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On uniform strict minima for vector-valued functions*

by

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Abstract: We introduce uniform strict minima for vector-valued functions and provide conditions for their Lipschitz continuity as functions of linear perturbations. We also investigate regularity properties of subdifferentials for cone convex vector-valued functions.

Keywords: uniform strict minima, K-convex mappings, metric subregularity, domination property

1. Introduction

For scalar-valued functions, the existence of sharp local minima is a crucial property regarding a number of issues, e.g. stability of optimization problems, convergence of algorithms, metric regularity of the subdifferential. Namely, for a real-valued function f, defined on normed space X, a local minimum x_0 is sharp of order $m \geq 1$ if there exist a positive constant $\kappa > 0$ and a neighbourhood V of x_0 such that

$$f(x) \geq f(x_0) + \kappa \|x - x_0\|^m \quad \text{for all } x \in V.$$

In this context, an important property is also constituted by the growth condition. For a real-valued function f, defined on normed vector space X, the growth condition of order $m \geq 1$ holds on a subset $S \subset \text{argmin} f$ if there exist an $\varepsilon > 0$ and a positive constant $\kappa > 0$ such that

$$f(x) \geq \inf f + \kappa d(x, S)^m \quad \text{for all } x \in S_\varepsilon = \bigcup_{s \in S} \mathbb{B}(s, \varepsilon).$$

These properties are used to study metric subregularity and regularity of subdifferentials of convex functions f in Aragón, Artacho and Geoffreoy (2008,

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2014), and to study metric subregularity and regularity of subdifferentials of lsc functions in Drusvyatskiy and Lewis (2013).

Our aim is to pursue a similar approach in the case of vector-valued functions. Main motivation comes from the fact that the vector counterpart of the growth condition, introduced in Bednarczuk (2007), is crucial for the stability of parametric vector optimization problems (see Bednarczuk, 2007, and the references therein). In the case of vector-valued functions, the counterpart of sharp local minima (called strict local minima) has been introduced in Bednarczuk (2002) and Jiménez (2002), and then exploited in Bednarczuk (2007) to study the stability of vector optimization problems.

In this paper we introduce the concept of uniform strict minima (local and global) of order $m \geq 1$ for vector-valued functions and study their continuity properties. Moreover, we provide sufficient conditions for metric subregularity of two kinds of subdifferentials in the case of $K-$convex mappings. These conditions are related to the growth condition for vector-valued functions as defined in Bednarczuk (2007).

The organization of the paper is as follows. In Section 1 we present basic definitions and properties. Section 2 is devoted to the study of strict local minima and their essential properties. In Section 3 we introduce the main concept of the paper, namely the uniform strict local minima of order 2. The notion of local domination property of order $m \geq 1$ is introduced in Section 4.

In Section 5 we investigate Lipschitz continuity of uniform strict local minima under perturbations of the function f of the form $f_a = f - a$, where a is a continuous linear operator from X into Y.

Some basic properties of K-convex mappings and two definitions of subdifferentials for $K-$conex mappings provided by Valadier (1972) and Isac and Postolica (1993), are given in Sections 6 and 7.

In Section 8 we study metric subregularity of subdifferentials, defined in Section 7. We show that the subdifferential for vector-valued $K-$convex functions, as defined in Valadier (1972), is metrically subregular at $(\bar{x}, \bar{a}) \in \mathrm{gph}\partial_I f$ if

$$f_{\bar{a}}(x) \notin f_{\bar{a}}(\bar{x}) + cd^2(x, (\partial_I f)^{-1}(\bar{a}))\mathbb{B}_Y - K \quad \text{for } x \neq \bar{x},\ x \in V \tag{1}$$

for some neighborhood V of $\bar{x}$ and constant $c > 0$. The analogous result for the subdifferential defined in Isac and Postolica (1993) requires an additional assumption. Let us note that in the case of a real-valued function the condition (1) reduces to condition (3.1) from Aragón, Artacho and Geoffroy (2008) and condition (2.1) from Aragón, Artacho and Geoffroy (2014).

2. Strict local minima

Let X and Y be normed spaces and $f : X \to Y$. Let $K \subset Y$ be a closed convex pointed cone in Y. By $\mathbb{B}$ ($\mathbb{B}_Y$) we denote the open unit ball in a normed space (in the normed space Y).

A point $\bar{x} \in X$ is a *local minimum of f with respect to K* if there exists a neighbourhood $V \subset X$ of $\bar{x}$ such that

$$(f(V) - f(\bar{x})) \cap (-K) = \{0\}, \text{ i.e. } f(V) \cap (f(\bar{x}) - K) = \{f(\bar{x})\}.$$

Equivalently, there is no $v \in V$ with $f(v) \neq f(\bar{x})$ such that

$$f(v) \in f(\bar{x}) - K.$$

Yet in other words, there is no $v \in V$ with $f(v) \neq f(\bar{x})$ such that

$$f(v) \leq_K f(\bar{x}),$$

where $y_1 \leq_K y_2$ (or $y_2 \geq_K y_1$) if $y_2 - y_1 \in K$.

A point $\bar{x} \in X$ is a *strict local minimum of order $m \geq 1$ of f with respect to K* (see Jiménez, 2002) if there exist a neighborhood $V \subset X$ of $\bar{x}$ and a constant $\kappa > 0$ such that

$$f(v) \notin f(\bar{x}) + \kappa\|\bar{x} - v\|^m \mathbb{B}_Y - K \text{ for every } v \in V, v \neq \bar{x}. \tag{2}$$

Clearly, each strict local minimum of order m is a local minimum of f, since $f(\bar{x}) - K \subset f(\bar{x}) + \kappa\mathbb{B}\|v - \bar{x}\|^m - K$.

When $Y = \mathbb{R}$ and $K = \mathbb{R}_+ := \{x \in \mathbb{R} \mid x \geq 0\}$, the relation (2) takes the form

$$f(v) \geq f(\bar{x}) + \kappa\|v - \bar{x}\|^m \quad \text{for} \quad v \in V \tag{3}$$

and the strict local minima of order $m \geq 1$ of a function $f : X \to \mathbb{R}$ coincide with sharp local minima of order $m \geq 1$ as defined, e.g., in Chapter 3 of Bonnans and Shapiro (2000).

