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The Bushell-Okrasiński inequality

*(Dedicated to the memory of
Peter J. Bushell (1934-2020) and Wojciech Okrasiński (1950-2020))*

Abstract We present an expository account of the Bushell-Okrasiński inequality, the motivation behind it, its history, and several generalisations. This inequality originally appeared in studies of nonlinear Volterra equations, but very soon gained interest of its own. The basic result has quickly been generalised and extended in different directions, strengthening the assertion, generalising the kernel and nonlinearity, providing the optimal prefactor, finding conditions under which it becomes an equality, and formulating variations valid for other than Lebesgue integrals. We review all of these aspects.

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1. Introduction Analysis is full of integral inequalities of many types and utility. Every young adept of the art has to learn and efficiently use results of Cauchy, Schwarz, Hölder, Jensen, Minkowski, Young, Sobolev, Poincaré, Friedrichs, Hardy, Chebyshev, and Opial, to name only a few classics. There is another type of inequality that resides somewhere between Hölder's and Jensen's. A result that can be thought of as a strengthening of the Chebyshev inequality. The Bushell-Okrasiński inequality, which in one of its basic forms for positive and increasing f, can be stated as

$$\int_0^x (x-s)^{\alpha-1} f(s)^\alpha ds \leq \left(\int_0^x f(s) ds\right)^\alpha, \quad \alpha \geq 1, \tag{1}$$

has been discovered in 1990 by Peter Bushell and Wojciech Okrasiński and published in their work on nonlinear Volterra integral equations [10]. The above result has quickly been included in Bullen's "Dictionary of Inequalities" ([6], p.35). Subsequently, many authors proceeded to investigate it further by relaxing assumptions, strengthening the claim, finding optimal constants, generalising to other than power-type kernels and nonlinearities, and translating it to fuzzy integrals. In this paper, we will look closely on historical development of this inequality and review some of its generalisations.

This review is structured as follows. First, we give some motivations behind Bushell-Okrasiński inequality (we will occasionally abbreviate it as BO inequality)and present its original proof. Then, we discuss Wolfang Walter's conjectures and their resolution in a joint work with Weckesser. Additionally, we present several different generalisations of the original result. We end the paper with a short detour into the land of fuzzy integrals, which also can enjoy some types of Bushell-Okrasiński inequality.

2. The original Bushell-Okrasiński inequality The main motivation behind the original Bushell-Okrasiński inequality was a study of nonlinear Volterra equations of the form

$$u(x) = \int_0^x k(x-s) g(u(s)) ds, \tag{2}$$

that arise in many important applications in porous media [21, 22] or shocks [18]. It is instructive to take a trip to the field of hydrology and see how the above Volterra integral equation can appear as a model of moisture imbibition. Suppose that a porous medium initially dry and half-infinite is subjected to water at $x = 0$. Then, in the absence of gravity, the capillary action is the only factor driving the evolution of the moisture $\theta = \theta(x,t)$ (that is, the percentage of representative volume filled with water). The mass conservation then leads to the following problem for the nonlinear diffusion equation known as the Richards equation with the diffusivity $D(\theta)$ [3],

$$\begin{cases} \dfrac{\partial \theta}{\partial t} = \dfrac{\partial}{\partial x}\left(D(\theta)\dfrac{\partial \theta}{\partial x}\right), & x>0, \quad t>0 \\ \theta(x,0) = 0, & \\ \theta(0,t) = 1, & \displaystyle\lim_{x\to 0^+} -D(\theta)\frac{\partial \theta}{\partial x} = 0, \end{cases} \tag{3}$$

where for simplicity we have chosen the appropriate physical units so that the resulting problem is nondimensional. Notice the no-flux boundary condition that tells us that no water is being injected into the medium - capillary forces do all the work. It is natural, both theoretically and experimentally, to look for self-similar solutions of the above in the form $\theta(x,t) = v(\eta)$ with $\eta := x/\sqrt{t}$. This gives an ordinary differential equation

$$-\frac{1}{2}\eta v' = (D(v)v')', \quad v(0) = 1, \quad (\cdot)' := \frac{d}{d\eta}. \tag{4}$$

Now, we can integrate the above over $[0,\eta]$ to obtain

$$\begin{aligned} -\frac{1}{2}\int_0^\eta s v'(s) ds &= \frac{1}{2}\left(-\eta v(\eta) + \int_0^\eta v(s) ds\right) \\ &= D(v(\eta))v'(\eta) - D(v(0))v'(0) = D(v(\eta))v'(\eta), \end{aligned} \tag{5}$$

where the second term on the right vanishes due to the no-flux condition. Notice also that the term on the left-hand side was integrated by parts. Furthermore, we can define the primitive of diffusivity D

$$G(u) := \int_0^u D(s)ds, \tag{6}$$

and with the help of which we have $D(v)v' = (G(v))'$. Therefore, a second integration yields

$$G(v(\eta)) = \frac{1}{2}\int_0^\eta (\eta - 2s)v(s)ds. \tag{7}$$

Finally, since D is positive, there is a well-defined inverse $g = G^{-1}$. Setting $v = g(u)$ brings us to an equation very similar to (2). For example, weakly singular kernels $k(s) = s^{\alpha-1}$ for some $\alpha \in (0,1)$ can arise in the study of anomalous diffusion [29, 30]. Different occurrences of (2) appear in shock propagation [19] and in axisymmetric water percolation problems [15].

