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CHARACTERISTIC EQUATIONS AND STABILITY OF LINEAR SYSTEMS WITH INVERSE STATE MATRICES

The inverse of Frobenius matrices and the characteristic equations of the inverse systems are investigated. It is shown that the inverse system of continuous-time linear system is asymptotically stable if and only if the standard system is asymptotically stable and the inverse system of discrete-time linear system is asymptotically stable if and only if the standard system is unstable. The considerations are illustrated by numerical examples.

INTRODUCTION

The linear systems and control systems have been the classical field of research and they have been considered in many books [1], [6], [7], [9]-[11]. The inverse systems of linear systems have been analyzed in [5].

The problem of stability for standard and positive linear systems has been considered in papers [2], [3], [8] and monographs [6], [7].

In this note the stability of continuous-time and discrete-time linear systems with inverse state matrices and the inverse of the Frobenius matrices, the characteristic equations of the inverse systems are investigated.

The paper is organized as follows. In section 2 some preliminaries concerning the standard autonomous continuous-time and discrete-time linear systems are recalled. The inverse of the Frobenius matrices are given in section 3. The characteristic equations of the inverse systems are analysed in section 4. In section 5 the stability of continuous-time and discrete-time linear systems with inverse state matrices is investigated. Concluding remarks are given in section 6.

The following notation will be used: $\Re$ - the set of real numbers, $\Re^{n\times m}$ - the set of $n\times m$ real matrices and $\Re^n = \Re^{n\times 1}$, Z_+ - the set of nonnegative integers, I_n - the $n\times n$ identity matrix.

1. PRELIMINARIES

Consider the standard autonomous continuous-time linear system

$$\dot{x}(t) = Ax(t),\ t \ge 0, \tag{2.1}$$

where $x(t)\in\Re^n$ is the state vector, $\dot{x}(t)=\frac{dx(t)}{dt}$ and $A\in\Re^{n\times n}$. It is assumed that the matrix A is nonsingular, i.e. $\det A \ne 0$.

Definition 2.1: The system

$$\dot{\bar{x}}(t) = \bar{A}\bar{x}(t),\ \bar{A} = A^{-1} \tag{2.2}$$

is called the inverse system of the system (2.1).

The inverse system (2.2) exists if and only if the matrix A of the system (2.1) is nonsingular.

The continuous-time linear system (2.1) is called asymptotically stable if

$$\lim_{t\to\infty} x(t) = 0 \text{ for any initial conditions } x_0 \in \Re^n. \tag{2.3}$$

In a similar way we define the asymptotic stability of the inverse system (2.2).

Now let us consider the standard discrete-time autonomous linear system

$$x_{i+1} = Ax_i,\ i\in Z_+ = \{0,1,...\}, \tag{2.4}$$

where $x_i \in \Re^n$ is the state vector and $A\in\Re^{n\times n}$. It is assumed that the matrix A is nonsingular, i.e. $\det A \ne 0$.

Definition 2.2: The system

$$\bar{x}_{i+1} = \bar{A}\bar{x}_i,\ \bar{A} = A^{-1} \tag{2.5}$$

is called the inverse system of the system (2.4).

The inverse system (2.5) exists if and only if the matrix A of the system (2.4) is nonsingular.

The discrete-time linear system (2.4) is called asymptotically stable if

$$\lim_{i\to\infty} x_i = 0 \text{ for any initial conditions } x_0 \in \Re^n. \tag{2.6}$$

In a similar way we define the asymptotic stability of the inverse system (2.5).

Theorem 2.1: Let λ_1, λ_2,..., λ_n be the eigenvalues of the matrix $A\in\Re^{n\times n}$, i.e.

$$\det[I_n\lambda - A] = (\lambda-\lambda_1)(\lambda-\lambda_2)...(\lambda-\lambda_n) \tag{2.7}$$

and $f(\lambda)$ be well-defined on the spectrum $\sigma_A = \{\lambda_1,\lambda_2,...,\lambda_n\}$ of the matrix A, i.e. $f(\lambda_k)$ are finite for $k=1,...,n$. Then $f(\lambda_k)$, $k=1,...,n$ are the eigenvalues of the matrix $f(A)$.

Proof: The proof is given in [4].

In particular case we have the following.