Proposition 1 *If $\bar{x} \in X$ is a strict local minimum of f of order $m \geq 1$ of f with respect to K, then $\bar{x}$ is a locally unique minimum in the sense that for any $v \in V \cap f(X)$*

$$f(v) = f(\bar{x}) \;\Rightarrow\; v = \bar{x}, \text{ i.e. } f(v) \neq f(\bar{x}) \text{ for } v \neq \bar{x}. \tag{4}$$

Proof By definition of strict local minimum, there exist a constant $\kappa > 0$ and a neighbourhood V of $\bar{x}$ such that for $v \neq \bar{x}$ we have

$$f(v) \notin f(\bar{x}) + \kappa\mathbb{B}\|v - \bar{x}\|^m - K.$$

This implies that $f(v) \neq f(\bar{x})$. □

Remark 1 *Let us note that, in general, there may exist other strict minima x in any neighborhood of $\bar{x}$ with different values $f(x) \neq f(\bar{x})$.*

3. Uniform strict local minima of order 2

Let $f : X \to Y$. Consider tilt (linear) perturbations of f of the form

$$f_a := f - a$$

where $a : X \to Y$ is a continuous linear operator, i.e. $a \in \mathcal{L}(X, Y)$.

Strict local minima are sensitive to the tilt perturbations of f. This is shown by the examples below.

Example 1 *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$, $K = \mathbb{R}^2_+$, $f(x) = (0, x_2{}^2)^T$. The point $\bar{x} = (0,0)^T$ is a strict local minimum of order 2 of f. Let us consider $a : \mathbb{R}^2 \to \mathbb{R}^2$, $\langle a, x\rangle = \begin{bmatrix} 0 & 0 \\ 0 & -r \end{bmatrix} x$ for some $r \neq 0$. The function $f_a(x) := f(x) - \langle a, x\rangle = (0, x_2^2 + rx_2)^T$ has a strict local minimum of order 2 at $x_a = (0, -\frac{r}{2})^T$.*

Proposition 2 *If $\bar{x}$ is a strict local minimum of order 1 of f, then $\bar{x}$ is a strict local minimum of order 1, for all functions f_a with a sufficiently small, i.e. there exist $\delta > 0$ such that $\bar{x}$ is a strict local minimum of order 1 of f_a for all linear operators $a \in \delta\mathbb{B}$.*

Proof It is enough to observe that for any $a \in \mathbb{B}(0, \delta)$, where $\delta < \frac{\kappa}{2}$, and any $v \in X$ we have

$$\langle a, v - \bar{x}\rangle + \frac{\kappa}{2}\|v - \bar{x}\|\mathbb{B}_Y - K \subset \kappa\|v - \bar{x}\|\mathbb{B}_Y - K.$$

Hence, if $f(v) - f(\bar{x}) \notin \kappa\|v - \bar{x}\|\mathbb{B}_Y - K$ for $v \in V$, then $f(v) - f(\bar{x}) \notin \langle a, v - \bar{x}\rangle + \frac{\kappa}{2}\|v - \bar{x}\|\mathbb{B}_Y - K$ for $v \in V$, and finally

$$f_a(v) - f_a(\bar{x}) \notin \kappa\|v - \bar{x}\|\mathbb{B}_Y - K \quad \text{for all} \quad v \in V,$$

which means that $\bar{x}$ is a strict local minimum of order 1 for the function f_a, where $a \in \mathbb{B}(0, \delta)$, with $\delta < \frac{\kappa}{2}$. □

The example below shows that this fact is not true for strict local minima of order 2.

Example 2 *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$, $K = \mathbb{R}^2_+$, $f(x) = (0, x_2{}^2)^T$. The point $\bar{x} = (0,0)^T$ is a strict local minimum of order 2 of f. Let us consider $a : \mathbb{R}^2 \to \mathbb{R}^2$, $\langle a, x\rangle = \begin{bmatrix} -r & 0 \\ 0 & 0 \end{bmatrix} x$ for some $r \neq 0$. The function $f_a(x) := f(x) - \langle a, x\rangle = (rx_1, x_2^2)^T$ has no local minima.*

This motivates the following definition.

Definition 1 *A point $\bar{x} \in X$ is a* uniform strict local minimum of order 2 of f with respect to K *if there exist a neighbourhood $V \subset X$ of $\bar{x}$ and a constant $\kappa > 0$ such that for each $a \in \mathcal{L}(X, Y)$ close to zero in the norm topology there exists $x_a \in V$ (for $a = 0$, $x_a = \bar{x}$) and*

$$f_a(v) \notin f_a(x_a) + \kappa\mathbb{B}_Y\|v - x_a\|^2 - K \quad \textit{for every} \quad v \in V,\ v \neq x_a. \tag{5}$$

The relation (5) means that for any $a \in \delta\mathbb{B}$, where $\delta > 0$ is a positive number, the function f_a has a strict local minimum $x_a \in V$ and, moreover, (2) is satisfied with the same V and $\kappa > 0$ for all $a \in \delta\mathbb{B}$. The word "uniform" in the terminology stresses the fact that all x_a are strict local minima with the same neighbourhood V.

In the case when $X = \mathbb{R}^n$ and $Y = \mathbb{R} \cup \{+\infty, -\infty\}$, the uniform strict local minima of order 2 coincide with the stable strong local minima of order 2 as defined in Drusvyatskiy and Lewis (2013), and the relation (5) takes the form

$$f_a(v) \geq f_a(x_a) + \kappa\|v - x_a\|^2 \quad \text{for every} \quad v \in V. \tag{6}$$

Proposition 3 *If $\bar{x} \in X$ is a uniform strict local minimum of order 2 of f with respect to K, then, for all $a \in \delta\mathbb{B}$, x_a (see Definition 1) is defined uniquely.*

Proof Let us assume that for a certain $a \in \mathcal{L}(X, Y)$ there are x_a, x'_a, $x_a \neq x'_a$, satisfying (5). Hence, for all $v \in V$, $v \neq x'_a$ we have

$$f(v) - \langle a, v\rangle \notin f(x'_a) - \langle a, x'_a\rangle + \kappa\mathbb{B}_Y\|v - x'_a\|^2 - K \tag{7}$$

and for all $v \in V$, $v \neq x_a$, we have

$$f(v) - \langle a, v\rangle \notin f(x_a) - \langle a, x_a\rangle + \kappa\mathbb{B}_Y\|v - x_a\|^2 - K. \tag{8}$$

By putting $v = x_a$ in (7), we get

$$f(x_a) - \langle a, x_a\rangle \notin f(x'_a) - \langle a, x'_a\rangle + \kappa\mathbb{B}_Y\|x'_a - x_a\|^2 - K,$$

and by putting $v = x'_a$ in (8), we get:

$$f(x'_a) - \langle a, x'_a\rangle \notin f(x_a) - \langle a, x_a\rangle + \kappa\mathbb{B}_Y\|x'_a - x_a\|^2 - K.$$

Hence,

$$f(x_a) - \langle a, x_a\rangle \notin f(x_a) - \langle a, x_a\rangle + 2\kappa\mathbb{B}_Y\|x'_a - x_a\|^2 - K,$$

which means that $0 \notin -\mathcal{K}$, a contradiction. □

The examples below illustrate the concept introduced in Definition 1.

