the stability range for φ_i^f, $i = 1,\dots,n$, that is $\left[\alpha\frac{\pi}{2}, 2\pi - \alpha\frac{\pi}{2}\right]$, can be quite narrow for α approaching the value of 2. Anyway, the first step in testing the system stability is the inspection of the argument condition, after which the modulus condition is checked.

Remark 6. It is interesting that the argument range condition in (15) is identical with that for the continuous-time fractional derivative systems [8,9]. However, the modulus condition for $|\lambda_i^f|$ must additionally be satisfied for (discrete-time) fractional difference systems.

In the sequel, all the illustrating examples will be processed in the Matlab environment.

Example 1. Consider the FD-based discrete-time state equation with

$$A_f = \begin{bmatrix} 0.2 & -0.5121 \\ 1 & -1 \end{bmatrix}$$

whose eigenvalues are $\lambda_{1,2}^f = -0.4 \pm 0.39i$. Now, $\varphi_1^f = 2.36885$ [rad], $\varphi_2^f = 3.91433$ [rad] and $|\lambda_{1,2}^f| = 0.558659$. For $\alpha = 0.7$ we have $\varphi_{1,2}^f \in [1.09955, 5.18363]$ and $|\lambda_{1,2}| < |w_{1,2}| = 1.42402$ and the system is stable. For $\alpha = 1.2$ we have $\varphi_{1,2}^f \in [1.8849, 4.3982]$ and $|\lambda_{1,2}| < |w_{1,2}| = 1.1373$ and the system is still stable. For $\alpha = 1.5$ we have $\varphi_{1,2}^f \in [2.35619, 3.92699]$ and $|\lambda_{1,2}| > |w_{1,2}| = 0.05062$ and the system is unstable.

4.2. Surjection, injection and bijection. It would be interesting now to analyse the mappings (7) and (8) in the framework related to the category of sets, that is to verify if and when the two transformations represent surjective, injective or bijective functions. This might sometimes lead to more quantitative results as compared to previous, qualitative ones based on the argument principle. Let us proceed with the transformation (7). Recall the contours $\mathcal{U} = \left\{e^{i\varphi}, 0 \le \varphi \le 2\pi\right\}$ and $\Omega = \left\{e^{i\varphi}(1-e^{i\varphi})^\alpha + 1, 0 \le \varphi \le 2\pi\right\}$ [10].

Firstly observe that when the contours $\mathcal{U}$ and $\Omega' = \Omega - 1$ enclosing the interiors of the stable domain and range sets, respectively, are incorporated into the two sets, then the function $w = z(1-z^{-1})^\alpha$ is both non-surjective and non-injective, even for $\alpha \in (0,1)$. In fact, in that case both $z = 0$ and $z = 1$ are mapped to $w = 0$, in addition to the contour $\mathcal{U}$ mapped to the contour Ω'. Well, luckily, we are definitely interested in the interiors enclosed by the contours.

With $|z| < 1$, we have an interesting result as below.

Theorem 5. Let $z \in \mathcal{C}$ and $|z| < 1$. Then the function $w = z(1-z^{-1})^\alpha$ is bijective for $\alpha \in (0,1)$.

Proof. Introduce the transformation $\underline{w} = z(|z|^{-1} - z^{-1})^\alpha$ and let $z = |z|e^{i\varphi}$. Then $\underline{w} = |z|^{1-\alpha}e^{i\varphi}(1-e^{i\varphi})^\alpha$ and, by virtue of (the proof of) Theorem 4 we have $\varphi = \varphi_{\underline{w}}$ and

$$\varphi_{\underline{w}} = \varphi_w = \varphi + \alpha(\pi - \varphi)/2. \tag{17}$$

Then

$$|\underline{w}| = |z|^{1-\alpha}\left(2\left|\sin\frac{\varphi_w - \alpha\frac{\pi}{2}}{2-\alpha}\right|\right)^\alpha \tag{18}$$

and for $|z| < 1$ and $\alpha \in (0,1)$ we have $|\underline{w}| < |w|$.

Now, Eqs. (17) and (18) define a bijection of the interior of the unit circle onto the interior range of $w = z(1-z^{-1})^\alpha$.