Theorem 2.2: If $\lambda_k = \alpha_k + j\beta_k$, $k = 1,...,n$ are the nonzero eigenvalues (not necessary distinct) of the matrix $A \in \Re^{n\times n}$ then $\bar{\lambda}_k = \lambda_k^{-1}$, $k = 1,...,n$ are the eigenvalues of the inverse matrix $\bar{A} = A^{-1}$.

2. INVERSE MATRICES OF THE FROBENIUS MATRICES

In this section the inverse matrices of the matrices in Frobenius canonical form will be given.

Theorem 3.1: The inverse matrix A_1^{-1} of the Frobenius matrix

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \quad (3.1)$$

has the form

$$A_1^{-1} = \begin{bmatrix} -\frac{a_1}{a_0} & -\frac{a_2}{a_0} & -\frac{a_3}{a_0} & \cdots & -\frac{a_{n-1}}{a_0} & -\frac{1}{a_0} \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}. \quad (3.2)$$

Proof: Using (3.1) and (3.2) we obtain

$$A_1 A_1^{-1} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \times \begin{bmatrix} -\frac{a_1}{a_0} & -\frac{a_2}{a_0} & -\frac{a_3}{a_0} & \cdots & -\frac{a_{n-1}}{a_0} & -\frac{1}{a_0} \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = I_n. \quad (3.3)$$

This completes the proof. □

In a similar way it is easy to show that the inverse matrix A_2^{-1} of

$$A_2 = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix} \quad (3.4)$$

has the form

$$A_2^{-1} = \begin{bmatrix} -\frac{a_1}{a_0} & 1 & 0 & \cdots & 0 \\ -\frac{a_2}{a_0} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{a_{n-1}}{a_0} & 0 & 0 & \cdots & 1 \\ -\frac{1}{a_0} & 0 & 0 & \cdots & 0 \end{bmatrix} \quad (3.5)$$

and

$$A_3 = \begin{bmatrix} -a_{n-1} & -a_{n-2} & \cdots & -a_1 & -a_0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \quad A_3^{-1} = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ -\frac{1}{a_0} & -\frac{a_{n-1}}{a_0} & \cdots & -\frac{a_2}{a_0} & -\frac{a_1}{a_0} \end{bmatrix}, \quad (3.6)$$

$$A_4 = \begin{bmatrix} -a_{n-1} & 1 & 0 & \cdots & 0 \\ -a_{n-2} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_1 & 0 & 0 & \cdots & 1 \\ -a_0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad A_4^{-1} = \begin{bmatrix} 0 & \cdots & 0 & 0 & -\frac{1}{a_0} \\ 1 & \cdots & 0 & 0 & -\frac{a_{n-1}}{a_0} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & -\frac{a_2}{a_0} \\ 0 & \cdots & 0 & 1 & -\frac{a_1}{a_0} \end{bmatrix}. \quad (3.7)$$

3. CHARACTERISTICS EQUATIONS OF THE INVERSE LINEAR SYSTEMS

Consider the standard autonomous linear system (2.1). The characteristic equation of the linear system has the form

$$p(\lambda) = \det[I_n\lambda - A] = \lambda^n + a_{n-1}\lambda^{n-1} + ... + a_1\lambda + a_0 = 0. \quad (4.1)$$

Theorem 4.1: If the equation (4.1) has nonzero real or complex conjugate roots λ_1, λ_2,..., λ_n then the equation

$$a_0\lambda^n + a_1\lambda^{n-1} + ... + a_{n-1}\lambda + 1 = 0 \qquad (4.2)$$

has the roots λ_1^{-1}, λ_2^{-1},..., λ_n^{-1}.

Proof: It is well-known [4] that the coefficients a_0, a_1,..., a_{n-1} of equation (4.1) with its roots λ_1, λ_2,..., λ_n are related by

$$\begin{aligned} a_{n-1} &= \lambda_1 + \lambda_2 + ... + \lambda_n, \\ a_{n-2} &= \lambda_1(\lambda_2 + \lambda_3 + ... + \lambda_n) + \lambda_2(\lambda_3 + \lambda_4 + ... + \lambda_n) \\ &\quad + ... + \lambda_{n-1}\lambda_n, \\ &\vdots \\ a_0 &= \lambda_1\lambda_2...\lambda_n. \end{aligned} \qquad (4.3)$$