Example 3 *Let $f : \mathbb{R} \to \mathbb{R}^2$, $K = \mathbb{R}^2_+$, $f(x) = \begin{cases} (1 + x, 0)^T & x \geq 0 \\ (0, -\frac{1}{x})^T & x < 0 \end{cases}$. The point $\bar{x} = 0$ is a strict local minimum of order $m \geq 1$ of f. Let us take $a : \mathbb{R} \to \mathbb{R}^2$, $V = (-\varepsilon, \varepsilon)$ for some $1 > \varepsilon > 0$, and constant $\kappa > 0$. We can assume that $\langle a, x\rangle = (a_1x, a_2x)^T$ and $\|a\| < \delta$, which means that $\sqrt{a_1^2 + a_2^2} < \delta$. Hence*

$$f_a(x) = \begin{cases} (x + 1 - a_1x, -a_2x)^T & x \geq 0 \\ (-a_1x, -\frac{1}{x} - a_2x)^T & x < 0. \end{cases}$$

We see that a point $x_a = \bar{x} = 0$ is a strict local minimum of f_a. This means that $\bar{x} = 0$ is a uniform strict local minimum of order 2 of f with respect to K.

Example 4 *Let $f : \mathbb{R} \to \mathbb{R}^2$, $K = \mathbb{R}^2_+$, $f(x) = (x^2, \frac{1}{3}x^2)^T$. Let us take $a : \mathbb{R} \to \mathbb{R}^2$, $\langle a, x\rangle = (a_1x, a_2x)^T$, $f_a(x) = (x^2 - a_1x, \frac{1}{3}x^2 - a_2x)^T$. We see that $x_a \neq x_{a'}$ for $a \neq a'$.*

4. Local domination property of order $m \geq 1$

In order to investigate the Lipschitzness of x_a with respect to a, we need the following concepts. Let US_f^m, St_f^m, S_f denote the set of all uniform strict local minima of order $m \geq 1$ of f, the set of all strict local minima of order $m \geq 1$ of f, and the set of all local minima of f, respectively. In the case of $m = 1$ we use the notation US_f, St_f, respectively.

DEFINITION 2 *Let $\bar{x} \in S_f$. We say that* the local domination property of order $m \geq 1$ *holds for f at $\bar{x}$ if there exist a neighbourhood V of $\bar{x}$ and a constant $\kappa > 0$ such that for any $v \in V$*

$$f(v) \in f(\bar{x}) + \kappa \|v - \bar{x}\|^m \mathbb{B}_Y + K,$$

$$\textit{i.e. } \exists\ k_v \in K,\ b_v \in \mathbb{B} \textit{ such that } \ f(v) = f(\bar{x}) + \kappa\|v - \bar{x}\|^m b_v + k_v.$$

DEFINITION 3 *Let $\bar{x} \in US_f^m$. We say that* the uniform local domination property of order $m \geq 1$ *holds for f at $\bar{x}$ if there exists a constant $\kappa > 0$ such that*

$$f_a(v) \in f_a(x_a) + \kappa\|v - x_a\|^m \mathbb{B}_Y + K, \quad \textit{for } v \in V,\ a \in \delta\mathbb{B}$$

i.e. $\exists\ k_v \in K,\ b_v \in \mathbb{B}$ *such that* $f_a(v) = f_a(x_a) + \kappa\|v - x_a\|^m b_v + k_v$.
where x_a, V and $\delta > 0$ are the same as those which follow from the fact that $\bar{x} \in US_f^m$ (see Definition 1).

In other words, the uniform local domination property of order $m \geq 1$ holds for f at $\bar{x} \in US_f^m$ if there exists a constant $\kappa > 0$ such that the local domination property of order $m \geq 1$ holds for all f_a at x_a with the same constant κ and for V and $\delta > 0$ resulting from the fact that $\bar{x} \in US_f^m$.

For scalar-valued functions, the uniform strict local minimizers of order 2 automatically have the uniform local domination property of order 2. For vector-valued functions, the uniform local domination property may not be satisfied as shown by the example below.

EXAMPLE 5 *Let* $f : \mathbb{R} \to \mathbb{R}^2$, $K = \mathbb{R}^2_+$, $f(x) = \begin{cases} (1, \frac{1}{x})^T & x > 0 \\ (1, 0)^T & x = 0 \\ (1, -\frac{1}{x})^T & x < 0 \end{cases}$. *Let us take* $a : \mathbb{R} \to \mathbb{R}^2$, $\langle a, x\rangle = (a_1 x, 0)^T$, $f_a(x) = \begin{cases} (1 - a_1 x, \frac{1}{x})^T & x > 0 \\ (1, 0)^T & x = 0 \\ (1 - a_1 x, -\frac{1}{x})^T & x < 0 \end{cases}$. *The point $x_a = 0$ is a strict minimum of order 2 of f_a, but f_a does not possess the domination property.*

5. The Lipschitz continuity of uniform strict minima

In this section we prove the Lipschitz continuity of x_a as a function of a bounded linear operator $a \in \mathcal{L}(X, Y)$ at $\bar{a} = 0$.

Theorem 1 *Let X and Y be normed spaces. Let $f : X \to Y$ and let $K \subset Y$ be a closed convex pointed cone.*

Let $\bar{x}$ be a uniform strict local minimum of order 2, $\bar{x} \in US^2_f$, with constants $\kappa > 0$ and $\delta > 0$ and let the uniform domination property of order 2 hold for f at $\bar{x}$ with the constant $\frac{\kappa}{2}$ (or $\kappa_0 < \kappa/2$).

Then the mapping $a \to x_a$ is Lipschitz around zero, i.e. there exists a constant $L > 0$ and $\delta > 0$ such that

$$\|x_a - x_{a'}\| \leq L\|a - a'\| \quad \textit{for every} \;\; a, a' \in B(0, \delta). \tag{9}$$

Proof Since $\bar{x}$ is a uniform strict local minimum, there exists a neighborhood V of $\bar{x}$ and a constant $\kappa > 0$ such that for any $a \in B(0, \delta)$ there exists a unique $x_a \in V$ satisfying the relation

$$f_a(v) \notin f_a(x_a) + \kappa\mathbb{B}\|x_a - v\|^2 - K \;\; \text{for all } v \in V, x_a \neq v. \tag{10}$$

Let us note that each x_a is a strict local minimum of order 2 of the function

$$f_a := f - a.$$

Let $a, a' \in B(0, \delta)$. Consider the corresponding strict local minima of order 2 $x_a, x_{a'} \in V$ of f_a, $f_{a'}$. We can assume that $x_a \neq x_{a'}$, since otherwise (9) is satisfied.