Unfortunately, for $|z| < 1$ and $\alpha \in (1,2)$ the function $w = z(1-z^{-1})^\alpha = \left(z^{1/\alpha} - z^{-1+1/\alpha}\right)^\alpha$ is only injective (and non-surjective) and this is because w tends to infinity as z tends to 0. This means that some part of the interior domain enclosed by the unit circle may be mapped to the exterior of Ω'. This is illustrated for $\alpha = 1.4$ in Fig. 2, where the stability and instability-related parts of the interior domain are shown in gray and white, respectively. It is interesting that for higher α the instability-related part of the interior domain gets larger. Of course, the stability criterion (15) is valid for the whole range of $\alpha \in (0,2)$.

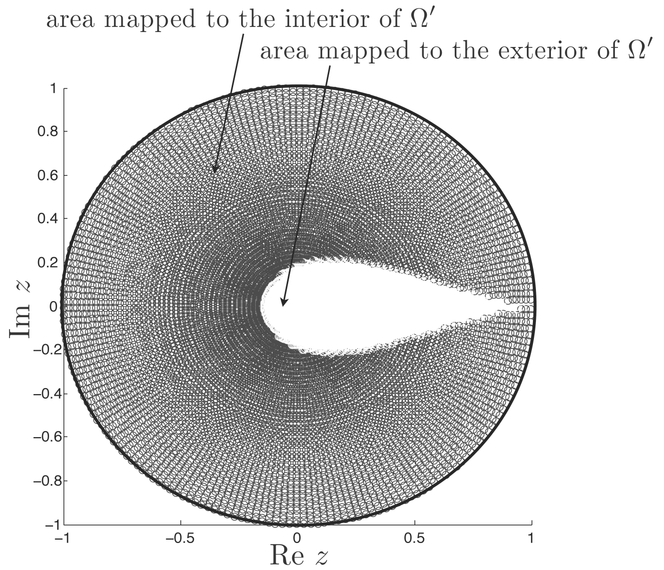

Fig. 2. Areas mapped to the interior and exterior of Ω', for $\alpha = 1.4$

4.3. Relationship to the continuous-time case. Accounting for the sampling interval T (see Remark 1), the stability criterion (15) is retained, with

$$|w_i| = \left(\frac{2}{T}\left|\sin\frac{\varphi_i^f - \alpha\frac{\pi}{2}}{2-\alpha}\right|\right)^{\alpha} \qquad i = 1, \ldots, n \tag{19}$$

which results immediately from the substitution of $T^\alpha A_f$ for A_f in Eqs. (11), (12), (13) and Theorem 4. Alternatively, Theorem 4 can be repeated here, with Eqs. (15) and (16) retained, but either the eigenvalues λ_i^f would concern the matrix $A_f T^\alpha$ rather than A_f or the transformation $\overline{w} = w/T^\alpha$ would be considered.

Plots of the stability contours for various T are shown in Fig. 3.

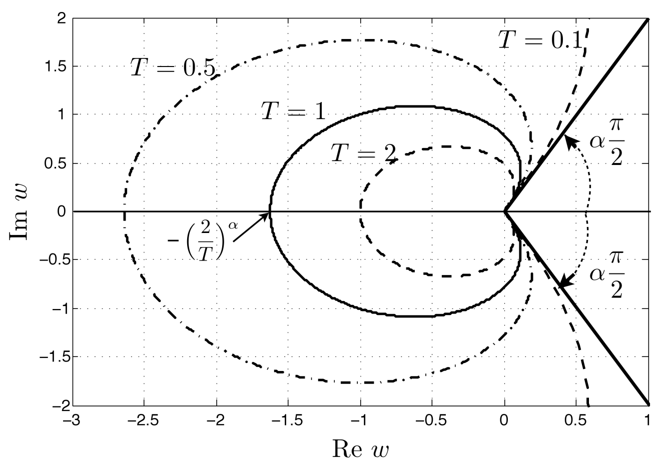

Fig. 3. Plots of FD stability contours for $\alpha = 0.7$ and specific values of T

Interestingly but not surprisingly, as T tends to 0, the well-known stability criterion for continuous-time fractional-order systems is obtained, in which only the argument condition is retained (compare [8, 9]). The above bridges the stability gap between the continuous-time and discrete-time fractional-order systems. The stability condition as in Eqs. (15) and (19) can thus be considered a general, unified, discrete-time/continuous-time stability criterion for fractional-order systems.

