# ON THE CROSSING NUMBERS OF JOIN PRODUCTS OF FIVE GRAPHS OF ORDER SIX WITH THE DISCRETE GRAPH

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**Abstract.** The main purpose of this article is broaden known results concerning crossing numbers for join of graphs of order six. We give the crossing number of the join product $G^* + D_n$, where the disconnected graph $G^*$ of order six consists of one isolated vertex and of one edge joining two nonadjacent vertices of the 5-cycle. In our proof, the idea of cyclic permutations and their combinatorial properties will be used. Finally, by adding new edges to the graph $G^*$, the crossing numbers of $G_i + D_n$ for four other graphs $G_i$ of order six will be also established.

**Keywords:** graph, drawing, crossing number, join product, cyclic permutation.

**Mathematics Subject Classification:** 05C10, 05C38.

## 1. INTRODUCTION

The *crossing number* $\mathrm{cr}(G)$ of a simple graph $G$ with the vertex set $V(G)$ and the edge set $E(G)$ is the minimum possible number of edge crossings in a drawing of $G$ in the plane. (For the definition of a *drawing* see [6].) It is easy to see that a drawing with minimum number of crossings (an optimal drawing) is always a *good* drawing, meaning that no edge crosses itself, no two edges cross more than once, and no two edges incident with the same vertex cross. Let $D$ ($D(G)$) be a good drawing of the graph $G$. We denote the number of crossings in $D$ by $\mathrm{cr}_D(G)$. Let $G_i$ and $G_j$ be edge-disjoint subgraphs of $G$. We denote the number of crossings between edges of $G_i$ and edges of $G_j$ by $\mathrm{cr}_D(G_i, G_j)$, and the number of crossings among edges of $G_i$ in $D$ by $\mathrm{cr}_D(G_i)$. It is easy to see that for any three mutually edge-disjoint subgraphs $G_i$, $G_j$, and $G_k$ of $G$, the following equations hold:

$$\mathrm{cr}_D(G_i \cup G_j) = \mathrm{cr}_D(G_i) + \mathrm{cr}_D(G_j) + \mathrm{cr}_D(G_i, G_j),$$
$$\mathrm{cr}_D(G_i \cup G_j, G_k) = \mathrm{cr}_D(G_i, G_k) + \mathrm{cr}_D(G_j, G_k).$$

 

It is well known that computing the crossing number of a graph is an NP-complete problem. The exact values of the crossing numbers are known only for some graphs or some families of graphs. The purpose of this article is to extend the known results concerning this topic. We also often use the Kleitman's result [4] on crossing numbers of the complete bipartite graphs $K_{m,n}$, that is,

$$\operatorname{cr}(K_{m,n}) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \quad \text{for } m \leq 6.$$

Using Kleitman's result [4], the crossing numbers for join of two paths, join of two cycles, and for join of path and cycle were studied in [7]. Moreover, the exact values for crossing numbers of $G + D_n$ and of $G + P_n$ for all graphs $G$ of order at most four are given in [9]. It is also important to note that the crossing numbers of the graphs $G + D_n$ are known for few graphs $G$ of order five and six, see e.g. [6, 8, 10–14]. In all these cases, the graph $G$ is usually connected and contains at least one cycle.

The methods presented in the paper are based on combinatorial properties of cyclic permutations. Some of the ideas and methods were used for the first time in [3, 11]. In [2,12,13], the properties of cyclic permutations are verified with the help of the software described in [1]. In our opinion, the methods used in [6,8,9] do not suffice for establishing the crossing number of the join product $G^* + D_n$. Some parts of proofs can be done with the help of software that generates all cyclic permutations in [1]. C++ version of the program is located also on the website `http://web.tuke.sk/fei-km/coga/`. The list with the short names of $6!/6 = 120$ cyclic permutations of six elements are collected in Table 1 of [12].

## 2. CYCLIC PERMUTATIONS AND CONFIGURATIONS

Let $G^*$ be the disconnected graph of order six consisting of one isolated vertex and of one edge joining two nonadjacent vertices of the 5-cycle and let $V(G^*) = \{v_1, v_2, \dots, v_6\}$. We consider the join product of the graph $G^*$ with the discrete graph $D_n$ on $n$ vertices. The graph $G^* + D_n$ consists of one copy of the graph $G^*$ and of $n$ vertices $t_1, t_2, \dots, t_n$, where each vertex $t_i$, $i = 1, 2, \dots, n$, is adjacent to every vertex of $G^*$. Let $T^i$, $1 \leq i \leq n$, denote the subgraph induced by the six edges incident with the vertex $t_i$. This means that the graph $T^1 \cup \dots \cup T^n$ is isomorphic with the complete bipartite graph $K_{6,n}$ and

$$G^* + D_n = G^* \cup K_{6,n} = G^* \cup \left( \bigcup_{i=1}^{n} T^i \right). \tag{2.1}$$

Let $D$ be a good drawing of the graph $G^* + D_n$. The *rotation* $\operatorname{rot}_D(t_i)$ of a vertex $t_i$ in the drawing $D$ is the cyclic permutation that records the (cyclic) counterclockwise order in which the edges leave $t_i$, as defined by Hernández-Vélez *et al.* [3]. We use the notation (123456) if the counterclockwise order of the edges incident with the vertex $t_i$ is $t_iv_1$, $t_iv_2$, $t_iv_3$, $t_iv_4$, $t_iv_5$, and $t_iv_6$. We have to emphasize that a rotation is a cyclic permutation. In a given drawing $D$, we separate all subgraphs $T^i$, $i = 1, \dots, n$, of the graph $G^* + D_n$ into three mutually disjoint subsets depending on how many times the considered $T^i$ crosses the edges of $G^*$ in $D$. For $i = 1, \dots, n$, $T^i \in R_D$ if $\mathrm{cr}_D(G^*, T^i) = 0$ and $T^i \in S_D$ if $\mathrm{cr}_D(G^*, T^i) = 1$. Every other subgraph $T^i$ crosses the edges of $G^*$ at least twice in $D$. For $T^i \in R_D$, let $F^i$ denote the subgraph $G^* \cup T^i$, $i \in \{1, 2, \dots, n\}$, of $G^* + D_n$ and let $D(F^i)$ be its subdrawing induced by $D$.