By (10)

$$f_a(x_{a'}) \notin f_a(x_a) + \kappa\mathbb{B}\|x_a - x_{a'}\|^2 - K. \tag{11}$$

We have

$$f_a(x_{a'}) = f(x_{a'}) - \langle a, x_{a'}\rangle = f(x_{a'}) - \langle a', x_{a'}\rangle + \langle a' - a, x_{a'}\rangle = f_{a'}(x_{a'}) + \langle a' - a, x_{a'}\rangle.$$

Analogously, $f_a(x_a) = f_{a'}(x_a) + \langle a' - a, x_{a'}\rangle$. Combining this with (11), we get

$$f_{a'}(x_{a'}) + \langle a' - a, x_{a'}\rangle \notin f_{a'}(x_a) + \langle a' - a, x_{a'}\rangle + \kappa\mathbb{B}\|x_a - x_{a'}\|^2 - K \tag{12}$$

and

$$\langle a' - a, x_{a'} - x_a\rangle \notin f_{a'}(x_a) - f_{a'}(x_{a'}) + \kappa\mathbb{B}\|x_a - x_{a'}\|^2 - K. \tag{13}$$

Let us note that we also have

$$\langle a' - a, x_{a'} - x_a\rangle \notin f_a(x_{a'}) - f_a(x_a) + \kappa\mathbb{B}\|x_a - x_{a'}\|^2 - K. \tag{14}$$

Since the uniform local domination property of order 2 holds at $\bar{x}$, we have

$$f_a(v) \in f_a(x_a) + \frac{\kappa}{2}\mathbb{B}_Y\|v - x_a\|^2 + K \;\; \text{for every} \;\; v \in V \tag{15}$$

and

$$f_a(x_{a'}) \in f_a(x_a) + \frac{\kappa}{2}\mathbb{B}_Y\|x_{a'} - x_a\|^2 + K.$$

By (15), there exist $b_a \in \mathbb{B}_Y$ and $k_a \in K$ such that

$$f_a(x_{a'}) - f_a(x_a) = \frac{\kappa}{2}\|x_{a'} - x_a\|^2 b_a + k_a. \tag{16}$$

From (13), by using (16), we get

$$\langle a' - a, x_{a'} - x_a\rangle \notin \frac{\kappa}{2}\|x_{a'} - x_a\|^2 b_a + \kappa\mathbb{B}\|x_a - x_{a'}\|^2 + k_a - K.$$

Clearly, $-K \subset k_a - K$, and consequently

$$\langle a' - a, x_{a'} - x_a\rangle \notin \kappa\|x_{a'} - x_a\|^2 b_a + \kappa\mathbb{B}\|x_a - x_{a'}\|^2 - K = b_a^1 - K. \tag{17}$$

Moreover,

$$\frac{\kappa}{2}\|x_{a'} - x_a\|^2\mathbb{B}_Y \subset \kappa\|x_{a'} - x_a\|^2 b_a + \kappa\mathbb{B}\|x_a - x_{a'}\|^2,$$

since, if $x \in \frac{\kappa}{2}\|x_{a'} - x_a\|^2\mathbb{B}_Y$, then

$$\|x\| < \frac{\kappa}{2}\|x_{a'} - x_a\|^2$$

and

$$\|b_a^1 - x\| < \|b_a^1\| + \|x\| < \frac{\kappa}{2}\|x_{a'} - x_a\|^2 + \frac{\kappa}{2}\|x_{a'} - x_a\|^2.$$

Consequently,

$$\langle a' - a, x_{a'} - x_a\rangle \notin \frac{\kappa}{2}\|x_{a'} - x_a\|^2\mathbb{B}_Y. \tag{18}$$

By (18),

$$\|\langle a' - a, x_{a'} - x_a\rangle\| \geq \frac{\kappa}{2}\|x_{a'} - x_a\|^2,$$

which gives

$$\|x_{a'} - x_a\| \leq \frac{2}{\kappa}\|a' - a\| \text{ for every } \ a, a' \in \mathbb{B}(0,\delta). \qquad \square$$

It follows from the proof that instead of taking $\kappa/2$ in the uniform local domination property of order 2 of f at $\bar{x}$ we can take any constant $\kappa_0 \leq \frac{\kappa}{2}$.

In the case of real-valued functions, the uniform strict local minima have been investigated in Drusvyatskiy and Lewis (2013). In Proposition 2.2 of Drusvyatskiy and Lewis (2013) it is proved that uniformly strict local minima are locally Lipschitz around zero. Let us observe that Theorem 1 reduces to Proposition 2.2. of Drusvyatskiy and Lewis (2013) in the real-valued case.

Corollary 1 *Let $\bar{x}$ be a uniform strict local minimum of order 2 of f with constant $\kappa > 0$ and neighborhood V. Under the assumptions of Theorem 1, the point $\bar{x}$ is a uniform strict local minimum of order 2 of f with any neighborhood $V_1 \subset V$.*

Proof By Theorem 1, there exist $\delta > 0$ and a constant $L > 0$ such that

$$\|x_a - \bar{x}\| \leq L\|a\|, \quad \text{for } a \in \mathbb{B}(0,\delta), \tag{19}$$

where x_a is a strict local minimum of f_a. Let $V_1 \subset V$ be a neighborhood of $\bar{x}$. Hence, $\mathbb{B}(\bar{x},\varepsilon) \subset V_1$ for some $\varepsilon > 0$. Now, by taking $\delta < \frac{\varepsilon}{L}$ from (19) we obtain that $\|x_a - \bar{x}\| < \varepsilon$. □

6. K-convex mappings

Let X and Y be linear spaces. Let $K \subset Y$ be a closed convex pointed cone in Y.

Let $\infty \notin Y$ denote the greatest element in the sense that $y \leq_K \infty$ and denote $Y^\bullet = Y \cup \{\infty\}$. We consider that $y + \infty = \infty$, $\infty + \infty = \infty$, $\lambda\infty = \infty$ for all $\lambda \in \mathbb{R}_+$.

Let $f : X \to Y^\bullet$. The domain of f is $\operatorname{dom} f := \{x \in X \ : \ f(x) \in Y\}$.