5. Introducing F-poles and F-zeros

It is obvious that (infinite-memory) FD-based LTI state-space systems have an infinite number of classical poles (and zeros). However, the stability condition (15) referring to a finite number of eigenvalues λ_i^f, $i = 1, ..., n$, implies that some other poles (and zeros) are desirable to be defined.

To this end, consider an FD-based state-space LTI system described by Eqs. (1) and (2), with the sampling period T possibly accounted for, and let us switch to the transfer function formulation explicitly operating on poles and zeros. Let us start with a SISO system, for which the transfer function can be written as

$$G(w) = C\left[wI - A_f\right]^{-1} B + D = \frac{num(w)}{den(w)}$$

$$= \frac{Cadj\left[wI - A_f\right]B + \det[wI - A_f]D}{\det[wI - A_f]} = k\frac{\prod\limits_{i=1}^{m}(w - \gamma_i)}{\prod\limits_{i=1}^{n}(w - \lambda_i^f)}, \tag{20}$$

where $w = z(1 - z^{-1})^\alpha$, k is the leading coefficient of $num(w)$, and λ_i^f, $i = 1, ..., n$, and γ_i, $i = 1, ..., m$, are some f-poles and f-zeros, respectively.

For a square MIMO system we use the MFD factorization

$$G(w) = C\left[wI - A_f\right]^{-1} B + D = A^{-1}(w)B(w) \tag{21}$$

with the matrix polynomials $A(w)$ and $B(w)$ being of full normal rank. Now, f-poles are the eigenvalues λ_i^f of A_f and f-zeros are f-poles of the inverse of $G(w)$, that is of the inverse of $B(w)$, that is transmission zeros of $G(w)$. Well, transmission f-zeros, in fact.

We leave the case of nonsquare MIMO systems for separate treatment as inverses of nonsquare polynomial matrices would have to be involved, which is quite another issue [14, 15].

We are in a position now to present a series of new useful definitions.

Definition 1. Consider the FD-based discrete-time state equation (1) with $\alpha \in (0, 2)$. Then the eigenvalues λ_i^f, $i = 1, ..., n$, of the matrix A_f are defined as f-poles of the FD-based state space system.

Definition 2. Consider the FD-based discrete-time state space system (1) and (2) with $\alpha \in (0, 2)$. Then f-zeros of the FD-based state space system are defined as f-poles of an inverse of the transfer function matrix $G(w)$.

Definition 3. The FD-based state space system (1) and (2) with $\alpha \in (0, 2)$ is called minimum phase if and only if its all f-zeros are stable. Otherwise the system is called nonminimum phase.

Let us present now the first useful result concerning minimum phase FD-based systems, which follows immediately from Definition 3 and Theorem 4.

Theorem 6 [fractional-order SISO systems] The FD-based discrete-time system (1) and (2) with $\alpha \in (0,2)$ is minimum phase if and only if

$$\varphi_i^\gamma \in \left[\alpha\frac{\pi}{2}, 2\pi - \alpha\frac{\pi}{2}\right] \quad \wedge \quad |\gamma_i| < |w_i| \quad i = 1, \ldots, n, \tag{22}$$

where $|\gamma_i|$ and φ_i^γ are the modulus and argument of f-zeros $\gamma_i, i = 1, ..., m$, and

$$|w_i| = \left(2\left|\sin\frac{\varphi_i^\gamma - \alpha\frac{\pi}{2}}{2-\alpha}\right|\right)^\alpha \qquad i = 1, \ldots, n. \tag{23}$$

Definition 4. (Square/nonsquare MIMO, integer/fractional-order systems). The system is called minimum phase if and only if it is stably invertible. Otherwise the system is called nonminimum phase.

A series of remarks is now due. Firstly, a rationale for the introduction of f-poles is obvious; in fact, they are decisive on the asymptotic stability of FD-based state space LTI systems. Secondly, f-poles help defining f-zeros and minimum/nonminimum phase FD-based systems, the issue being important for the design of closed-loop stable FD-based control systems, in particular minimum-variance/perfect control of FD-based state space LTI systems. And thirdly, f-zeros contribute to the formulation of the general Definition 4 of minimum/nonminimum phase systems, which is valid for all SISO/MIMO, integer/fractional-order systems and which is a unification of the definition used for integer-order systems.