According to the arguments in the proof of the main Theorem 3.4, if we would like to obtain a drawing $D$ of $G^* + D_n$ with the smallest number of crossings, then the set $R_D$ must be nonempty. Thus, we will only consider drawings of the graph $G^*$ for which there is a possibility of obtaining a subgraph $T^i \in R_D$. Let us discuss all possible drawings of $G^*$. Since the graph $G^*$ consists of the edge disjoint subgraphs $C_4$ and $P_3$ (for brevity, we write $C_4(G^*)$ and $P_3(G^*)$), we only need to consider possibilities of crossings between subdrawings of subgraphs $C_4(G^*)$ and $P_3(G^*)$. Of course, the edges of the cycle $C_4(G^*)$ can cross itself in the considered subdrawings. Let us first consider a good subdrawing of $G^*$ in which the edges of $C_4(G^*)$ do not cross each other. In this case, we obtain three non isomorphic drawings shown in Figure 1(a), (b), and (c). If we consider a good subdrawing of $G^*$ in which the edges of $C_4(G^*)$ cross each other, then the edges of $P_3(G^*)$ do not cross the edges of $C_4(G^*)$ only in one case that is shown in Figure 1(d). If the edges of $C_4(G^*)$ are crossed at least once by the edges of $P_3(G^*)$, then there are next four possibilities due to the considered good subdrawing of $G^*$ and they are shown in Figure 1(e), (f), (g), and (h). The vertex notation of the graph $G^*$ in Figure 1 will be justified later.

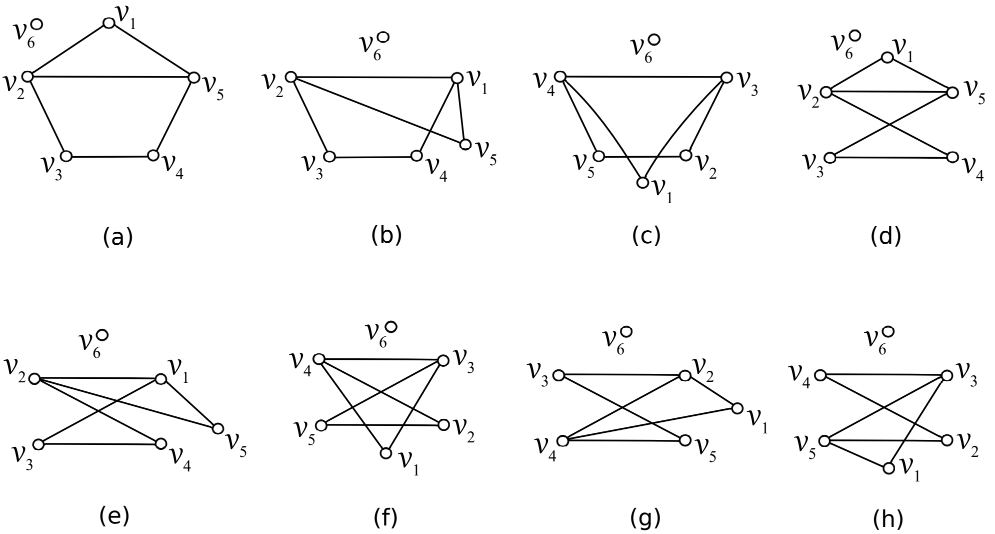


**Fig. 1.** Eight possible non isomorphic drawings of the graph $G^*$

First, let us assume a good drawing $D$ of the graph $G^* + D_n$ in which the edges of $G^*$ do not cross each other. In this case, without loss of generality, we can choose the vertex notation of the graph $G^*$ in such a way as shown in Figure 1(a). Our aim is to list all possible rotations $\mathrm{rot}_D(t_i)$ which can appear in $D$ if the edges of $T^i$ do not cross the edges of $G^*$. Since there is only one subdrawing of $F^i \setminus \{v_6\}$ represented by the rotation (15432), there are five possibilities to obtain the subdrawing of $F^i$ depending on in which region the edge $t_i v_6$ is placed. We denote these five configurations by $\mathcal{A}_k$, for $k = 1, \dots, 5$. For our considerations over the number of crossings of $G^* + D_n$, it does not play a role in which of the regions is unbounded. So we can assume the configurations of $F^i$ drawn as shown in Figure 2. In the rest of the paper, we represent a cyclic permutation by the permutation with 1 in the first position. Thus the configurations $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$, $\mathcal{A}_4$, and $\mathcal{A}_5$ are represented by the cyclic permutations (156432), (154362), (165432), (154632), and (154326), respectively. Of course, in a fixed drawing of the graph $G^* + D_n$, some configurations from $\mathcal{M} = \{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5\}$ need not appear. So we denote by $\mathcal{M}_D$ the set of all configurations of $\mathcal{M}$ that appear in $D$.