The flagship example of nonlinearity is the power function $g(u) = u^{1/p}$ for some $p > 1$ that models the diffusivity of many porous media (this is the Brooks-Correy model of moisture transport in soil [4]). As can be easily observed, in this case the integral equation (2) has a trivial solution $u \equiv 0$. More generally, for a nonlinearity satisfying $g(0) = 0$ the trivial solution is always present. However, when g is *non-Lipschitz* then a non-trivial solution might exist (the Lipschitz condition rules out this case). Investigating these solutions is the main objective of Bushell's and Okrasiński's paper [10] as well as several other authors throughout the last decades (for example [5, 7, 8, 16, 26, 27, 28, 31, 32]). For a thorough review of this account in view of numerical methods, the Reader is referred to [1]. For the aforementioned root-type nonlinearity $g(u) = u^{1/p}$ one can easily verify that a non-trivial solution of (2) with a kernel $k(s) = s^{\alpha-1}$ and some $\alpha > 0$ is the power function

$$u(x) = x^{\frac{\alpha p}{p-1}} B\left(\alpha, 1 + \frac{\alpha}{p-1}\right)^{\frac{p}{p-1}}, \tag{8}$$

where $B(\cdot,\cdot)$ is the Euler beta function. There are many approaches that find the necessary and sufficient conditions on k and g for which (2) has non-trivial solutions and reviewing all of them would take us too far from the main theme of this paper. However, we briefly note that Bushell's and Okrasiński's approach is to use the monotone iteration method with sub- and supersolutions defined in functional cones (see, for example, [39]). In the main argument, the authors construct an *explicit* solution of the following associated integral equation

$$w(x) = \int_0^x g(w(s)^\alpha)^{\frac{1}{\alpha}} ds, \tag{9}$$

and show that it can exist if and only if $1 \leq \alpha < \alpha_c$ for some critical value α_c. This assertion is then carried over to the case of (2) and the crucial link between these two nonlinear Volterra equations is supplemented by the Bushell-Okrasiński inequality.

In many talks between the author and W. Okrasiński he always stressed that the inequality was just a "passing auxiliary lemma" needed to show necessary conditions for existence. Originally, W. Okrasiński did not realise that it can have a value of its own. This is probably the reason that he together with P. Bushell did not try to polish the result and provide a stronger assertion. This was later done by other authors and, to some extent, by Bushell, to which we will turn in next sections. Now, we present the original proof of Bushell-Okrasiński inequality[1]

THEOREM 2.1 (THE ORIGINAL BO INEQUALITY, LEMMA 2 IN [10]) Let $f \in C[0,X]$, $X>0$, be a non-decreasing and non-negative function. If $\alpha \geq 1$, then

$$\int_0^x (x-s)^{\alpha-1} f(s) ds \leq \left(\int_0^x f(s)^{\frac{1}{\alpha}} ds \right)^{\alpha}. \tag{10}$$

PROOF Our aim is to show (10) first for natural α by mathematical induction, then by Hölder inequality extend the result to all rational numbers, and finally by density argument arrive at α real. Fix f as in the assumptions and define for $n \in \mathbb{N}$ let

$$I_n(x) := \left(\int_0^x f(s) ds \right)^n - n \int_0^x (x-s)^{n-1} f(s)^n ds. \tag{11}$$

We can compute the derivative of the above quantity

$$\begin{aligned} I_n'(x) &= n f(x) \left(\int_0^x f(s) ds \right)^{n-1} - n(n-1) \int_0^x (x-s)^{n-2} f(s)^n ds \\ &= n f(x) I_{n-1}(x) + n(n-1) \int_0^x \left(f(x) - f(s)\right)(x-s)^{n-2} f(s)^{n-1} ds, \end{aligned} \tag{12}$$

which is valid for all $n \geq 1$. For $n=1$, we trivially have $I_1(x) = 0$, while for the next step,

$$I_2'(x) = 2 \int_0^x \left(f(x) - f(s)\right) f(s) ds \geq 0, \tag{13}$$

because f is non-decreasing. Therefore, $I_2(x) \geq I_2(0) = 0$. Now, we assume that $I_{n-1}(x) \geq 0$ for $n > 2$. We immediately have $I_n(x) \geq 0$ since manifestly

[1] In this review we will state all important results as "Theorems" in contrast, for example, with [10] where the BO inequality is designated as "Lemma".