Definition 4 *We say that $f : X \to Y^\bullet$ is K-convex if $\forall\, x_1, x_2 \in X$, $\lambda \in [0,1]$,*

$$f(\lambda x_1 + (1-\lambda)x_2) \leq_K \lambda f(x_1) + (1-\lambda)f(x_2).$$

Equivalently, f is K-convex if $\forall\, x_1, x_2 \in X\, f$, $\lambda \in [0,1]$,

$$f(\lambda x_1 + (1-\lambda)x_2) \in \lambda f(x_1) + (1-\lambda)f(x_2) - K.$$

As in the scalar case, any local minimum of a $K-$ convex function is a global minimum in the sense that there is no $x \in X$, $f(x) \neq f(x_0)$ such that

$$f(x) \in f(x_0) - K.$$

Below, we prove the similar fact for strict minima.

Proposition 4 *Let X and Y be normed spaces. Let $f : X \to Y$ be $K-$convex. If $\bar{x}$ is a strict local minimum of order $m \geq 1$ of f with constant $\kappa > 0$, then $\bar{x}$ is a global strict minimum of order $m \geq 1$ of f with constant $\kappa > 0$ in the sense that*

$$f(x) \notin f(\bar{x}) + \kappa\|x - \bar{x}\|^m \mathbb{B} - K \quad \textit{for all} \quad x \in X. \tag{20}$$

Proof Suppose that (20) does not hold, i.e. there exists a $z \in X$ such that

$$f(z) = f(\bar{x}) + \kappa\|z - \bar{x}\|^m b - k, \quad \text{where} \quad b \in \mathbb{B},\ k \in K. \tag{21}$$

Consider $w(\lambda) := \lambda z + (1-\lambda)\bar{x}$, $0 \leq \lambda \leq 1$. By the $K-$convexity of f we have

$$f(\lambda z + (1-\lambda)\bar{x}) = \lambda f(z) + (1-\lambda)f(\bar{x}) - k_\lambda, \quad k_\lambda \in K.$$

Hence, by (21), for all $0 \le \lambda \le 1$ we have

$$\begin{aligned} f(\lambda z + (1-\lambda)\bar{x}) &= \lambda f(z) + (1-\lambda) f(\bar{x}) - k_\lambda \\ &= f(\bar{x}) + \lambda(f(z) - f(\bar{x})) - k_\lambda \\ &= f(\bar{x}) + \lambda(\kappa\|z-\bar{x}\|^m b - k) - k_\lambda \\ &= f(\bar{x}) + \lambda\kappa\|z-\bar{x}\|^m b - \lambda k - k_\lambda \\ &= f(\bar{x}) + \lambda\kappa\|z-\bar{x}\|^m b - \bar{k}_\lambda, \text{ where } \bar{k}_\lambda = \lambda k - k_\lambda \in -K \\ &= f(\bar{x}) + \kappa\|\lambda(z-\bar{x})\|^m b - \bar{k}_\lambda \\ &= f(\bar{x}) + \kappa\|(\lambda z + (1-\lambda)\bar{x}) - \bar{x}\|^m b - \bar{k}_\lambda. \end{aligned}$$

This contradicts the fact that $\bar{x}$ is a strict local minimum of order m of f. □

Corollary 2 *Let $f : X \to Y$ be a $K-$convex function f. If $\bar{x}$ is a uniform local minimum of order $m \ge 1$ of f with constants $\kappa > 0$ and $\delta > 0$ and a neighbourhood V of $\bar{x}$, then $\bar{x}$ is a uniform (global) minimum of order $m \ge 1$ of f with constants $\kappa > 0$ and $\delta > 0$, i.e. for $a \in \mathbb{B}(0,\delta)$ there exists $x_a \in V$ such that*

$$f(x) - \langle a, x\rangle \notin f(x_a) - \langle a, x_a\rangle + \kappa\|x - x_a\|^m \mathbb{B} - K \quad \text{for all} \quad x \in X. \tag{22}$$

Proof As already observed, any $x_a \in V$ satisfying (22) is a strict local minimum of the function f_a. Since f_a is a $K-$convex function, by Proposition 4, x_a is a global strict minimum of f_a, hence (22) holds. □

7. Subgradients of vector-valued functions

Let $f : X \to Y^\bullet$ be a K-convex vector-valued function, taking values in a Banach space Y. Definitions of subdifferentials for K-convex vector-valued functions at a point have been proposed by Valadier (1972) and Isac and Postolica (1993), and investigated in Papageorgiu (1983), Stamate (2003), and Zălinescu et al. (2003).

Definition 5 *(Valadier, 1972; Zălinescu et al., 2003) A linear continuous operator $a \in \mathcal{L}(X,Y)$ is an* ideal subgradient *of f at $\bar{x} \in domf$ if*

$$f(x) \ge_K f(\bar{x}) + \langle a, x - \bar{x}\rangle \quad \text{for all } x \in X \tag{23}$$

By $\partial_I f(\bar{x})$ we denote the set of all ideal subgradients of f (23) at $\bar{x}$.

Definition 6 *(Isac and Postolica, 1993) A linear continuous operator $a \in \mathcal{L}(X,Y)$ is a* Pareto-subgradient *of f at $\bar{x} \in domf$ if*

$$f(x) \notin f(\bar{x}) + \langle a, x - \bar{x}\rangle - K \quad \text{for all } x \in X \text{ such that } f_a(x) \ne f_a(\bar{x}). \tag{24}$$

By $\partial_P f(\bar{x})$ we denote the set of all Pareto-subgradients of f at $\bar{x}$.

The terminology above is motivated by the following proposition:

PROPOSITION 5 *Let X and Y be normed spaces and let $K \subset Y$ be a closed convex and pointed cone in Y. Let $f : X \to Y$ be $K-$convex. The following relations hold:*

(i) $\bar{x} \in X$ is a (global) minimum of f if and only if $0 \in \partial_P f(\bar{x})$,

(ii) $\bar{x} \in X$ is a (global) ideal minimum of f in the sense that $f(x) \geq_K f(\bar{x})$ for all $x \in X$ if and only if $0 \in \partial_I f(\bar{x})$.

PROOF The proof follows immediately from the definitions. □

COROLLARY 3 *In case (ii) $\bar{x}$ is the unique minimum in the sense that if there exists another ideal minimum x_1, then $f(\bar{x}) = f(x_1)$.*
If K is not pointed, which means that $K \cap (-K) = linK \neq \{0\}$ and $\bar{x}$, x_1 are ideal minima, then $f(\bar{x}) - f(x_1) \in linK$, where $linK$ is the lineality space of K.

PROPOSITION 6 $a \in \partial_I f(\bar{x}) \Rightarrow a \in \partial_P f(\bar{x})$.