Multiplying (4.2) by a_0^{-1} we obtain

$$\lambda^n + \frac{a_1}{a_0}\lambda^{n-1} + ... + \frac{a_{n-1}}{a_0}\lambda + \frac{1}{a_0} = 0. \qquad (4.4)$$

Using (4.3) we have

$$\begin{aligned} \frac{1}{a_0} &= \frac{1}{\lambda_1\lambda_2...\lambda_n} = \lambda_1^{-1}\lambda_2^{-1}...\lambda_n^{-1}, \\ \frac{a_{n-1}}{a_0} &= \frac{\lambda_1 + \lambda_2 + ... + \lambda_n}{\lambda_1\lambda_2...\lambda_n} = \lambda_2^{-1}\lambda_3^{-1}...\lambda_n^{-1} + \lambda_1^{-1}\lambda_3^{-1}...\lambda_n^{-1} \\ &\quad + ... + \lambda_1^{-1}\lambda_2^{-1}...\lambda_{n-1}^{-1}, \\ &\vdots \\ \frac{a_1}{a_0} &= \lambda_1^{-1} + \lambda_2^{-1} + ... + \lambda_n^{-1}. \end{aligned} \qquad (4.5)$$

Therefore, λ_1^{-1}, λ_2^{-1},..., λ_n^{-1} are the roots of the equation (4.2). This completes the proof. □

The proof of Theorem 4.1 for the matrices in Frobenius form follows immediately from Theorem 3.1.

The characteristic equation of the Frobenius matrix (3.1) has the form (4.1) and its eigenvalues are λ_1, λ_2,..., λ_n. The characteristic equation of the inverse Frobenius matrix (3.2) multiplied by a_0 has the form (4.2) and its eigenvalues are λ_1^{-1}, λ_2^{-1},..., λ_n^{-1}.

Example 4.1: The equation

$$\lambda^3 + 5\lambda^2 + 9\lambda + 5 = (\lambda + 1)(\lambda + 2 - j)(\lambda + 2 + j) = 0 \qquad (4.6)$$

has the roots $\lambda_1 = -1$, $\lambda_2 = -2 + j$, $\lambda_3 = -2 - j$, and the equation

$$5\lambda^3 + 9\lambda^2 + 5\lambda + 1 = (\lambda + 1)\left(\lambda - \frac{1}{-2 + j}\right)\left(\lambda - \frac{1}{-2 - j}\right) = 0 \qquad (4.7)$$

has the roots $\lambda_1^{-1} = -1$, $\lambda_2^{-1} = -\frac{1}{2}$, $\lambda_3^{-1} = -\frac{1}{3}$.

4. STABILITY OF THE INVERSE LINEAR SYSTEMS

Consider the standard autonomous continuous-time linear system (2.1) with the characteristic equation (4.1) and its inverse system (2.2) with the characteristic equation (4.2).

Theorem 5.1: The inverse system (2.2) is asymptotically stable if and only if the system (2.1) is asymptotically stable.

Proof: It is well-known that the system (2.1) (and (2.2)) is asymptotically stable if and only if $\text{Re}\,\lambda_k = -\alpha_k < 0$ for all eigenvalues $\lambda_k = -\alpha_k + j\beta_k$, $k = 1,..., n$ of the matrix A ($\overline{A} = A^{-1}$). By Theorem 2.2 the eigenvalues $\overline{\lambda}_k$, $k = 1,..., n$ of the matrix $\overline{A}$ are related with the eigenvalues λ_k of the matrix A by the equality

$$\overline{\lambda}_k = \frac{1}{\lambda_k} = \frac{1}{-\alpha_k + j\beta_k} = \frac{-\alpha_k - j\beta_k}{\alpha_k^2 + \beta_k^2} = -\overline{\alpha}_k - j\overline{\beta}_k, \quad k = 1,..., n, \qquad (5.1)$$

where

$$\overline{\alpha}_k = \frac{\alpha_k}{\alpha_k^2 + \beta_k^2}, \quad \overline{\beta}_k = \frac{\beta_k}{\alpha_k^2 + \beta_k^2}. \qquad (5.2)$$

From (5.2) it follows that $\text{Re}\,\overline{\lambda}_k = -\overline{\alpha}_k < 0$ if and only if $\text{Re}\,\lambda_k = -\alpha_k < 0$. Therefore, the inverse system (2.2) is asymptotically stable if and only if the system (2.1) is asymptotically stable. □