$I_n'(x) \geq 0$ due to inductive assumption and non-decreasing of f. Therefore, $I_n(x) \geq 0$ for all $n \in \mathbb{R}$, that is,

$$n \int_0^x (x-s)^{n-1} f(s)^n ds \leq \left(\int_0^x f(s) ds \right)^n, \tag{14}$$

which implies (10) for $\alpha = n$ with f replaced by $f^{1/n}$ which is also positive and non-decreasing.

Now, fix $p > 1$ and take the conjugate exponent $q^{-1} = 1 - p^{-1}$. By Hölder inequality we have

$$\int_0^x (x-s)^{\frac{n-1}{p}} f(s)^{\frac{n}{p}} f(s)^{\frac{1}{q}} \leq \left(\int_0^x (x-s)^{n-1} f(s)^n ds \right)^{\frac{1}{p}} \left(\int_0^x f(s) ds \right)^{\frac{1}{q}}. \tag{15}$$

If we now put $\alpha = 1 + (n-1)/p$, due to the arbitrariness of p, we obtain (10) for any $\alpha \in \mathbb{Q}$. From the density of rational numbers in $\mathbb{R}$ we obtain the Bushell-Okrasiński inequality for all real $\alpha \geq 1$. This concludes the proof. ■

Remark 2.1 The original result, that is, Lemma 2 in [10] also contains the following inequality

$$\left(\frac{\beta - \alpha}{\beta - 1} \right)^{\beta - 1} \left(\int_0^x f(s)^{\frac{1}{\beta}} ds \right)^{\beta} \leq \int_0^x (x-s)^{\alpha-1} f(s) ds, \quad 0 < x \leq 1, \tag{16}$$

for $\beta > \alpha$ without the requirement on f to be non-decreasing. However, the proof is a simple consequence of Hölder's inequality and thus we omit it here. This justifies the claim that (10) is a *reverse Hölder type inequality* (see [2]).

We close this section with some remarks concerning nonlocal operators, in particular fractional integrals. For a comprehensive treatment of this subject, the reader is invited to consult [20]. The notion of generalising derivatives to not necessarily integer order has been present in mathematics since the beginning of the calculus itself (an interesting historical account can be found in [25]). Many different approaches have been employed and culminated in the definition of the Riemann-Liouville fractional integral and derivative.

Definition 2.2 The *Riemann-Liouville fractional integral* of order $\alpha > 0$ of a locally integrable function $f : [0, X] \mapsto \mathbb{R}$ is given by

$$I_a^\alpha f(x) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \int_a^x (x-s)^{\alpha-1} f(s) ds, & \alpha > 0 \\ f(x), & \alpha = 0. \end{cases} \quad x \geq a. \tag{17}$$

Moreover, let $\alpha > 0$ and $n = \lceil \alpha \rceil = \min \{ m \in \mathbb{N} : m \geq n \}$ ($\lceil \cdot \rceil$ is the ceiling function). Then *Riemann-Liouville fractional derivative* of order α is defined by

$$D_a^\alpha f(x) = \frac{d^n}{dx^n} I_a^{n-\alpha} f(x), \quad x \geq a. \tag{18}$$

We immediately can notice the similarity of the fractional integral and the left-hand side of Bushell-Okrasiński inequality. Fix $p \geq 1$ and let $q = \alpha p > 1$ along with the conjugate exponent $q^{-1} + q'^{-1} = 1$. Then, from (10) we obtain for $0 < x \leq X \leq 1$ and f positive non-decreasing,

$$I_0^\alpha f(x) \leq \frac{1}{\Gamma(\alpha)} \left(\int_0^x f(s)^{\frac{1}{\alpha}} ds \right)^\alpha \leq \frac{1}{\Gamma(\alpha)} \left(\int_0^1 f(s)^{\frac{q}{\alpha}} ds \right)^{\frac{\alpha}{q}} \left(\int_0^1 1^{q'} ds \right)^{\frac{1}{q'}}$$
$$= \frac{1}{\Gamma(\alpha)} \left(\int_0^1 f(s)^p ds \right)^{\frac{1}{p}}, \quad 1 \leq p \leq \infty, \tag{19}$$

which follows from Hölder's inequality. Therefore, Bushell-Okrasiński inequality implies that the fractional integral is a bounded linear operator for such functions (for a general $L^p(0,1)$ space, see [20], Lemma 2.1). In fact, we will see below in (20) that the optimal constant in the bound is equal to $1/\Gamma(1+\alpha)$.