PROOF Let us take some $a \in \partial_I f(\bar{x})$, which means that $f(x) - f(\bar{x}) + \langle a, x - \bar{x}\rangle = k$ for some $k \in K$. We have that $f(x) - f(\bar{x}) + \langle a, x - \bar{x}\rangle \notin -K$, because $K \cap -K = \{0\}$. □

Clearly, in general, we do not have the inclusion $\partial_P f(\bar{x}) \subset \partial_I f(\bar{x})$, as shown by the example below.

EXAMPLE 6 *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x) = \begin{cases} (0,0)^T & x = (0,0)^T \\ (1, x_2^2)^T & x \neq (0,0)^T \end{cases}$. Let us take $a : \mathbb{R}^2 \to \mathbb{R}^2$, $\langle a, x\rangle = (0, \frac{1}{10}x_2)^T$.*

$$f_a(x) = \begin{cases} (0,0)^T & x = (0,0)^T \\ (1, x_2^2 - \frac{1}{10}x_2)^T & x \neq (0,0)^T \end{cases}.$$

We have $a \in \partial_P f(0)$ and $a \notin \partial_I f(0)$.

PROPOSITION 7 *$\partial_I f(\bar{x})$ is a convex set.*

PROOF Let us take some $\lambda \in [0,1]$ and $a_1, a_2 \in \partial_I f(\bar{x})$. For all $x \in X$

$$f(x) - f(\bar{x}) - \langle a_1, x - \bar{x}\rangle = k_1, \;\; f(x) - f(\bar{x}) - \langle a_2, x - \bar{x}\rangle = k_2,$$

where $k_1, k_2 \in K$. Since $k = \lambda k_1 + (1-\lambda)k_2 \in K$, we have

$f(x) - f(\bar{x}) - \langle \lambda a_1 + (1-\lambda)a_2, x - \bar{x}\rangle = k.$ □

On the other hand, $\partial_P f(\bar{x})$ needs not to be a convex set, as is shown by the example below.

EXAMPLE 7 *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$,*

$$f(x) = \begin{cases} 0 & x = 0 \\ (1 + x_1^2, 1 + x_2^2)^T & x \neq 0 \end{cases}.$$

Let us take linear operators $a, \bar{a} : \mathbb{R}^2 \to \mathbb{R}^2$

$$a = \begin{bmatrix} 10 & 0 \\ 0 & 0 \end{bmatrix} \qquad \bar{a} = \begin{bmatrix} 0 & 0 \\ 0 & 10 \end{bmatrix}.$$

We have $f_a(x) = (1 + x_1^2 - 10x_1, 1 + x_2^2)^T$, $f_{\bar{a}}(x) = (1 + x_1^2, 1 + x_2^2 - 10x_2)^T$. We can see that $a, \bar{a} \in \partial_P f(0)$, but $\frac{1}{2}a + \frac{1}{2}\bar{a} \notin \partial_P f(0)$, which means that $\partial_P f(0)$ is not a convex set.

8. Metric regularity of subdifferentials

In this section we investigate the regularity properties of the set-valued mappings ideal and Pareto subdifferentials. More precisely, we investigate the relationships between metric subregularity of the subdifferential of f around point $(\bar{x}, \bar{a})$ and the behavior of the function f around $\bar{x}$. For any subset C of a normed space, we put $d(x, C) := \inf_{y \in C} \|x - y\|$ and $\operatorname{gph} \partial f := \{(x, a) | a \in \partial f(x)\}$.

Definition 7 *A mapping $F : X \rightrightarrows Y$ is* metrically subregular *at $\bar{x}$ for $\bar{y}$, $\bar{y} \in F(\bar{x})$, if there is a positive constant κ along with neighborhoods V of $\bar{x}$ and U of $\bar{y}$ such that*

$$d(x, F^{-1}(\bar{y})) \le kd(\bar{y}, F(x) \cap U) \quad \textit{for all } x \in V. \tag{25}$$

The following theorem provides the sufficient conditions for metric subregularity for subdifferential.

Theorem 2 *Let X and Y be normed spaces and K be a closed pointed cone in Y. Let $f : X \to Y$ be a K-convex mapping. Let $\bar{x} \in X$ and $\bar{a} \in \mathcal{L}(X, Y)$ be such that $\bar{a} \in \partial_I f(\bar{x})$.*

Then, $\partial_I f$ is metrically subregular at $\bar{x}$ for $\bar{a}$ if there exist a neighborhood V of $\bar{x}$ and a positive constant c such that

$$f(v) \notin f(\bar{x}) + \langle \bar{a}, v - \bar{x} \rangle + cd^2(v, (\partial_I f)^{-1}(\bar{a}))\mathbb{B}_Y - K \quad \textit{for all } v \in V,\ v \neq \bar{x}. \tag{26}$$

Proof Assume that (26) holds. Consider any $v \in V$, $v \neq \bar{x}$ and take any $a \in \partial_I f(v)$ (if $\partial_I f(v) = \emptyset$, there is nothing to prove).

Choose $\varepsilon > 0$. Since $(\partial_I f)^{-1}(\bar{a}) \neq \emptyset$, there is $x_\varepsilon \in (\partial_I f)^{-1}(\bar{a})$ such that

$$\|v - x_\varepsilon\| \le d(v, (\partial_I f)^{-1}(\bar{a})) + \varepsilon. \tag{27}$$

We have

$$f(w) \ge_K f(x_\varepsilon) + \langle \bar{a}, w - x_\varepsilon \rangle \quad \text{for all } w \in X.$$

For $w = \bar{x}$, there is $k_1 \in K$, and we have

$$f(\bar{x}) - f(x_\varepsilon) = \langle \bar{a}, \bar{x} - x_\varepsilon \rangle + k_1.$$

Since $a \in \partial_I f(v)$, $f(w) \ge_K f(v) + \langle a, w - v \rangle$ for $w \in X$, and there exists $k_2 \in K$ such that

$$\langle a, v - x_\varepsilon \rangle = f(v) - f(x_\varepsilon) + k_2.$$

Hence

$$\begin{aligned}\langle a-\bar{a}, v-x_\varepsilon\rangle &= \langle a, v-x_\varepsilon\rangle - \langle \bar{a}, v-\bar{x}\rangle - \langle \bar{a}, \bar{x}-x_\varepsilon\rangle \\ &= f(v) - f(\bar{x}) - \langle \bar{a}, v-\bar{x}\rangle + k_1 + k_2.\end{aligned} \tag{28}$$

Since $v \in V$, by (26),

$$f(v) - f(\bar{x}) - \langle \bar{a}, v-\bar{x}\rangle \notin cd^2(v, (\partial_I f)^{-1}(\bar{a}))\mathbb{B}_Y - K.$$