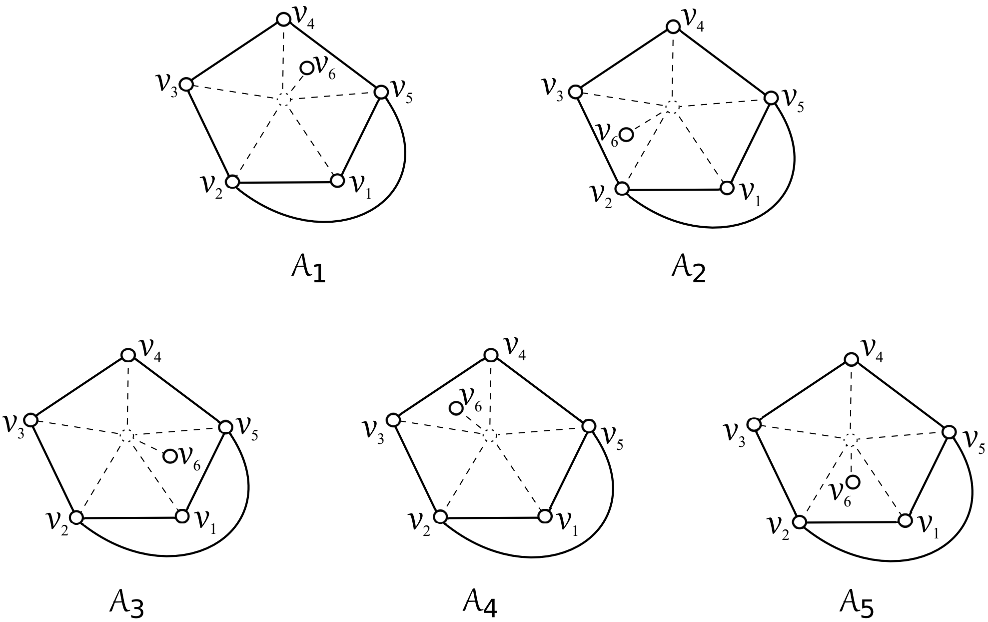


**Fig. 2.** Drawings of all five possible configurations of the subgraph $F^i$

We remark that if two different subgraphs $F^i$ and $F^j$ with configurations from $\mathcal{M}_D$ cross in a drawing $D$ of $G^* + D_n$, then only the edges of $T^i$ cross the edges of $T^j$. Thus, we will deal with the minimum numbers of crossings between two different subgraphs $F^i$ and $F^j$ depending on their configurations. Let $\mathcal{X}$, $\mathcal{Y}$ be the configurations from $\mathcal{M}_D$. We denote by $\mathrm{cr}_D(\mathcal{X}, \mathcal{Y})$ the number of crossings in $D$ between $T^i$ and $T^j$ for different $T^i, T^j \in R_D$ such that $F^i$, $F^j$ have configurations $\mathcal{X}$, $\mathcal{Y}$, respectively. Finally, let $\mathrm{cr}(\mathcal{X}, \mathcal{Y}) = \min\{\mathrm{cr}_D(\mathcal{X}, \mathcal{Y})\}$ over all good drawings of the graph $G^* + D_n$ with $\mathcal{X}, \mathcal{Y} \in \mathcal{M}_D$. Our aim is to establish $\mathrm{cr}(\mathcal{X}, \mathcal{Y})$ for all pairs $\mathcal{X}, \mathcal{Y} \in \mathcal{M}$. In particular, the configurations $\mathcal{A}_1$ and $\mathcal{A}_2$ are represented by the cyclic permutations (156432) and (154362), respectively. Since the minimum number of interchanges of adjacent elements of (156432) required to produce cyclic permutation (154362) is two, we need at least four interchanges of adjacent elements of (156432) to produce cyclic permutation $\overline{(154362)} = (126345)$.[1)] So any subgraph $T^j$ with the configuration $\mathcal{A}_2$ of $F^j$ crosses the edges of $T^i$ with the configuration $\mathcal{A}_1$ of $F^i$ at least four times, that is, $\mathrm{cr}(\mathcal{A}_1, \mathcal{A}_2) \geq 4$. The same reason gives

$$\mathrm{cr}(\mathcal{A}_1, \mathcal{A}_3) \geq 5, \quad \mathrm{cr}(\mathcal{A}_1, \mathcal{A}_4) \geq 5, \quad \mathrm{cr}(\mathcal{A}_1, \mathcal{A}_5) \geq 4, \quad \mathrm{cr}(\mathcal{A}_2, \mathcal{A}_3) \geq 4,$$
$$\mathrm{cr}(\mathcal{A}_2, \mathcal{A}_4) \geq 5, \quad \mathrm{cr}(\mathcal{A}_2, \mathcal{A}_5) \geq 5, \quad \mathrm{cr}(\mathcal{A}_3, \mathcal{A}_4) \geq 4, \quad \mathrm{cr}(\mathcal{A}_3, \mathcal{A}_5) \geq 5,$$
$$\mathrm{cr}(\mathcal{A}_4, \mathcal{A}_5) \geq 4.$$

Clearly, also $\mathrm{cr}(\mathcal{A}_x, \mathcal{A}_x) \geq 6$ for any $x = 1, \ldots, 5$. The resulting lower bounds for the number of crossings of configurations from $\mathcal{M}$ are summarized in the symmetric Table 1 (here, $\mathcal{A}_x$ and $\mathcal{A}_y$ are configurations of the subgraphs $F^i$ and $F^j$, where $x, y \in \{1, 2, 3, 4, 5\}$).

**Table 1.** The necessary number of crossings between $T^i$ and $T^j$ for the configurations $\mathcal{A}_x$, $\mathcal{A}_y$

| $-$ | $\mathcal{A}_1$ | $\mathcal{A}_2$ | $\mathcal{A}_3$ | $\mathcal{A}_4$ | $\mathcal{A}_5$ |
|---|---|---|---|---|---|
| $\mathcal{A}_1$ | 6 | 4 | 5 | 5 | 4 |
| $\mathcal{A}_2$ | 4 | 6 | 4 | 5 | 5 |
| $\mathcal{A}_3$ | 5 | 4 | 6 | 4 | 5 |
| $\mathcal{A}_4$ | 5 | 5 | 4 | 6 | 4 |
| $\mathcal{A}_5$ | 4 | 5 | 5 | 4 | 6 |

Assume a good drawing $D$ of the graph $G^* + D_n$ with at least one crossing among edges of the graph $G^*$ (in which there is a subgraph $T^i \in R_D$). In this case, without loss of generality, we can choose the vertex notations of the graph $G^*$ in such a way as shown in Figure 1 (b)–(h). In all mentioned cases, we are able to use the same idea as above, i.e., we obtain the same five rotations for $t_i$ with $T^i \in R_D$, and also the same corresponding lower-bounds for numbers of crossings between two configurations of $F^i$ and $F^j$ as in Table 1.

## 3. THE CROSSING NUMBER OF $G^* + D_n$

Recall that two vertices $t_i$ and $t_j$ of $G^* + D_n$ are *antipodal* in a drawing of $G^* + D_n$ if the subgraphs $T^i$ and $T^j$ do not cross. A drawing is *antipodal-free* if it has no antipodal vertices. In the proof of the main theorem, the following statements related to some restricted drawings will be helpful.

[1)] Let $T^x$ and $T^y$ be two different subgraphs represented by their $\mathrm{rot}(t_x)$ and $\mathrm{rot}(t_y)$ of length $m$, $m \geq 3$. If the minimum number of interchanges of adjacent elements of $\mathrm{rot}(t_x)$ required to produce $\mathrm{rot}(t_y)$ is at most $z$, then $\mathrm{cr}_D(T^x, T^y) \geq \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor - z$. Details have been worked out by Woodall [15].

Let us first note that if $D$ is a good and antipodal-free drawing of $G^* + D_n$ with the vertex notation of the graph $G^*$ in such a way as shown in Figure 1(a), and $T^i \in R_D$ such that $F^i$ has configuration $\mathcal{A}_x \in \mathcal{M}_D$, then $\mathrm{cr}_D(G^* \cup T^i, T^l) \geq 3$ for any $T^l$, $l \neq i$, see Figure 2. Therewith, there are possibilities of obtaining a subgraph $T^k \in S_D$ with $\mathrm{cr}_D(T^i, T^k) = 2$ only for the cases of the configurations $\mathcal{A}_1$ and $\mathcal{A}_2$ of $F^i$.