3. Improvements and generalisations

3.1. Walter's conjectures As we have mentioned above, authors of the original 1990 paper [10] treated the Bushell-Okrasiński inequality as a side lemma that was needed to investigate the nontrivial solutions of the nonlinear Volterra integral equation (2). Very soon, in fact, almost immediately after the publication, the inequality gained some attention in the mathematical community. In December 1990 during the 6th International Conference on General Inequalities in Oberwolfach, Germany, Wolfgang Walter posed two conjectures related to strengthening of the original inequality (the proceedings [36] appeared in 1992). The first one is based on the observation that in the original proof a stronger result is obtained for $\alpha \in \mathbb{N}$ as in (14) where the factor n appears on the left-hand side. Then, in the proof for $\alpha \in \mathbb{Q}$ this fact is used only partially, and the stronger inequality is lost. Walter asked whether it is possible to find a simpler proof of the Bushell-Okrasinski inequality with the improved assertion that for all $\alpha \geq 1$ we have

$$\alpha \int_0^x (x-s)^{\alpha-1} f(s)^\alpha ds \leq \left(\int_0^x f(s) ds \right)^\alpha. \tag{20}$$

It is also natural to ask whether the above is satisfied for $0 < \alpha < 1$ provided that the function f is nonnegative and decreasing. Furthermore, the question arises of whether the assumption that f is defined in an interval $[0, X]$ with $X \leq 1$ is necessary at all. All of these claims have been successfully proved in a joint paper with V. Weckesser published in 1993 (the paper was submitted in November 1991 and revised in August 1992). In fact these authors proved a much more general result with a proof that is based on approximations by step functions and Monotone Convergence Theorem.

THEOREM 3.1 (GENERALIZED BO INEQUALITY, THEOREM 1 IN [37]) Suppose that $f : [0, X] \mapsto [0, \infty)$ for $X > 0$, $g : [0, \infty) \mapsto [0, \infty)$, and $k \in L^1[0, X]$. Define

$$K(x) := \int_0^x k(s)ds, \quad h_c(y) := g(cy) - K(c)g(y), \tag{21}$$

for $0 < c \leq X$. Then, if either

$$f \text{ is non-decreasing}, \quad g \text{ is convex}, \quad h_c \text{ is nonnegative and non-decreasing}, \tag{I}$$

or

$$f \text{ is non-increasing}, \quad g \text{ is concave}, \quad h_c \text{ is nonnegative and non-increasing}, \tag{II}$$

the following generalized Bushell-Okrasiński inequality is satisfied

$$\int_0^x k(x-s)g(f(s))ds \leq g\left(\int_0^x f(s)ds\right), \quad 0 < x \leq X. \tag{22}$$

Before we proceed to the proof, we relate the original Walter's conjectures to the above theorem.

COROLLARY 3.2 The inequality (20) is satisfied for all $0 < x \leq X$ with an arbitrary $X > 0$ when

- $f : [0, X] \mapsto [0, \infty)$ is non-decreasing and $\alpha \geq 1$, or
- $f : [0, X] \mapsto [0, \infty)$ is non-increasing and $0 < \alpha \leq 1$.

PROOF We have $k(x) = \alpha x^{\alpha-1}$ and therefore $K(x) = x^\alpha$. Further, $g(y) = y^\alpha$ and thus for $0 < c \leq X$ we have

$$h_c(y) = g(cy) - K(c)g(y) = (cy)^\alpha - c^\alpha y^\alpha = 0, \tag{23}$$

and the condition for h_c is satisfied both in (I) and (II). Since the power function g is concave for $0 < \alpha \leq 1$ and convex for $\alpha \geq 1$ the proof is completed using Theorem 3.1. ■

We can now proceed to the proof of Walter and Weckesser's result.

PROOF (of Theorem 3.1). We will focus only on (I) case; the proof of the other is similar. Due to Beppo Levi's Theorem of Monotone Convergence, it is sufficient to consider (22) for f which are step functions

$$f(x) = \sum_{i=1}^{n-1} a_i \chi_{[x_{i-1}, x_i)}(x) + a_n \chi_{[x_{n-1}, x_n]}(x), \tag{24}$$

for a partition

$$0 = x_0 < x_1 < x_2 < ... < x_n = x, \tag{25}$$

where χ_A is a characteristic function of a measurable set A. If the integral on the left in (22) is denoted by L_n, a simple calculation gives

$$L_n = \sum_{i=1}^{n} \left(K(x - x_{i-1}) - K(x - x_i)\right) g(a_i). \tag{26}$$