Example 2. Consider three FD-based discrete-time state space SISO systems (A_{f1}, B, C_1, D), (A_{f2}, B, C_1, D) and (A_{f1}, B, C_2, D) with

$$A_{f1} = \begin{bmatrix} 0.6 & -1 \\ 1 & -1 \end{bmatrix}, \qquad A_{f2} = \begin{bmatrix} 0.8 & -1.17 \\ 1 & -1 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad C_1 = [1 \quad -0.95],$$

$$C_2 = [1 \quad -1.05], \qquad D = 0, \qquad \alpha = 0.95$$

The system (A_{f1}, B, C_1, D). Eigenvalues of the matrix A_{f1} are $\lambda_{1,2}^f = -0.2 \pm 0.6i$, so $\varphi_{1,2}^f \in [1.49226, 4.79093]$ and $|\lambda_{1,2}| = 0.63245 < |w_{1,2}| = 0.7552$ and the system is stable. The f-zero of the system is $\gamma_1 = -0.05$, so $\varphi_1^\gamma \in [1.49226, 4.79093]$ and $|\gamma_1| < |w_1| = 1.9318$ and the system is minimum phase.

The system (A_{f2}, B, C_1, D). Eigenvalues of the matrix A_{f2} are $\lambda_{1,2}^f = -0.1 \pm 0.6i$, so $\varphi_{1,2}^f \in [1.49226, 4.79093]$ and $|\lambda_{1,2}| = 0.60828 > |w_{1,2}| = 0.47822$ and the system is unstable. However, f-zero of the system is $\gamma_1 = -0.05$, so $\varphi_1^\gamma \in [1.49226, 4.79093]$ and $|\gamma_1| < |w_1| = 1.9318$ and the system is minimum phase.

The system (A_{f2}, B, C_2, D). Eigenvalues of the matrix A_{f2} are as above, so the system is unstable. The f-zero of the system is $\gamma_1 = 0.05$, so $\varphi_1^\gamma \notin [1.49226, 4.79093]$ and the system is nonminimum phase.

Example 3. Consider two FD-based discrete-time state space MIMO systems (A_f, B_1, C, D), (A_f, B_2, C, D)

$$A_f = \begin{bmatrix} 1.56 & -2.536 & 0.96 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 1 & 0.2 \\ 1 & -1.5 \\ -0.3 & 1 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 1 & 0.2 \\ 1 & -1.1 \\ -0.3 & 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -0.6 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \alpha = 0.95.$$

The system (A_f, B_1, C, D). After MFD factorization as in Eqn. (21) we obtain the roots of $\det(A(w))$ as follows $\lambda_{1,2}^f = -0.2 \pm 0.6i$, $\lambda_3^f = -0.04$. Since $\varphi_{1,2,3}^f \in [1.49226, 4.79093]$, $|\lambda_{1,2}| = 0.63245 < |w_{1,2}| = 0.7552$, $|\lambda_3| < |w_3| = 1.9319$, then the system is stable. The root of $\det(B(w))$ is $\gamma_1 = -1.53781$, so $\varphi_1^\gamma \in [1.49226, 4.79093]$ and $|\gamma_1| < |w_1| = 1.9318$ and the system is minimum phase.

The system (A_f, B_2, C, D). The poles of the system are as above, so the system is stable. However, the root of $\det(B(w))$ is $\gamma_1 = -2.1540$, so $|\gamma_1| > |w_1| = 1.9318$ and the system is nonminimum phase.

Remark 7. It is important that f-poles and f-zeros defined for the discrete-time FD-based systems can be easily extended now to continuous-time fractional-derivative LTI state space systems, with a certain s-domain transformation being an analogue of our w-domain. Note that the lack of the modulus condition in the Matignon criterion [8,9], might have delayed possible introduction of f-poles and f-zeros for continuous-time systems.

6. Stabilizing controls for FD-based systems

6.1. Linear state control. Consider a linear state control problem

$$u(t) = -Kx(t) \tag{24}$$

subject to the state equation (1). Immediately, the closed-loop stability can now be tested using our stability criterion (Theorems 3 or 4) applied to the matrix $(A_f - BK)$. We will show how the closed-loop stability can be affected by a choice of K, in terms of open-loop stability/instability and minimum/nonminimum phase behavior of the systems under control, with the two properties apparently related to f-poles and f-zeros.