Since

$$\begin{aligned}cd^2(v, (\partial_I f)^{-1}(\bar{a}))\mathbb{B}_Y &\subset cd^2(v, (\partial_I f)^{-1}(\bar{a}))\mathbb{B}_Y - K \\ &\subset cd^2(v, (\partial_I f)^{-1}(\bar{a}))\mathbb{B}_Y - K + k_1 + k_2,\end{aligned}$$

by (28), we get

$$\langle a-\bar{a}, v-x_\varepsilon\rangle \notin cd^2(v, (\partial_I f)^{-1}(\bar{a}))\mathbb{B}_Y.$$

From this and by (27),

$$\|a-\bar{a}\|(d(v, (\partial_I f)^{-1}(\bar{a})) + \varepsilon) \geq \|a-\bar{a}\|\|v-x_\varepsilon\|$$
$$\geq \|\langle a-\bar{a}, v-x_\varepsilon\rangle\| \geq cd^2(v, (\partial_I f)^{-1}(\bar{a})).$$

Thus, passing to the limit when ε goes to zero, we obtain

$$d^2(v, (\partial_I f)^{-1}(\bar{a})) \leq \frac{1}{c}\|a-\bar{a}\| d(v, (\partial_I f^{-1})(\bar{a})). \tag{29}$$

If $d(v, (\partial_I f)^{-1}(\bar{a})) = 0$, then $v \in (\partial_I f)^{-1}(\bar{a})$ and $d(\bar{a}, \partial_I f(v)) = 0$ and the conclusion follows. If $d(v, (\partial_I f)^{-1}(\bar{a})) \neq 0$, then, by (29),

$$d(v, (\partial_I f)^{-1}(\bar{a})) \leq \frac{1}{c}\|a-\bar{a}\|,$$

and, since $a \in \partial_I f(v)$ is chosen arbitrarily, we obtain

$$d(v, (\partial_I f)^{-1}(\bar{a})) \leq \frac{1}{c} d(\bar{a}, \partial_I f(v)).$$

Since this inequality holds for all U such that $\bar{a} \in U$, we get that $\partial_I f$ is metrically subregular at $\bar{x}$ for $\bar{a}$. □

The above theorem and condition (26) reduce to Theorem 3.3 and condition 3.1 of Aragón, Artacho and Geoffroy (2014).

Theorem 2 allows for providing the sufficient conditions for metric subregularity in terms of strict local minima.

Corollary 4 *Let X and Y be normed spaces and K be a closed convex pointed cone in Y. Let $f : X \to Y$ be a K-convex mapping, $\bar{x} \in X$, $\bar{a} \in \mathcal{L}(X, Y)$, $\bar{a} \in \partial_I f(\bar{x})$. If $x_{\bar{a}} \in St^2_{f_{\bar{a}}}$, i.e. there exist a neighborhood V of $x_{\bar{a}}$ and a positive constant $c > 0$ such that*

$$f_{\bar{a}}(v) \notin f_{\bar{a}}(x_{\bar{a}}) + c\|v - x_{\bar{a}}\|^2\mathbb{B}_Y - K \textit{ for all } v \in V,\ v \neq x_{\bar{a}}, \tag{30}$$

then $\partial_I f$ is metrically subregular at $\bar{x}$ for $\bar{a}$.

Proof By (30),

$$f_{\bar{a}}(v) \notin f_{\bar{a}}(\bar{x}) + [f_{\bar{a}}(x_{\bar{a}}) - f_{\bar{a}}(\bar{x})] + c\|v - x_{\bar{a}}\|^2 \mathbb{B}_Y - K \quad \forall v \in V,\ v \neq x_{\bar{a}}. \quad (31)$$

Since $\bar{a} \in \partial_I f(\bar{x})$, we have

$$f_{\bar{a}}(w) - f_{\bar{a}}(\bar{x}) = k_w \in K \quad \text{for all} \quad w \in X.$$

By Proposition 5, $x_{\bar{a}}$ is a global ideal minimum of $f_{\bar{a}}$, there must be $f_{\bar{a}}(x_{\bar{a}}) = f_{\bar{a}}(\bar{x})$, and since $x_{\bar{a}}$ is a global strict minimum of order 2 of $f_{\bar{a}}$, there must be $x_{\bar{a}} = \bar{x}$. Consequently, by (31),

$$f_{\bar{a}}(v) \notin f_{\bar{a}}(\bar{x}) + c\|v - \bar{x}\|^2 \mathbb{B}_Y - K \quad \forall v \in V,\ v \neq \bar{x}. \quad (32)$$

Since $d(v, (\partial_I f)^{-1}(\bar{a})) \leq \|v - \bar{x}\|$, by (32),

$$f_{\bar{a}}(v) \notin f_{\bar{a}}(\bar{x}) + cd^2(v, (\partial_I f)^{-1}(\bar{a}))\mathbb{B}_Y - K \quad \forall v \in V,\ v \neq \bar{x}. \quad (33)$$

By Theorem 2, the conclusion follows. □

For the Pareto subdifferential the following result holds:

Theorem 3 *Let X and Y be normed spaces and K be a closed pointed cone in Y. Let $f : X \to Y$ be a K-convex mapping. Let $\bar{x} \in X$ and $\bar{a} \in \mathcal{L}(X, Y)$ be such that $\bar{a} \in \partial_P f(\bar{x})$.*

Then, $\partial_P f$ is metrically subregular at $\bar{x}$ for $\bar{a}$ if the following two conditions hold:

(i) there exist a neighborhood V of $\bar{x}$ and a positive constant $c > 0$ such that

$$f(v) \notin f(\bar{x}) + \langle \bar{a}, v - \bar{x}\rangle + cd^2(v, (\partial_P f)^{-1}(\bar{a}))\mathbb{B}_Y - K \quad \textit{for all} \quad v \in V,\, v \neq \bar{x}, \quad (34)$$

(ii) there exists a neighbourhood U of $\bar{a}$ such that for all $a \in U$ we have

$$f_a(w) \in f_a(v) + \frac{c}{2} d^2(v, (\partial_P f)^{-1}(\bar{a}))\mathbb{B}_Y + K \ \textit{ for all } w \in (\partial_P f)^{-1}(\bar{a})),\ v \in S_{f_a}. \quad (35)$$

Proof We need to show that that there exist a constant $M > 0$ and a neighbourhood V of $\bar{x}$ and a neighbourhood U of $\bar{a}$ such that

$$d(v, (\partial_P f)^{-1}(\bar{a})) \leq M \cdot d(\bar{a}, \partial_P f(v) \cap U).$$