Similarly,

$$I_n := \int_0^x f(s)ds = \sum_{i=1}^{n} a_i(x_i - x_{i-1}), \tag{27}$$

hence, the right-hand side of (22), denoted by R_n, satisfies $R_n = g(I_n)$. Now, notice that for $n = 1$, that is, for constant functions, we have $x_1 = x$, and

$$L_1 - R_1 = g(a_1)\left(K(x) - K(0)\right) - g(a_1 x) = h_{a_1}(x) \geq 0, \tag{28}$$

by the assumption on h_c. We can now utilise mathematical induction. Assume that $L_n \leq R_n$. We claim then that this inequality holds for $n+1$. Without any loss of generality, we can assume that the $n+1$-th step of the function f can arise as a partition of the interval $[x_{n-1}, x_n]$. Pick any $0 < c < x_n - x_{n-1}$ and $y \geq 0$. Now, the update to the non-decreasing step function f is given by

$$f(x) = \sum_{i=1}^{n-1} a_i \chi_{[x_{i-1}, x_i)}(x) + a_n \chi_{[x_{n-1}, x_n - c)} + (a_n + y)\chi_{[x_n - c, x_n]}. \tag{29}$$

It is now straightforward to calculate the relevant integrals. It follows that only the last interval makes the difference, that is, since we still have $x_n = x$, and

$$\begin{aligned} \Delta L_{n+1} :&= L_{n+1} - L_n = \left(g(a_n + y) - g(a_n)\right) \int_{x-c}^{x} k(x-s)ds \\ &= \left(g(a_n + y) - g(a_n)\right) K(c). \end{aligned} \tag{30}$$

Similarly,

$$\Delta R_{n+1} := R_{n+1} - R_n = g(I_n + cy) - g(I_n). \tag{31}$$

Since g is convex and trivially $a_n c \leq I_n$ we have further $\Delta R_{n+1} \leq g(c(a_n + y)) - g(ca_n)$. Next, by the fact that h_c is non-decreasing

$$\begin{aligned} \Delta L_{n+1} - \Delta R_{n+1} &\leq \left(g(a_n + y) - g(a_n)\right) K(c) - g(c(a_n + y)) + g(ca_n) \\ &= h_c(a_n) - h_c(a_n + y) \leq 0. \end{aligned} \tag{32}$$

Therefore, by the inductive assumption that $L_n \leq R_n$ we have $L_{n+1} = L_n + \Delta L_{n+1} \leq R_n + \Delta R_{n+1} = R_{n+1}$ and the proof is complete. ∎

As we have seen, the proof of Walter and Weckesser is of a completely different nature than Bushell and Okrasiński's. It can be regarded as elementary, which allows for a substantial improvement of the claim. We have seen in the above corollary that taking K and g as power functions, the general inequality (22) reduces to the stronger version of the original (20). Quite recently, T. Małolepszy and J. Matkowski, asked a somewhat reverse question: is this the only choice that yields Bushell-Okrasiński inequality? (see [24]). They show several results concerning that topic. One of which states that assuming $X > 1$, (I), and $K(x) = x^\alpha$, the only choice for the other function is very restricted, that is $g(y) = g(1)y^\alpha$ or $g \equiv 0$ for $\alpha \geq 1$. Interestingly, for $0 < \alpha < 1$, the only option allowed is a trivial g.

3.2. Equality in (14) Having an inequality of the type (20) there naturally arises a question about its sharpness. This can be answered quickly positively by taking a constant function f for which the inequality becomes an equality. But is this the only case where it occurs? It was shown in [37] that (notice that here $x = 1$)

$$\alpha \int_0^1 (1-t)^{\alpha-1} f(t) dt = \int_0^1 f(t) dt \quad \Longleftrightarrow \quad f \equiv \text{const.} \tag{33}$$

Encouraged by this example, W. Walter asked whether the same conclusion holds true for the Bushell-Okrasiński inequality

$$\alpha \int_0^1 (1-t)^{\alpha-1} f(t)^\alpha dt = \left(\int_0^1 f(t) dt \right)^\alpha \quad \Longleftrightarrow \quad f \equiv \text{const.} \tag{34}$$

It is relatively easy to prove that an equality in (14) occurs only for constant f when $\alpha \in \mathbb{N}$. The proof follows the same route as the original one by Bushell and Okrasiński for their inequality. One has just to inductively verify conditions for which I_n defined in (11) is equal to 0. This method was generalised to P. Bushell and A. Carbery in [9]. They have proved that the equality in (14) for all $\alpha \geq 1$ (in fact, in some generalised version of it) occurs only if for some $0 \leq x_0 < X$ we have

$$f(x) = \begin{cases} 0, & 0 < x \leq x_0, \\ C, & x_0 < x \leq X, \end{cases} \tag{35}$$

where $C > 0$ is a constant. This solves the open problem posed by Walter.

3.3. Reversed Jensen type inequalities In subsequent years following [10] and [36] several other generalisations and improvements of Bushell-Okrasiński inequality appeared in the literature. For example, Y. Egorov in 2000 gave another elementary functional-analytic proof of Walter's conjecture [13] in the case of continuous functions for a slightly stronger inequality as in (14). P. Bushell himself went further into the direction of investigating the reversed Jensen inequality. In a paper with A. Carbery [9] they stated the following result (actually, they have proved a much general inequality). The proof is different from Walter and Weckesser's and in the following we present its key features.