**Lemma 3.1.** *Let $D$ be a good and antipodal-free drawing of the graph $G^* + D_n$, $n > 2$, with $\mathrm{cr}_D(G^*) = 0$ and with the vertex notation of the graph $G^*$ in such a way as shown in Figure* 1(a). *Let $T^i \in R_D$ be a subgraph such that $F^i$ has configuration $\mathcal{A}_x \in \mathcal{M}_D$, $x \in \{1, 2\}$. If there is a subgraph $T^k \in S_D$ with $\mathrm{cr}_D(T^i, T^k) = 2$, then*

a) $\mathrm{cr}_D(T^i \cup T^k, T^l) \geq 4$ *for any subgraph $T^l$, $l \neq i, k$;*
b) $\mathrm{cr}_D(G^* \cup T^i \cup T^k, T^l) \geq 7$ *for any subgraph $T^l \in S_D$, $l \neq k$.*

*Proof.* Let us assume the configuration $\mathcal{A}_1$ of $F^i$, and recall that it is represented by the cyclic permutation (156432). The unique subdrawing $D(F^i)$ of the subgraph $F^i$ contains five regions with the vertex $t_i$ on their boundaries, see Figure 2. If there is a subgraph $T^k \in S_D$ with $\mathrm{cr}_D(T^i, T^k) = 2$, then the vertex $t_k$ must be placed in the region with three vertices $v_4$, $v_5$, and $v_6$ of the graph $G^*$ on its boundary. This enforces that the edge $v_4v_5$ of the graph $G^*$ must be crossed by the edge $t_kv_2$ and $\mathrm{cr}_D(T^i, T^k) = 2$ only for $T^k$ with $\mathrm{rot}_D(t_k) = (152436)$. For more details, see the considered subdrawing of $G^* \cup T^k$ in Figure 3.

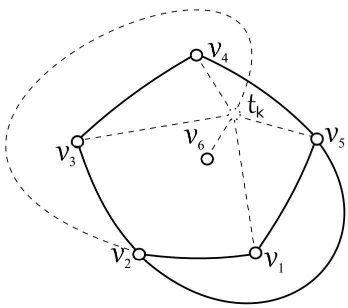


**Fig. 3.** The drawing of $G^* \cup T^k$ for $T^k \in S_D$ with $\mathrm{rot}_D(t_k) = (152436)$

a) Let $T^k \in S_D$ be a subgraph with $\mathrm{cr}_D(T^i, T^k) = 2$ and let $T^l$ be any subgraph with $l \neq i, k$. As $\mathrm{cr}_D(K_{6,3}) \geq 6$, we obtain

$$\mathrm{cr}_D(T^i \cup T^k, T^l) \geq 4.$$

b) Let $T^k \in S_D$ be a subgraph with $\mathrm{cr}_D(T^i, T^k) = 2$. If there is $T^l \in S_D$ with $\mathrm{cr}_D(T^k, T^l) = 1$, then the vertex $t_l$ must be placed in the region of $D(G^* \cup T^k)$ with four vertices $v_1$, $v_2$, $v_3$, and $v_6$ of $G^*$ on its boundary. This forces that no edge of the graph $G^* \cup T^k$ is crossed by an edge $t_lv_j$, for $j = 1, 2, 3, 6$. As $\mathrm{cr}_D(T^k, T^l) = 1$, the subgraph $T^l$ is represented by $\mathrm{rot}_D(t_l) = (164325)$ if the edge $v_1v_2$ is crossed by the edge $t_lv_5$, or by $\mathrm{rot}_D(t_l) = (156342)$ if the edge $v_2v_3$ is crossed by $t_lv_4$. Since the minimum number of interchanges of adjacent elements of (156432) required to produce cyclic permutations (164325) and (156342) is one, the subgraph $T^l$ must cross the edges of $T^i$ at least five times. Thus,

$$\mathrm{cr}_D(G^* \cup T^i \cup T^k, T^l) \geq 1 + 5 + 1 = 7.$$

To finish the proof, let us assume that $\mathrm{cr}_D(T^k, T^l) \geq 2$ for each $T^l \in S_D$, $l \neq k$. Clearly, the case $\mathrm{cr}_D(T^i, T^l) \geq 4$ enforces the desired result

$$\mathrm{cr}_D(G^* \cup T^i \cup T^k, T^l) \geq 1 + 4 + 2 = 7.$$

Further, if $\mathrm{cr}_D(T^i, T^l) = 2$, then $\mathrm{rot}_D(t_k) = \mathrm{rot}_D(t_l)$ and $\mathrm{cr}_D(T^k, T^l) \geq 6$, see [15]. Similarly,

$$\mathrm{cr}_D(G^* \cup T^i \cup T^k, T^l) \geq 1 + 3 + 3 = 7$$

is fulfilling if $\mathrm{cr}_D(T^k, T^l) \geq 3$ and $\mathrm{cr}_D(T^i, T^l) = 3$. So, let us consider a subgraph $T^l \in S_D$ with respect to the restrictions $\mathrm{cr}_D(T^k, T^l) = 2$ and $\mathrm{cr}_D(T^i, T^l) = 3$. If $\mathrm{cr}_D(T^k, T^l) = 2$, then the vertex $t_l$ must be also placed in the region of $D(G^* \cup T^k)$ with four vertices of $G^*$ on its boundary and no edge of the graph $G^* \cup T^k$ is crossed by an edge $t_l v_j$, for $j = 2, 6$. It is not difficult to show that $\mathrm{rot}_D(t_l)$ is only one of (153642), (143256), (146325), (165342), and (164352). Since the minimum number of interchanges of adjacent elements of (156432) required to produce any of the five cyclic permutations listed is two, this in turn implies that $\mathrm{cr}_D(T^i, T^l) \geq 4$, which contradicts the fact that $\mathrm{cr}_D(T^i, T^l) = 3$ in $D(T^i \cup T^k \cup T^l)$.

Due to the symmetry of the configurations $\mathcal{A}_1$ and $\mathcal{A}_2$, we are able to use the same arguments for the configuration $\mathcal{A}_2$ of $F^i$, and this completes the proof. □

Now, let us turn to a good drawing $D$ of the graph $G^* + D_n$ with at least one crossing among edges of the graph $G^*$. For $T^i \in R_D$, we obtain the same configurations $\mathcal{A}_1, \ldots, \mathcal{A}_5$ of $F^i$ together with the corresponding cyclic permutations as in the case when $\mathrm{cr}_D(G^*) = 0$. For $T^l \in S_D$, in Figure 1, it is possible to verify that $\mathrm{cr}_D(T^i, T^l) = 2$ only for the configuration $\mathcal{A}_1$ of $F^i$ obtained from the drawings of $G^*$ in Figure 1(b) and (e). The same holds for the configurations $\mathcal{A}_1$ and $\mathcal{A}_2$ of $F^i$ obtained from the drawing of $G^*$ in Figure 1(d).