Example 4. Consider the FD-based SISO system (1), (2), with $\alpha = 0.9$ and

$$A_f = \begin{bmatrix} 0.1 & -0.9425 \\ 1 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & -0.95 \end{bmatrix}, \qquad D = 0$$

which is an open-loop stable minimum phase system with f-poles $\lambda_{1,2}^f = 0.55 \pm 0.8i$ and f-zero $\gamma_1 = 0.95$. Choosing $K_1 = [-0.5, 0.2]$ provides the closed loop stability, with the closed-loop system eigenvalues $\lambda_{1,2}^c = -0.2 \pm 0.7088i$ and respective $|w_{1,2}| = 0.7863$. However, choosing $K_2 = [-0.5, 0.3]$ makes the system unstable. On the other hand, with $B = [1, 0.5]^T$ the open-loop system is nonminimum phase and neither K_1 nor K_2 can stabilize the closed-loop system, but $K_3 = [0.5, 0.3]$ does stabilize it.

It is worth mentioning that even though the simple linear state controller is nonfractional, its application example illustrates the need for the proper formal characterization of the FD-based system to be controlled, in terms of an open-loop stability and minimum phase behavior. In fact, the design of any controller should account for the two properties of the system, and these are apparently related with f-poles and f-zeros.

6.2. Perfect state control. Implementation of perfect control, a special case of (deterministic) predictive control with a prediction horizon equal to a system delay, can only be made for NFFD/FFD-based systems.

Now, equating the state one-step predictor as in Eq. (6) to its reference x_{ref} one obtains a prefect state control law

$$u(t) = B^+ \left[x_{ref} - \left(A_f + \frac{\alpha}{N} I \right) x(t) + \frac{1}{N} \sum_{j=2}^{J} P_j(\alpha) x(t-j+1) \right], \tag{25}$$

where B^+ is either a right/left inverse or the Moore-Penrose pseudoinverse of B (depending on rank-related properties of B).

A classical perfect control system, whether state or output control one, is asymptotically stable provided the system under control is minimum phase [15]. A similar statement can be made for perfect state control (25) of NFFD/FFD-based state space systems, which we have confirmed in a plethora of simulation examples, with the minimum phase system understood here as having all stable (approximate) f-zeros.

Unfortunately, no rigorously proven result can be obtained here, just due to the approximate nature of f-poles and f-zeros involved and a possible unreliable inference on NFFD/FFD-based system stability and minimum phase property in the boundary areas (compare Fig. 6 in Part I [10]).

Example 5. Consider two FD-based discrete-time state space systems (A_f, B, C_1, D) and (A_f, B, C_2, D) with

$$A_f = \begin{bmatrix} 0.2 & -0.5121 \\ 1 & -1.1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$
$$C_1 = [-0.823 \quad 1], \qquad C_2 = [-0.824 \quad 1],$$
$$D = 0, \qquad \alpha = 0.84.$$

The system (A_f, B, C_1, D)*:* f-poles $\lambda_{1,2}^f = -0.45 \pm 0.3i$, f-zero $\gamma_1 = -1.78620$; *the system* (A_f, B, C_2, D)*:* f-poles $\lambda_{1,2}^f = -0.45 \pm 0.3i$, f-zero $\gamma_1 = -1.7930$; $\varphi_1^\gamma = \pi$, $|w_i| = 1.7900$ – so the system (A_f, B, C_1, D) is minimum phase and the system (A_f, B, C_2, D) is nonminimum phase. Fig. 4 presents time plots of control signals for perfect control ($x_{ref} = \mathbf{1}(t)$), which is stable for the system (A_f, B, C_1, D), see Fig. 4a, but unstable for the system (A_f, B, C_2, D), see Fig. 4b, thus confirming the minimum/nonminimum phase behavior analysis for the two systems.