THEOREM 3.3 (COROLLARY 2 IN [9]) Let f be non-decreasing, positive function on $[0, X]$ and $g : [0, \infty) \to [0, \infty)$ be a convex function with $g(0) = 0$. For any positive and integrable function k define

$$K(x) = \int_0^x k(s)ds, \quad 0 < x \leq X. \tag{36}$$

Further, suppose that

$$g\left(\frac{y}{c}\right) K(cx) \leq g(y)K(x), \quad 0 < x < X, \quad y > 0, \quad 0 < c < 1. \tag{37}$$

Then,

$$\int_0^x k(x-s)g(f(s))ds \leq K(x)g\left(\frac{1}{x}\int_0^x f(s)ds\right), \quad 0 < x \leq X. \tag{38}$$

PROOF Fix $x \in (0, X]$. After substitution $x \to s - x$ the inequality (38) becomes

$$\int_0^x k(s)g(f(x-s))ds \leq K(x)g\left(\frac{1}{x}\int_0^x f(x-s)ds\right), \tag{39}$$

Since the function $[0, x] \ni s \to x - s \in [0, x]$ is decreasing, so is the function $[0, x] \ni s \to f(x - s)$. Therefore, we can put $w(s) := f(x - s)$, which renders our claimed inequality as

$$\int_0^x k(s)g(w(s))ds \leq K(x)g\left(\frac{1}{x}\int_0^x w(s)ds\right), \tag{40}$$

where the function $w = w(x)$ is *non-increasing* and positive. Now we define an auxiliary function $h = h(x)$ such that

$$h(x) := \frac{1}{x}\int_0^x w(s)ds \geq w(x) > 0, \tag{41}$$

and $h(0) = w(0)$.

Take any $\epsilon > 0$ and use (37) with $c = y/(y + \epsilon)$ to obtain

$$g(y+\epsilon)K\left(\frac{xy}{y+\epsilon}\right) \leq g(y)K(x). \tag{42}$$

By subtracting from (37) we arrive at the following

$$(g(y+\epsilon) - g(y))\, K\left(\frac{xy}{y+\epsilon}\right) \leq g(y)\left(K(x) - K\left(x - \frac{x}{y+\epsilon}\epsilon\right)\right), \tag{43}$$

since every convex function is differentiable y-a.e. we can divide by ϵ and pass to the limit, obtaining

$$yg'(y)K(x) \leq xg(y)k(x), \tag{44}$$

since, by definition, $K'(x) = k(x)$. Then, if we define

$$\Delta(x) = xg(h(x))k(x) - h(x)g'(h(x))K(x), \tag{45}$$

and

$$\varphi(x) = K(x)g(h(x)) - \int_0^x k(s)g(w(s))ds, \tag{46}$$

a straightforward computation yields $\varphi(0) = 0$, and

$$\varphi'(x) = k(x)f(x)\left(\frac{g(h(x))}{h(x)} - \frac{g(w(x))}{w(x)}\right) + \frac{\Delta(x)}{xh(x)}\left(h(x) - w(x)\right). \tag{47}$$

From here and from the definition of $h(x)$ we can also observe that $\varphi'(0) = 0$ since the first term in the above vanishes due to continuity of g while for the second we have

$$\begin{aligned} \lim_{x\to 0}\frac{\Delta(x)}{x} &= \lim_{x\to 0}\left(g(h(x))k(x) - h(x)g'(h(x))\frac{K(x)}{x}\right) \\ &= g(w(0))k(0) - w(0)g'(w(0))\lim_{x\to 0}\frac{1}{x}\int_0^x k(s)ds \\ &= g(w(0))k(0) - w(0)g'(w(0))k(0). \end{aligned} \tag{48}$$

Hence, $\Delta(x)/x$ is bounded, and the second term in (47) vanishes when $x \to 0$. Therefore, $\varphi'(0) = 0$.

Furthermore, from (44) we have that $\Delta(x) \geq 0$ and hence the second term in (47) is non-negative. On the other hand, the function $g(y)/y$ is non-decreasing due to convexity of g, as can be verified by computing derivatives. From this we conclude that $\varphi(x) \geq 0$ which immediately proves that

$$\int_0^x k(s)g(w(s))ds \leq K(x)g(h(x)), \tag{49}$$

which is (40). By the definition of w, this inequality is equivalent to (38). The proof is complete. ■

There is an interesting corollary of this. Taking $g(y) = y^\alpha$ and $K(x) = x^\beta$ with $1 \leq \alpha \leq \beta$ yields

$$\beta\int_0^x (x-s)^{\beta-1}f(s)^\alpha ds \leq x^{\beta-\alpha}\left(\int_0^x f(s)ds\right)^\alpha, \quad 0 < x \leq 1 \tag{50}$$

which is the "β-generalization" of the Bushell-Okrasiński inequality. Note that Walter and Weckesser's result (22) would give a weaker inequality, lacking the $x^{\beta-\alpha}$ factor on the right-hand side.