**Lemma 3.2.** *Let $D$ be a good and antipodal-free drawing of the graph $G^* + D_n$, $n > 2$, with $\mathrm{cr}_D(G^*) \neq 0$ and with the vertex notation of the graph $G^*$ shown in Figure* 1(b), (d) *or* (e). *Let $T^i \in R_D$ be a subgraph such that $F^i$ has configuration $\mathcal{A}_x \in \mathcal{M}_D$. If there is a subgraph $T^k \in S_D$ with $\mathrm{cr}_D(T^i, T^k) = 2$, then*

a) $\mathrm{cr}_D(T^i \cup T^k, T^l) \geq 4$ *for any subgraph $T^l$, $l \neq i, k$;*
b) $\mathrm{cr}_D(G^* \cup T^i \cup T^k, T^l) \geq 7$ *for any subgraph $T^l \in S_D$, $l \neq k$.*

Due to the use of similar arguments as in the proof of the previous Lemma 3.1, the proof of Lemma 3.2 can be omitted.

**Collorary 3.3.** *Let $D$ be a good and antipodal-free drawing of $G^* + D_n$, for $n > 2$, with the corresponding vertex notations of the graph $G^*$ in such a way as shown in Figure 1. If $T^i$, $T^j \in R_D$ are different subgraphs such that $F^i$, $F^j$ have different configurations from any of the sets $\{\mathcal{A}_1, \mathcal{A}_2\}$, $\{\mathcal{A}_2, \mathcal{A}_3\}$, $\{\mathcal{A}_3, \mathcal{A}_4\}$, $\{\mathcal{A}_4, \mathcal{A}_5\}$, and $\{\mathcal{A}_5, \mathcal{A}_1\}$, then*

$$\mathrm{cr}_D(T^i \cup T^j, T^k) \geq 6 \quad \textit{for any } T^k \in S_D,$$

*i.e.,*

$$\mathrm{cr}_D(G^* \cup T^i \cup T^j, T^k) \geq 7 \quad \textit{for any } T^k \in S_D.$$

*Proof.* Let $D$ be a good and antipodal-free drawing of $G^* + D_n$ with the subdrawing of $G^*$ as shown in Figure 1(a) and let us assume the configurations $\mathcal{A}_1$ of $F^i$, and $\mathcal{A}_2$ of $F^j$. If there is a subgraph $T^k \in S_D$ with $\mathrm{cr}_D(T^i, T^k) = 2$, then the subgraph $G^* \cup T^k$ can be represented only by the cyclic permutation $(152436)$. Note that the configuration $\mathcal{A}_2$ is represented by $(154362)$. Since the minimum number of interchanges of adjacent elements of $(154362)$ required to produce cyclic permutation $(152436)$ is two, we obtain $\mathrm{cr}_D(T^j, T^k) \geq 4$. Hence,

$$\mathrm{cr}_D(G^* \cup T^i \cup T^j, T^k) \geq 1 + 2 + 4 = 7.$$

We can apply the same idea for the case, if there is a subgraph $T^k \in S_D$ with $\mathrm{cr}_D(T^j, T^k) = 2$. It remains to consider the case where $\mathrm{cr}_D(T^i, T^k) \geq 3$ and $\mathrm{cr}_D(T^j, T^k) \geq 3$, which yields that

$$\mathrm{cr}_D(G^* \cup T^i \cup T^j, T^k) \geq 1 + 3 + 3 = 7$$

trivially holds for any such $T^k \in S_D$. The proof proceeds in the similar way also for the remaining pairs of configurations, and also for others considered drawings of $G^*$ in Figure 1 with $\mathrm{cr}_D(G^*) \geq 1$. This completes the proof. □

**Theorem 3.4.** $\mathrm{cr}(G^* + D_n) = 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2\left\lfloor \frac{n}{2} \right\rfloor$ *for* $n \geq 1$.

*Proof.* In Figure 4 there are the drawings of $G^* + D_n$ with $6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2\left\lfloor \frac{n}{2} \right\rfloor$ crossings. Thus,

$$\mathrm{cr}(G^* + D_n) \leq 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2\left\lfloor \frac{n}{2} \right\rfloor.$$

We prove the reverse inequality by induction on $n$. The graph $G^* + D_1$ is planar; hence, $\mathrm{cr}(G^* + D_1) = 0$. The possibility of adding of a subgraph $T^k \notin R_D \cup S_D$ with two crossings into the subdrawing of $\mathcal{A}_1$ in Figure 2 forces $\mathrm{cr}(G^* + D_2) \leq 2$. The graph $G^* + D_2$ contains a subgraph that is a subdivision of the graph $K_6 \setminus e$ obtained by removing one edge from the complete graph $K_6$. It was proved in [5] that $\mathrm{cr}(K_6 \setminus e) = 2$. So, the result is true for $n = 1$ and $n = 2$. Suppose now that, for some $n \geq 3$, there is a drawing $D$ with

$$\mathrm{cr}_D(G^* + D_n) < 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2\left\lfloor \frac{n}{2} \right\rfloor \tag{3.1}$$

and that

$$\mathrm{cr}(G^* + D_m) \geq 6\left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor + 2\left\lfloor \frac{m}{2} \right\rfloor \quad \text{for any positive integer } m < n. \tag{3.2}$$