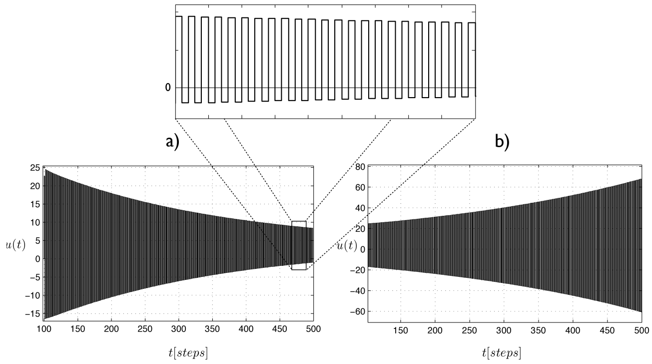

Fig. 4. Plots of control signals for FD-based perfect regulation

7. NFFD/FFD-based systems re-revisited

Unfortunately, no analytical stability result like for FD can be obtained for NFFD/FFD-based systems. However, having the FD result as in Theorem 4 we can essentially simplify a numerical procedure for testing the stability of the NFFD-based system. In fact, we do not have to scan φ within the whole range $0 \le \varphi \le 2\pi$, but only in some (close) vicinity to the reference argument φ_i, $i = 1, ..., n$, obtained as in Theorem 4. Now, a numerical searching procedure based on static minimization of some e.g. quadratic criterion with respect to φ_i can be easily arranged for the NFFD-based system, with its starting point φ_i^0 being the reference φ_i, $i = 1, ..., n$, obtained from $\varphi_i^f = \varphi_i^w$. We have tested evolutionary vs. `fminsearch`-based Matlab algorithms and the latter has been found quite sufficient owing to very close (or 'almost' identical) values of φ_i^0 and optimum φ_i^{opt}. Finally, the NFFD-based system is stable if and only if the condition (15) is satisfied, where

$$|w_i| = |e^{i\varphi_i^{opt}} \Psi(\varphi_i^{opt})| \qquad i = 1, \ldots, n. \tag{26}$$

with $\Psi(\varphi)$ calculated as in Theorem 3 under specification *i)*.

However, arranging for such a minimization procedure for the FFD-based system may be not that easy due to the specific shape of the stability contour for φ being close to 0 and 2π (compare Fig. 6 in Part I [10]). In this case we have to shift to the transformation (7) and, in a minimization procedure to guess a starting point for φ_i, which may sometimes be quite distant from φ_i^{opt} to be obtained. This increases the execution time of the minimization algorithm and may also cause the necessity to use an evolutionary optimization algorithm.

Example 6. Consider the NFFD/FFD/FD-based discrete-time state space system with

$$A_f = \begin{bmatrix} 0.6 & -1.45 \\ 1 & -1 \end{bmatrix}$$

and $\alpha = 0.77$. Testing the stability of the FD-based system via Theorem 4 reveals that the arguments $\varphi_1^f = \varphi_1^w = 1.7894$ rad,

$\varphi_2^f = \varphi_2^w = 4.4937$ rad are within the range $\left[\alpha\frac{\pi}{2}, 2\pi - \alpha\frac{\pi}{2}\right]$ and $|\lambda_1^f| = |\lambda_2^f| = 0.92195$, $|w_1| = |w_2| = 0.92874$, which means that the system is asymptotically stable.

When testing the stability of the NFFD-based system, `fminsearch`-based minimization of $(\varphi_i^f - \varphi_i^w)^2$, $i = 1, 2$, with respect to φ_i is started with $\varphi_1^0 = 0.9430$ rad, $\varphi_2^0 = 5.3402$ rad obtained from Theorem 4 and fast convergence of the minimization procedure to $\varphi_1^{opt} = 0.9099$ rad, $\varphi_2^{opt} = 5.3732$ rad is gained. Since the obtained $|w_1^{opt}| = |w_2^{opt}| = 0.91142$ is lower than $|\lambda_1^f| = |\lambda_2^f| = 0.92195$, then the NFFD-based system is unstable. Of course, we have purposefully selected the example system in order to illustrate possible (small) differences in stability results for FD an NFFD-based systems in some boundary areas (compare Fig. 6 of Part I). Let us emphasize at last that the rationale for using the above minimization criterion is that, unlike for FD-systems, we cannot analytically calculate φ_i from $\varphi_i^f = \varphi_i^w$ and therefore numerical minimization of a distance of φ_i^f from φ_i^w has to be applied.