Another interesting approach to generalisation of (14) was given by S. M. Malamud in 2001 (see [23]). The author observed that the Bushell-Okrasiński inequality is a strengthening of the classical Chebyshev's result[2] for increasing $f \geq 0$

$$\alpha \int_0^1 (1-s)^{\alpha-1} f(s)^\alpha ds \leq \left(\alpha \int_0^1 (1-s)^{\alpha-1} ds\right) \left(\int_0^1 f(s)^\alpha ds\right) = \int_0^1 f(s)^\alpha ds. \tag{51}$$

The point is that by Hölder's inequality we always have $\int_0^1 f^\alpha ds \geq (\int_0^1 f ds)^\alpha$. Malamud proved the following general result

$$\frac{\displaystyle\int_0^1 g(f(s))w(s)d\Phi(s)}{\displaystyle\int_0^1 w(s)d\Phi(s)} \leq g\left(\frac{\displaystyle\int_0^1 f(s)w(s)ds}{\displaystyle\int_0^1 w(s)ds}\right), \tag{52}$$

when w is the weight and there are certain restrictions placed on g and Φ. The above reduces to the Bushell-Okrasiński inequality for $g(x) = x^\alpha$, $\Phi(x) = 1-(1-x)^\alpha$, and $w \equiv 1$. Moreover, it can also be thought of as an inverse to Jensen's inequality. Similarly to Walter and Weckesser's result, the proof proceeds by approximation of step functions. Also, the question of whether the equality in the above occurs only for constant functions remains unanswered. For simplicity, we will consider only the unweighted case, that is, $w \equiv 1$. The general inequality can be proved by the technique of approximation with step functions, however, the proof is more involved and the assumptions are more restrictive.

Theorem 3.4 (Theorem 2.1 in [23]) Suppose that f is a positive non-decreasing function, Φ is a positive function of bounded variation equal to 1, and g is positive, non-decreasing, convex, and differentiable function. Then, provided that

$$g'(y)\frac{\Phi(1)-\Phi(x)}{1-x} \leq g'(y(1-x)), \quad 0 < x < 1, \quad 0 < \lambda < \infty, \tag{53}$$

we have

$$\int_0^1 g(f(s))d\Phi(s) \leq g\left(\int_0^1 f(s)ds\right). \tag{54}$$

Proof Without the loss of generality we approximate the monotone function by an increasing sequence of step-functions

$$f_n(x) = \sum_{i=1}^n a_i \chi_{(\frac{i-1}{n},\frac{i}{n})}(x). \tag{55}$$

[2] Here, we mean the Chebyshev inequality for positive increasing f and positive decreasing g: $\int_0^1 f(x)g(x)dx \leq \left(\int_0^1 f(x)dx\right)\left(\int_0^1 g(x)dx\right)$.

For this choice, the inequality simply becomes

$$\sum_{i=1}^{n} g(a_i)\left(\Phi\left(\frac{i}{n}\right)-\Phi\left(\frac{i-1}{n}\right)\right) \le g\left(\frac{1}{n}\sum_{i=1}^{n} a_i\right). \tag{56}$$

Next, we introduce φ as the difference between the left and right-hand side of the above as a function of the largest value of f, that is,

$$\begin{aligned}\varphi(y) := g\left(\frac{y}{n}+\frac{1}{n}\sum_{i=1}^{n-1} a_i\right) - g(y)\left(\Phi\left(1\right)\right.\\ \left.-\Phi\left(\frac{n-17130}{n}\right)\right) - \sum_{i=1}^{n-1} g(a_i)\left(\Phi\left(\frac{i}{n}\right)-\Phi\left(\frac{i-1}{n}\right)\right).\end{aligned} \tag{57}$$

Then, taking the derivative gives

$$\begin{aligned}\varphi'(y) &= \frac{1}{n}g'\left(\frac{y}{n}+\frac{1}{n}\sum_{i=1}^{n-1} a_i\right) - g'(y)\left(\Phi\left(1\right)-\Phi\left(\frac{n-1}{n}\right)\right)\\ &\ge \frac{1}{n}g'\left(y\left(1-\frac{n-1}{n}\right)\right) - g'(y)\left(\Phi\left(1\right)-\Phi\left(\frac{n-1}{n}\right)\right),\end{aligned} \tag{58}$$

since g is convex. Furthermore, by our assumption (53) we conclude that $\varphi'(y) \ge 0$. Therefore, φ increases and we can consider (56) for the worst case, that is, for $a_n = a_{n-1}$ (since by assumption we always have $a_n \ge a_{n-1}$). But then, by redefining the function φ for $x = a_{n-1}$ we reduce the inequality to the case where $g_{n-1} = g_{n-2}$. Continue in this way yields the obvious $\sum_{i=1}^{n}\left(\Phi(i/n)-\Phi((i-1)/n)\right) = 1$. The general case follows from the Monotone Convergence Theorem. ■