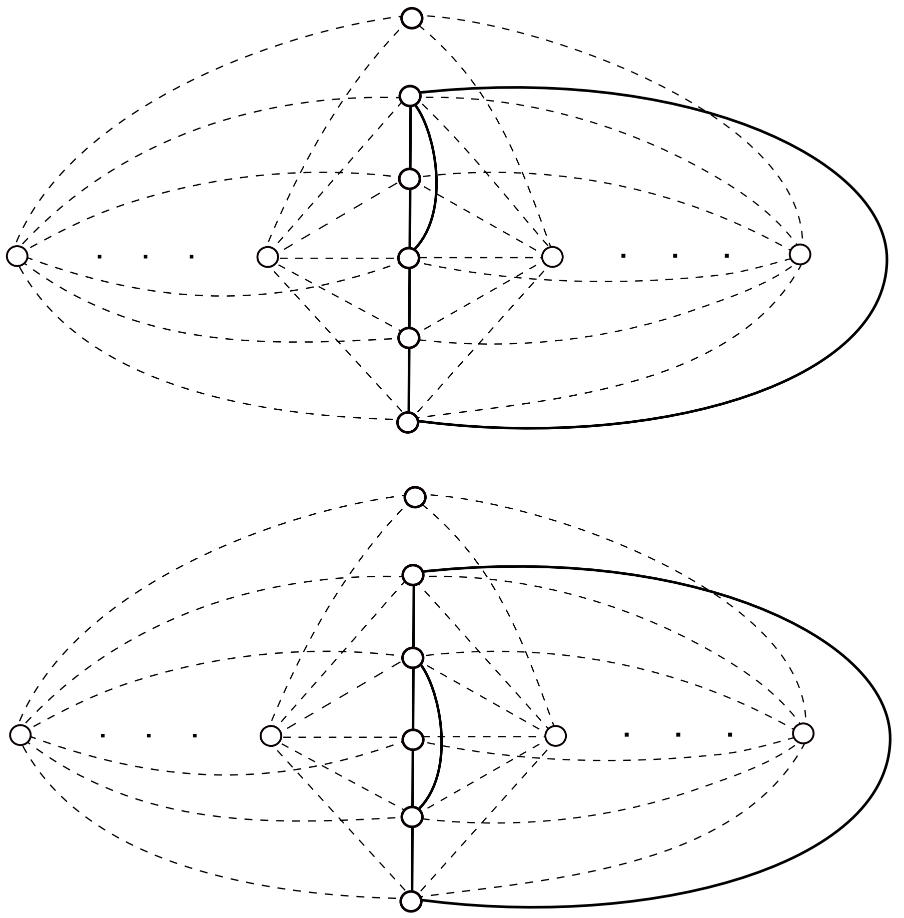

**Fig. 4.** Two good drawings of $G^* + D_n$ with $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2\lfloor \frac{n}{2} \rfloor$ crossings

We claim that the considered drawing $D$ must be antipodal-free. For a contradiction suppose, without loss of generality, that $\mathrm{cr}_D(T^{n-1}, T^n) = 0$. If at least one of $T^{n-1}$ and $T^n$, say $T^n$, does not cross $G^*$, it is not difficult to verify in Figure 1 that $T^{n-1}$ must cross $G^* \cup T^n$ at least twice, that is, $\mathrm{cr}_D(G^*, T^{n-1} \cup T^n) \geq 2$. By [4], we already know that $\mathrm{cr}(K_{6,3}) = 6$, which yields that each $T^k$, $k = 1, 2, \ldots, n-2$, crosses the edges of the subgraph $T^{n-1} \cup T^n$ at least six times. So, for the number of crossings in $D$ we have

$$\begin{aligned}
\mathrm{cr}_D(G^* + D_n) &= \mathrm{cr}_D\left(G^* + D_{n-2}\right) + \mathrm{cr}_D(T^{n-1} \cup T^n) + \mathrm{cr}_D(K_{6,n-2}, T^{n-1} \cup T^n) \\
&\quad + \mathrm{cr}_D(G^*, T^{n-1} \cup T^n) \\
&\geq 6\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + 2\left\lfloor \frac{n-2}{2} \right\rfloor + 6(n-2) + 2 \\
&= 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2\left\lfloor \frac{n}{2} \right\rfloor.
\end{aligned}$$

This contradiction with the assumption (3.1) confirms that $D$ is antipodal-free. Moreover, if $r = |R_D|$ and $s = |S_D|$, the assumption (3.2) together with the well-known fact $\mathrm{cr}(K_{6,n}) = 6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ imply that, in $D$, there is at least one subgraph $T^i$ which does not cross the edges of $G^*$, that is, $r \geq 1$. More precisely:

$$\mathrm{cr}_D(G^*) + \mathrm{cr}_D(G^*, K_{6,n}) \leq \mathrm{cr}_D(G^*) + 0r + 1s + 2(n-r-s) < 2\left\lfloor \frac{n}{2} \right\rfloor,$$

i.e.,

$$0r + s + 2(n-r-s) < 2\left\lfloor \frac{n}{2} \right\rfloor. \tag{3.3}$$

This forces that $2r + s > 2n - 2\lfloor \frac{n}{2} \rfloor$ and $r > n - r - s$. Moreover, if $n = 3$, then $r \geq 2$. For $r = 3$,

$$\mathrm{cr}_D(G^* + D_3) \geq \mathrm{cr}_D(T^1 \cup T^2 \cup T^3) \geq 13$$

holds by summing the corresponding three values of Table 1. For $r = 2$ and $s = 1$, we can assume that $G^* + D_3 = G^* \cup T^1 \cup T^2 \cup T^3$, where $T^1, T^2 \in R_D$ and $T^3 \in S_D$. It was discussed above that, in any good drawing of $G^* + D_n$, $T^i$ crosses $T^k$ at least twice if $T^i \in R_D$ and $T^k \in S_D$. Hence,

$$\mathrm{cr}_D(G^* + D_3) \geq \mathrm{cr}_D(T^1 \cup T^2) + \mathrm{cr}_D(T^1 \cup T^2, T^3) \geq 4 + 2 + 2 = 8.$$

These contradictions with the assumption (3.1) confirms that $n \geq 4$. Now, for $T^i \in R_D$, we will discuss the existence of possible configurations of subgraph $F^i = G^* \cup T^i$ in the drawing $D$ and we show that in all cases the contradiction with the assumption (3.1) is obtained.