Assume that (34) and (35) hold. Consider any $v \in V$, $v \neq \bar{x}$ and take any $a \in \partial_P f(v) \cap U$ (if $\partial_P f(v) \cap U = \emptyset$, there is nothing to prove). Since $a \in \partial_P f(v)$, we have

$$f(w) \notin f(v) + \langle a, w - v\rangle - K \text{ for all } w \in X, \quad f_a(w) \neq f_a(v),$$

which means that $v \in Sf_a$.
Choose $\varepsilon > 0$. Since $(\partial_P f)^{-1}(\bar{a}) \neq \emptyset$, there is $x_\varepsilon \in (\partial_P f)^{-1}(\bar{a})$ such that

$$\|v - x_\varepsilon\| \leq d(v, (\partial_P f)^{-1}(\bar{a})) + \varepsilon. \tag{36}$$

By purely algebraic manipulations, we have

$$\begin{aligned} &\langle a, v - x_\varepsilon\rangle - \langle \bar{a}, v - \bar{x}\rangle - \langle \bar{a}, \bar{x} - x_\varepsilon\rangle = \\ &= [f(v) - f(\bar{x}) - \langle \bar{a}, v - \bar{x}\rangle] + [f(x_\varepsilon) - f(v) + \langle a, v - x_\varepsilon\rangle] + \\ &+ [f(\bar{x}) - f(x_\varepsilon) - \langle \bar{a}, \bar{x} - x_\varepsilon\rangle]. \end{aligned} \tag{37}$$

Since $\bar{a} \in \partial_P f(\bar{x})$ and $\bar{a} \in \partial_P f(x_\varepsilon)$, we have $\bar{x} \in Sf_{\bar{a}}$, and $x_\varepsilon \in Sf_{\bar{a}}$ and $d(x_\varepsilon, (\partial_P f)^{-1}(\bar{a})) = 0$. Hence, by (35), there must be

$$f_{\bar{a}}(\bar{x}) = f_{\bar{a}}(x_\varepsilon)$$

and the relation (37) takes the form

$$\begin{aligned} &\langle a, v - x_\varepsilon\rangle - \langle \bar{a}, v - \bar{x}\rangle - \langle \bar{a}, \bar{x} - x_\varepsilon\rangle = \\ &= [f(v) - f(\bar{x}) - \langle \bar{a}, v - \bar{x}\rangle] + [f(x_\varepsilon) - f(v) + \langle a, v - x_\varepsilon\rangle]. \end{aligned} \tag{38}$$

Since

$$\langle a - \bar{a}, v - x_\varepsilon\rangle = \langle a, v - x_\varepsilon\rangle - \langle \bar{a}, v - \bar{x}\rangle - \langle \bar{a}, \bar{x} - x_\varepsilon\rangle,$$

and $v \in V$, by (34),

$$f(v) - f(\bar{x}) - \langle \bar{a}, v - \bar{x}\rangle \notin cd^2(v, (\partial_I f)^{-1}(\bar{a}))\mathbb{B}_Y - K,$$

and so, from (38) we get

$$\langle a - \bar{a}, v - x_\varepsilon\rangle \notin cd^2(v, (\partial_I f)^{-1}(\bar{a}))\mathbb{B}_Y + f(x_\varepsilon) - f(v) + \langle a, v - x_\varepsilon\rangle - K.$$

Since $a \in U$ and $v \in Sf_a$, by (35) we have

$$f_a(x_\varepsilon) = f_a(v) + \frac{c}{2} d^2(v, (\partial_P f)^{-1}(\bar{a}))b + k$$

for some $b \in \mathbb{B}_Y$ and $k \in K$. Since $f_a(\cdot) = f(\cdot) - \langle a, \cdot\rangle$, for $a \in U$ we get

$$\langle a - \bar{a}, v - x_\varepsilon\rangle \notin \frac{c}{2} d^2(v, (\partial_I f)^{-1}(\bar{a}))b + k + cd^2(v, (\partial_I f)^{-1}(\bar{a}))\mathbb{B}_Y - K.$$

Since

$$\frac{c}{2} d^2(v, (\partial_I f)^{-1}(\bar{a}))\mathbb{B}_Y \subset \frac{c}{2} d^2(v, (\partial_I f)^{-1}(\bar{a}))b + cd^2(v, (\partial_I f)^{-1}(\bar{a}))\mathbb{B}_Y,$$

for $a \in U$ we get

$$\langle a - \bar{a}, v - x_\varepsilon\rangle \notin \frac{c}{2} d^2(v, (\partial_I f)^{-1}(\bar{a}))\mathbb{B}_Y.$$

From this, and by (36), for $a \in U$ we get

$$\|a-\bar{a}\|(d(v,(\partial_P f)^{-1}(\bar{a}))+\varepsilon) \geq \|a-\bar{a}\|\|v-x_\varepsilon\|$$
$$\geq \|\langle a-\bar{a}, v-x_\varepsilon\rangle\| \geq \frac{c}{2}d^2(v,(\partial_P f)^{-1}(\bar{a})).$$

Thus, passing to the limit when ε goes to zero, we obtain

$$d^2(v,(\partial_P f)^{-1}(\bar{a})) \leq \frac{2}{c}\|a-\bar{a}\|d(v,(\partial_P f^{-1})(\bar{a})). \tag{39}$$

If $d(v,(\partial_P f)^{-1}(\bar{a}))=0$, then $v \in (\partial_P f)^{-1}(\bar{a})$ and $d(\bar{a},\partial_P f(v))=0$, and the conclusion follows. If $d(v,(\partial_I f)^{-1}(\bar{a})) \neq 0$, then, by (39),

$$(d(v,(\partial_P f)^{-1}(\bar{a})) \leq \frac{2}{c}\|a-\bar{a}\|,$$

and since $a \in \partial_P f(v) \cap U$ is chosen arbitrarily, we obtain

$$d(v,(\partial_P f)^{-1}(\bar{a})) \leq \frac{2}{c}d(\bar{a},\partial_P f(v)\cap U) \quad \text{for} \quad v \in V,$$

which proves that $\partial_P f$ is metrically subregular at $\bar{x}$ for $\bar{a}$ with $M=\frac{c}{2}$. $\square$

By Proposition 5, for a $K-$convex function f we have

$$(\partial_P f)^{-1}(0)=S,$$

where S is the set of all global minima of f. Hence, for $\bar{a}=0$ the formula (34) takes the form

$$f(v) \notin f(\bar{x})+cd^2(v,S)\mathbb{B}_Y-K \quad for \quad all \quad v \in V,\ v \neq \bar{x}. \tag{40}$$

Condition (40) is the quadratic growth condition for f at $\bar{x}$, as defined in Bednarczuk (2007). In the scalar case, the analogous relation was noted in Aragón Artacho and Geoffroy (2008).

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