Finally, when testing the stability of the FFD-based system, we perform `fminsearch`-based minimization of $(\varphi_i^\lambda - \varphi_i^v)^2$ with respect to φ_i, where the reference $\varphi_i^\lambda = \arg\lambda_i$, $i = 1, 2$. The minimization procedure is started with $\varphi_1^0 = 0.8442$ rad, $\varphi_2^0 = 5.4390$ rad, which are the values of φ_i^v, (being the arguments of $\lambda_i^v = \lambda_i$, $i = 1, 2$), as no better guesses are available. The procedure is (more slowly) convergent to $\varphi_1^{opt} = 0.9380$ rad and $\varphi_2^{opt} = 5.3451$ rad. Since the obtained value of $|v_1| = |v_2| = 1.2082$ is higher than $|\lambda_1^v| = |\lambda_2^v| = 1.2042$, then the FFD-based system is asymptotically stable. Of course, we could easily select such example systems where the FFD-based system would be unstable, whereas the FD and NFFD-based ones would be stable. For the present example, we have verified the above stability results for NFFD and FFD-based systems via independent computations according to Theorem 3.

Remark 8. Up to date, the above proposed numerical procedures for testing the asymptotic stability of NFFD/FFD-based systems are the simplest ones out of all the available tests. This is owing to the procedures' close relation to the ultimate FD result of Theorem 4.

Remark 9. The above stability testing procedures for NFFD/FFD-based systems can as well be used to check if the system is stably invertible, that is minimum phase. In both cases, f-poles and f-zeros involved can be considered a sort of approximation of those defined for FD-based systems.

8. Conclusions

This paper has offered a series of original results on the asymptotic stability of discrete-time fractional-difference (FD) systems and their finite-memory approximations, namely finite FD (FFD) and, in particular, normalized finite FD (NFFD). In Part I, sufficient stability conditions for FFD and NFFD-based systems have been illustrated to cover, for a specific example, only some 50% of the actual stability area. Therefore, new, computationally effective, necessary and sufficient conditions for the asymptotic stability of NFFD/FFD/FD-based LTI state-space systems have been given in the general, unified framework. In Part II, the main result of the paper has been presented as a new, simple, general analytical criterion for the asymptotic stability of the FD-based system. It is not until this paper that the celebrated Matignon stability criterion for the continuous-time fractional-derivative systems [8, 9] has been paralleled for the discrete-time fractional-difference ones. The latter case has appeared more complicated, involving both argument and modulus conditions. However, the form of the latter stability criterion has directly given rise to the introduction of a new quality in the stability analysis of closed-loop FD/FFD/NFFD-based LTI state space systems, that is f-poles and f-zeros. In addition to the f-poles' obvious contribution to the stability analysis, the f-zeros have been employed to redefine, for the first time, the minimum phase property for FD-based systems, with transparent implications to their closed-loop stability analysis. The new stability criterion for FD-based systems has been finally used to offer simple numerical procedures for testing the stability of FFD and, in particular, NFFD-based systems. Selected simulation examples have illustrated the achievements of the paper.

Our current and future research involves predictive control of FFD/NFFD-based systems, the topic directly applying our open/closed-loop stability analysis, with the f-poles and f-zeros employed.

Appendix A. Proof of Theorem 4

For comparison, we initially refer to the two alternative mappings (7) and (8), to finally choose the former one as the most appropriate for the purpose.

Accounting for Eq. (7) rewrite Eq. (10) as

$$v = |v|e^{i\varphi^v} = |w|e^{i\varphi^w} + 1,$$

where

$$|w| = \left(2\left|\sin\frac{\varphi}{2}\right|\right)^\alpha \quad \text{and} \quad \varphi^w = \varphi + \alpha\frac{\pi - \varphi}{2},$$

with the two latter equations obtained via fundamental trigonometric identities, namely

$$|w| = \left(\sqrt{(1-\cos\varphi)^2 + \sin^2\varphi}\right)^\alpha = \left(\sqrt{2(1-\cos\varphi)}\right)^\alpha = \left(2\left|\sin\frac{\varphi}{2}\right|\right)^\alpha$$

and

$$\varphi^w = \varphi + \alpha \arctan\frac{\sin\varphi}{1-\cos\varphi} = \varphi + \alpha \arctan\left(\tan\frac{\pi-\varphi}{2}\right) = \varphi + \alpha\frac{\pi-\varphi}{2}.$$

Note that since $0 \le \varphi \le 2\pi$ then $\alpha\frac{\pi}{2} \le \varphi^w \le 2\pi - \alpha\frac{\pi}{2}$.