*Case 1.* $\mathrm{cr}_D(G^*) = 0$. Without loss of generality, we can choose the vertex notation of the graph $G^*$ in such a way as shown in Figure 1(a). Thus, we will deal with the configurations belonging to the nonempty set $\mathcal{M}_D$, i.e., we will discuss over all possible subsets of the set $\mathcal{M}_D$ in the following subcases:

*Subcase 1a.* $\{\mathcal{A}_x, \mathcal{A}_y, \mathcal{A}_z\} \subseteq \mathcal{M}_D$ with $x + 2 \equiv y + 1 \equiv z \pmod 5$. Without lost of generality, let us consider three different subgraphs $T^{n-2}$, $T^{n-1}$, $T^n \in R_D$ such that $F^{n-2}$, $F^{n-1}$ and $F^n$ have different configurations $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$, respectively. Then,

$$\mathrm{cr}_D(G^* \cup T^{n-2} \cup T^{n-1} \cup T^n, T^i) \geq 14$$

is fulfilling for any $T^i \in R_D$ with $i \neq n-2, n-1, n$ by summing the values in all columns in the considered three rows of Table 1. Moreover,

$$\mathrm{cr}_D(G^* \cup T^{n-2} \cup T^{n-1} \cup T^n, T^l) \geq 1 + 6 + 3 = 10$$

holds for any subgraph $T^l \in S_D$ by Corollary 3.3 and by the fact that the edges of $T^l$ have to cross the edges of $T^n$ at least thrice. Further, it is not difficult to verify that

$$\mathrm{cr}_D(G^* \cup T^{n-2} \cup T^{n-1} \cup T^n, T^l) \geq 6$$

is also true for each $T^l \notin R_D \cup S_D$. If

$$\mathrm{cr}_D(G^* \cup T^{n-2} \cup T^{n-1} \cup T^n, T^l) < 6,$$

then

$$\mathrm{cr}_D(T^{n-2}, T^l) = \mathrm{cr}_D(T^{n-1}, T^l) = \mathrm{cr}_D(T^n, T^l) = 1.$$

Hence, the cyclic permutation associated with $\overline{\mathrm{rot}_D(t_l)}$ must be obtained from all three cyclic permutations which represent the configurations $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{A}_3$ by only one exchange of adjacent elements. But, the only cyclic permutation obtained from both permutations associated with $\mathcal{A}_1$ and $\mathcal{A}_2$ is (154632) and this permutation is not possible to obtain from the permutation associated with $\mathcal{A}_3$. This forces that

$$\mathrm{cr}_D(T^{n-2} \cup T^{n-1} \cup T^n, T^l) \geq 4$$

and therefore,

$$\mathrm{cr}_D(G^* \cup T^{n-2} \cup T^{n-1} \cup T^n, T^l) \geq 6$$

for any such $T^l$. As

$$\mathrm{cr}_D(T^{n-2} \cup T^{n-1} \cup T^n) \geq 13$$

holds by summing of three corresponding values of Table 1 between the considered configurations $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$, by fixing the subgraph $G^* \cup T^{n-2} \cup T^{n-1} \cup T^n$, we have

$$\begin{aligned}
\mathrm{cr}_D(G^* + D_n) &= \mathrm{cr}_D(K_{6,n-3}) + \mathrm{cr}_D(K_{6,n-3}, G^* \cup T^{n-2} \cup T^{n-1} \cup T^n) \\
&\quad + \mathrm{cr}_D(G^* \cup T^{n-2} \cup T^{n-1} \cup T^n) \\
&\geq 6\left\lfloor \frac{n-3}{2} \right\rfloor \left\lfloor \frac{n-4}{2} \right\rfloor + 14(r-3) + 10s + 6(n-r-s) + 13 \\
&= 6\left\lfloor \frac{n-3}{2} \right\rfloor \left\lfloor \frac{n-4}{2} \right\rfloor + 6n + 4(2r+s) - 29 \\
&\geq 6\left\lfloor \frac{n-3}{2} \right\rfloor \left\lfloor \frac{n-4}{2} \right\rfloor + 6n + 4\left(2n - 2\left\lfloor \frac{n}{2} \right\rfloor + 1\right) - 29 \\
&\geq 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2\left\lfloor \frac{n}{2} \right\rfloor.
\end{aligned}$$

This contradicts the assumption of $D$.

*Subcase 1b.* $\{\mathcal{A}_x, \mathcal{A}_y\} \subseteq \mathcal{M}_D$ with $x + 1 \equiv y \pmod 5$ and if $\mathcal{A}_z \in \mathcal{M}_D$, $z \neq x, y$, then neither $y + 1 \equiv z \pmod 5$ nor $z + 1 \equiv x \pmod 5$. Without lost of generality, let us consider two different subgraphs $T^{n-1}$, $T^n \in R_D$ such that $F^{n-1}$ and $F^n$ have mentioned configurations $\mathcal{A}_x$ and $\mathcal{A}_y$, respectively. Then,

$$\mathrm{cr}_D(G^* \cup T^{n-1} \cup T^n, T^i) \geq 10$$

holds for any $T^i \in R_D$ with $i \neq n-1, n$ also by summing the values in Table 1, and Corollary 3.3 forces

$$\mathrm{cr}_D(G^* \cup T^{n-1} \cup T^n, T^l) \geq 1 + 6 = 7$$

for any $T^l \in S_D$. Hence, by fixing the subgraph $G^* \cup T^{n-1} \cup T^n$, we have

$$\begin{aligned}\mathrm{cr}_D(G^*+D_n) &\geq 6\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor+10(r-2)+7s+4(n-r-s)+4\\ &= 6\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor+4n+3(2r+s)-16\\ &\geq 6\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor+4n+3\left(2n-2\left\lfloor\frac{n}{2}\right\rfloor+1\right)-16\\ &\geq 6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor+2\left\lfloor\frac{n}{2}\right\rfloor.\end{aligned}$$

This also contradicts the assumption of $D$.

*Subcase 1c.* $1 \leq |\mathcal{M}_D| \leq 2$ and if $\mathcal{M}_D = \{\mathcal{A}_x, \mathcal{A}_y\}$ for $x, y \in \{1,\ldots,5\}$, $x \neq y$, then neither $x+1 \not\equiv y \pmod 5$ nor $x-1 \not\equiv y \pmod 5$. Now, let us first suppose that $\mathcal{A}_y \in \mathcal{M}_D$ for some $y \in \{3,4,5\}$. Without lost of generality, we can assume that $T^n \in R_D$ with the configuration $\mathcal{A}_3$ of the subgraph $F^n$. The subdrawing of $F^n$ induced by $D$ is shown in Figure 2. Thus, we can easy to verify that there is no $T^l \in S_D$ with $\mathrm{cr}_D(T^n, T^l) \leq 2$. Moreover, $\mathrm{cr}_D(T^n, T^i) \geq 5$ holds for any $T^i \in R_D$, $i \neq n$, by the remaining values in Table 1. Thus, by fixing of the graph $G^* \cup T^n$, we have

$$\begin{aligned}\mathrm{cr}_D(G^*+D_n) &\geq 6\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor+5(r-1)+4s+3(n-r-s)+0\\ &= 6\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor+3n+(2r+s)-5\\ &\geq 6\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor+3n+\left(2n-2\left\lfloor\frac{n}{2}\right\rfloor+1\right)-5\\ &\geq 6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor+2\left\lfloor\frac{n}{2}\right\rfloor.\end{aligned}$$