Example 5.1: Consider the standard autonomous continuous-time linear system (2.1) with the state matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -12 & -16 & -7 \end{bmatrix}. \qquad (5.3)$$

The system is stable since (5.3) has the eigenvalues $\lambda_1 = \lambda_2 = -2$, $\lambda_3 = -3$. The inverse matrix of (5.3) has the form

$$\overline{A} = A^{-1} = \begin{bmatrix} -\frac{4}{3} & -\frac{7}{12} & -\frac{1}{12} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad (5.4)$$

and is also stable since its eigenvalues are $\overline{\lambda}_1 = \lambda_1^{-1} = \overline{\lambda}_2 = \lambda_2^{-1} = -\frac{1}{2}$, $\overline{\lambda}_3 = \lambda_3^{-1} = -\frac{1}{3}$.

Now let us consider the standard autonomous discrete-time linear system (2.4) with the characteristic equation (4.1) and its inverse system (2.5) with the characteristic equation (4.2).

Theorem 5.2: The inverse system (2.5) is asymptotically stable if and only if the system (2.4) is unstable.

Proof: It is well-known that the system (2.4) (and (2.5)) is asymptotically stable if and only if $|\lambda_k| < 1$ for all eigenvalues λ_k ($\overline{\lambda}_k$), $k = 1,..., n$ of the matrix A ($\overline{A} = A^{-1}$). By Theorem 2.2 the module of eigenvalues $\overline{\lambda}_k$, $k = 1,..., n$ of the matrix $\overline{A}$ with module of eigenvalues λ_k of the matrix A are related by the equality

$$\left|\bar{\lambda}_k\right| = \frac{1}{\left|\lambda_k\right|},\ k = 1,\dots,n. \qquad (5.5)$$

From (5.5) it follows that $\left|\bar{\lambda}_k\right| < 1$ if and only if $\left|\lambda_k\right| > 1$. Therefore, the inverse system (2.4) is asymptotically stable if and only if the system (2.5) is unstable. □

Example 5.2: Consider the standard autonomous discrete-time linear system (2.5) with the state matrix

$$A = \begin{bmatrix} -\frac{13}{12} & 1 & 0 \\ -\frac{9}{24} & 0 & 1 \\ -\frac{1}{24} & 0 & 0 \end{bmatrix}. \qquad (5.6)$$

The system is stable since (5.6) has the eigenvalues $\lambda_1 = -\frac{1}{2}$, $\lambda_2 = -\frac{1}{3}$, $\lambda_3 = -\frac{1}{4}$. The inverse matrix of (5.6) has the form

$$\bar{A} = A^{-1} = \begin{bmatrix} 0 & 0 & -24 \\ 1 & 0 & -26 \\ 0 & 1 & -9 \end{bmatrix} \qquad (5.7)$$

and the inverse system is unstable since the eigenvalues of (5.7) are $\bar{\lambda}_1 = \lambda_1^{-1} = -2$, $\bar{\lambda}_2 = \lambda_2^{-1} = -3$, $\bar{\lambda}_3 = \lambda_3^{-1} = -4$.

CONCLUDING REMARKS

The inverse Frobenius matrices (Theorem 3.1) and the characteristic equations of the inverse systems (Theorem 4.1) have been investigated. The stability of continuous-time and discrete-time linear systems with inverse state matrices has been analyzed. It has been shown that the inverse system of continuous-time linear system is asymptotically stable if and only if the standard system is asymptotically stable (Theorem 5.1) and the inverse system of discrete-time linear system is asymptotically stable if and only if the standard system is unstable (Theorem 5.2). The considerations have been illustrated by numerical examples.

The presented approach can be extended to positive and fractional linear systems and electrical circuits.

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Równanie charakterystyczne oraz stabilność układów liniowych z odwrotnością macierzy stanu

Rozważane będzie odwrotność macierzy Frobeniusa oraz równanie charakterystyczne układów o macierzy odwrotnej. Wykazano, że liniowy układ ciągły o macierzy odwrotnej jest asymptotycznie stabilny wtedy i tylko wtedy gdy układ podstawowy jest asymptotycznie stabilny oraz że liniowy układ dyskretny o macierzy odwrotnej jest asymptotycznie stabilny wtedy i tylko wtedy gdy układ podstawowy jest niestabilny. rozważania zobrazowano przykładami numerycznymi.

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