The presence of the unity in the transformation (8) precludes simple analytical manipulations on v, in particular φ^v, so that is why we have switched to the transformation (7), with $w = v - 1$. Shifting the original unit circle to the left by 1 (Fig. 5) we end up with eigenvalues

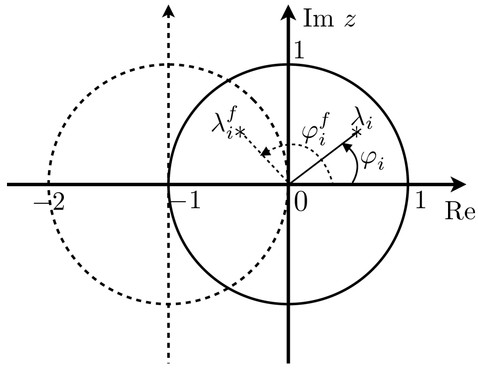

Fig. 5. Shifting the reference argument

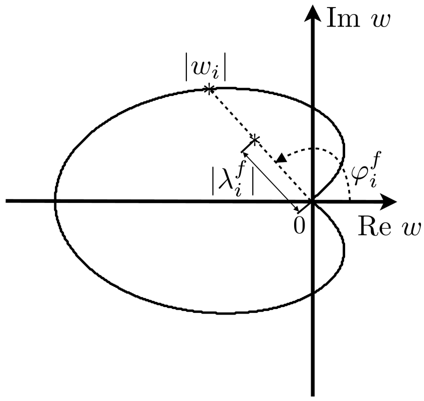

Fig. 6. Concluding on the system stability when φ_i^f is within the range $\left[\alpha\frac{\pi}{2}, 2\pi - \alpha\frac{\pi}{2}\right]$

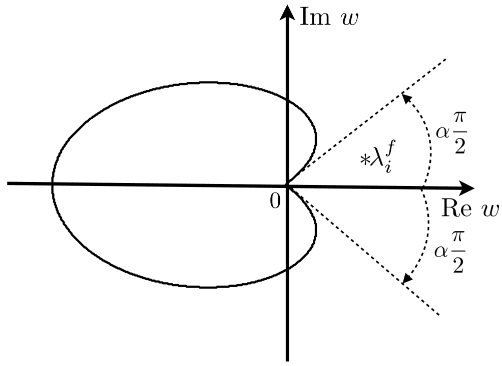

Fig. 7. Concluding on the system stability when φ_i^f is outside the range $\left[\alpha\frac{\pi}{2}, 2\pi - \alpha\frac{\pi}{2}\right]$

λ_i^f of the matrix $A_f = A - I$ instead of using the matrix A, with $\lambda_i^f = \lambda_i - 1$, $i = 1, \ldots, n$. The respective arguments φ_i^f and φ_i are shown in Fig. 5. Now, instead of operation on the reference argument φ_i^v we can operate on the reference one $\varphi_i^f = \arg \lambda_i^f = \arctan(\mathrm{Im}\ \lambda_i^f/\mathrm{Re}\ \lambda_i^f) = \arctan(\mathrm{Im}\ \lambda_i/(\mathrm{Re}\ \lambda_i - 1))$, $i = 1, \ldots, n$, which should be equal to $\varphi_i^w = \varphi_i + \alpha(\pi - \varphi_i)/2$ as obtained before. The reference φ_i, that is $\varphi_i = (2\varphi_i^f - \pi\alpha)/(2 - \alpha)$ can now be obtained from the reference $\varphi_i^f = \varphi_i^w$ in order to be put into the modulus $|w| = |w_i| = \left(2 \sin \frac{\varphi_i}{2}\right)^\alpha$, $i = 1 \ldots, n$. The proof is completed if we note from Theorem 3 that the asymptotic stability is obtained when $|\lambda_i^f| < |w_i|$, $i = 1, \ldots, n$, which should hold true for all eigenvalues λ_i^f whose arguments φ_i^f are within the range $\left[\alpha\frac{\pi}{2}, 2\pi - \alpha\frac{\pi}{2}\right]$ (see Fig. 6). In fact, when any of the arguments φ_i^f is outside that range (see Fig. 7), the system is unstable and the modulus condition in (15) need not be verified. Also note that, by virtue of Theorem 3, the criterion (15) is valid for $\alpha \in (0, 2)$.

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