Hence, the discussed drawing contradicts the assumption of $D$ again. Now, assume $\mathcal{M}_D = \{\mathcal{A}_y\}$ for only one $y \in \{1,2\}$. Without lost of generality, we can consider the configuration $\mathcal{A}_1$ of $F^n$. Let us denote

$$S_D(T^n) = \{T^l \in S_D : \mathrm{cr}_D(F^n, T^l) = 3\},$$

and $s_1 = |S_D(T^n)|$. Remark that $S_D(T^n)$ is a subset of $S_D$, and $s_1 \leq s$, that is, $s - s_1 \geq 0$. Hence, we will discuss two possibilities:

1) If $r > s_1$, that is, $r - 1 \geq s_1$, then by fixing of the graph $G^* \cup T^n$ we have

$$\begin{aligned}&\mathrm{cr}_D(G^*+D_n)\\ &\geq 6\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor+6(r-1)+3s_1+4(s-s_1)+3(n-r-s)+0\\ &\geq 6\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor+5(r-1)+4s+3(n-r-s)\\ &\geq 6\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor+(2r+s)+3n-5\\ &\geq 6\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor+\left(2n-2\left\lfloor\frac{n}{2}\right\rfloor+1\right)+3n-5\\ &\geq 6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor+2\left\lfloor\frac{n}{2}\right\rfloor.\end{aligned}$$

This contradicts the assumption of $D$.

2) Let us assume that $r \leq s_1$, that is, $0 \leq r-1 \leq s_1 - 1$. Let $T^l$ be a subgraph from nonempty set $S_D(T^n)$. As $\mathcal{M}_D = \{\mathcal{A}_1\}$, we have

$$\mathrm{cr}_D(G^* \cup T^n \cup T^l, T^i) \geq 6 + 2 = 8$$

for any $T^i \in R_D$, $i \neq n$. In the proof of Lemma 3.1, it was showed that the subgraph $F^l$ can be represented only by the cyclic permutation $(152436)$. Thus,

$$\mathrm{cr}_D(G^* \cup T^n \cup T^l, T^k) \geq 1 + 2 + 6 = 9$$

holds for any $T^k \in S_D(T^n)$, $k \neq l$. Again by Lemma 3.1, we can verify that

$$\mathrm{cr}_D(G^* \cup T^n \cup T^l, T^k) \geq 7$$

is fulfilling for any $T^k \in S_D$. Moreover,

$$\mathrm{cr}_D(G^* \cup T^n \cup T^l, T^k) \geq 2 + 4 = 6$$

for any $T^k \notin R_D \cup S_D$ provided by $\mathrm{cr}_D(K_{6,3}) \geq 6$ and $\mathrm{cr}_D(T^n, T^l) = 2$. Since $n - r - s \leq r - 1 \leq s_1 - 1$, by fixing of the graph $G^* \cup T^n \cup T^l$, we have

$$\begin{aligned}
&\mathrm{cr}_D(G^* + D_n) \\
&\geq 6\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + 8(r-1) + 9(s_1-1) + 7(s-s_1) + 6(n-r-s) + 3 \\
&\geq 6\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + 8(r-1) + 7(s_1-1) + 7(s-s_1) + 7(n-r-s) + 3 \\
&= 6\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + r + 7n - 12 \\
&\geq 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2\left\lfloor \frac{n}{2} \right\rfloor.
\end{aligned}$$

This also contradicts the assumption of $D$.

*Case 2.* $\mathrm{cr}_D(G^*) \geq 1$. In all considered cases, without loss of generality, we can choose the corresponding vertex notation of the graph $G^*$ in such a way as shown in Figure 1(b)–(h). Further, using Lemma 3.2 and Corollary 3.3, we are able to use the same idea and the same arguments as in Case 1, i.e., we obtain the same configurations, and also the same corresponding lower-bounds of numbers of crossings between two configurations as in Table 1.

Thus, it was shown in all mentioned cases that there is no good drawing $D$ of the graph $G^* + D_n$ with fewer than $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2\lfloor \frac{n}{2} \rfloor$ crossings. This completes the proof of the main theorem. □

## 4. FOUR OTHER GRAPHS

Finally, at least into one subdrawing in Figure 4, we are able to add some edges to the graph $G^*$ without additional crossings, and we obtain four new graphs $G_i + D_n$. The graphs $G_i$, $i = 1, 2, 3, 4$, are shown in Figure 5. Therefore, the drawings of the graphs $G_1 + D_n$, $G_2 + D_n$, $G_3 + D_n$, and $G_4 + D_n$ with $6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 2\lfloor\frac{n}{2}\rfloor$ crossings are obtained. On the other hand, $G^* + D_n$ is a subgraph of each $G_i + D_n$, and therefore, $\mathrm{cr}(G_i + D_n) \geq \mathrm{cr}(G^* + D_n)$ for any $i = 1, 2, 3, 4$. Thus, the next results are obvious.

**Collorary 4.1.** $\mathrm{cr}(G_i + D_n) = 6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + 2\left\lfloor\frac{n}{2}\right\rfloor$ *for* $n \geq 1$, *where* $i = 1, 2, 3, 4$.

Remark that the crossing numbers of the graphs $G_1 + D_n$ and $G_2 + D_n$ was already obtained in [12] also using the vertex rotation, and the crossing number of the graph $G_3 + D_n$ was established in [6] without using the vertex rotation.

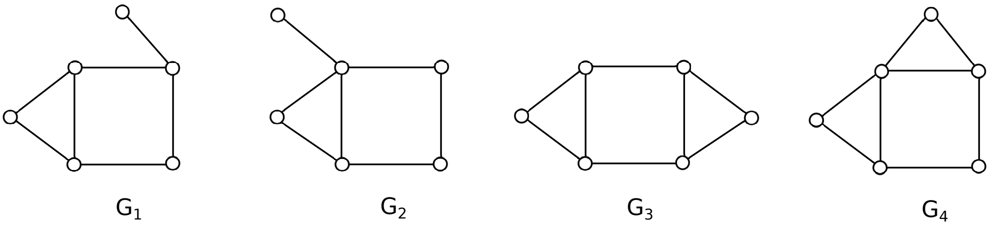


**Fig. 5.** Four graphs $G_1$, $G_2$, $G_3$, and $G_4$ obtained by adding new edges to the graph $G^*$

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