We can note how the Malamud's result (54) corresponds to Walter and Weckesser's result (22). Recall the definition of the function $h_c(y)$ in (21). If we assume that it is differentiable, then it is non-decreasing when $h_c'(y) \ge 0$. But this requirement is exactly the same as (53) with $c = 1-x$ and $k(t) = \Phi'(1-t)$ provided the latter derivative exists. Therefore, we can think that Walter and Weckesser's result requires less regularity than Malamud's for the generalised Bushell-Okrasiński inequality to hold. Note, however, that (52) is more general than (22).

We end this section by mentioning some other approaches to generalising Bushell-Okrasiński inequality. In 1995 H. Heinig and L. Maligranda proved that for f, Φ positive and non-decreasing with $\lim_{s\to a^+}\Phi(s) = 0$ it holds that

$$\int_a^b f(b-s)^\alpha d\left(\Phi(s)\right)^\alpha \left(\le \int_a^b f(b-s)d\Phi(s)\right)^\alpha, \quad \alpha \ge 1, \tag{59}$$

which is (14) for $\Phi(s) = s$, $a = 0$, and $b = x$. Note that Malamud's inequality (54) includes this case, since (53) is satisfied since $g(s) = s^\alpha$ is convex. Further generalisations have been given in [2].

4. Bushell-Okrasiński inequality for fuzzy integrals Lately, a number of researchers have initiated the programme of extending the Bushell-Okrasiński inequality onto some other than Lebesgue types of integrals. In 2008 a Sugeno type fuzzy integral has been considered by H. Román-Flores, A. Flores-Franulič, and Y. Chalco-Cano [33]. In order to present this interesting result first we have to introduce some concepts concerning fuzzy measures (for a comprehensive treatment, see [38]).

DEFINITION 4.1 Let Σ be the σ-algebra of subsets of $\mathbb{R}$. Then, a function $\mu : \Sigma \mapsto [0, \infty]$ is a *fuzzy measure* if

- $\mu(\emptyset) = 0$,
- it is monotone,
- it is continuous from above and below.

In particular, the difference with the classical measure is the fact that in the fuzzy setting, we relax the requirement of additivity in favour of monotonicity from both sides. If f is a non-negative real-valued function we define its α-level set by $\{f > \alpha\} := \{x \in \mathbb{R} : f(x) > \alpha\}$ with $\alpha > 0$. Moreover, if μ is a fuzzy measure we define

$$\mathcal{F}(\mathbb{R}) := \{f : \mathbb{R} \mapsto [0, \infty) : f \text{ is measurable}\} \tag{60}$$

This lets us define the Sugeno fuzzy integral.

DEFINITION 4.2 (SUGENO INTEGRAL [35]) Let μ be the fuzzy measure on $(\Sigma, \mathbb{R})$. For $f \in \mathcal{F}$ and $A \in \Sigma$ the Sugeno integral (or fuzzy integral) is defined as

$$⨍_A f d\mu = \sup_{\alpha \geq 0} \left[\min\left(\alpha, \mu(A \cap \{f \geq \alpha\})\right)\right] \tag{61}$$

It is interesting to observe that Sugeno integrals do not enjoy some properties of the Lebesgue integrals. For example, they are not linear operators. However, many types of inequalities can also be proved for Sugeno integrals. For example, A. Flores-Franulič and H. Román-Flores [14] showed the following Chebyshev inequality for strictly increasing continuous functions

$$⨍_0^1 fg d\mu \geq \left(⨍_0^1 f d\mu\right)\left(⨍_0^1 g d\mu\right), \tag{62}$$

where μ is the Lebesgue measure. In similar spirit, K. Sadarangani and J. Caballero [11] proved the following type of Chebyshev inequality for Sugeno integrals

$$\mu\left(x \in A : f(x) > \alpha\right) \leq \frac{1}{\alpha^2} ⨍_A f^2 d\mu, \quad 0 < \alpha \leq 1, \tag{63}$$

for $\mu : \sigma \mapsto [0,1]$ being a fuzzy measure and positive $f \in \mathcal{F}$. It is important to note that the above inequality is valid if and only if $0 < \alpha \leq 1$. The proof is a of completely different nature than in the Lebesgue case, since one cannot utilise the linearity of the integral operator. The Bushell-Okrasiński type inequality is also valid for Sugeno integrals. In [33] authors showed that for positive, continuous, and increasing functions f we have

$$\alpha ⨍_0^1 s^{\alpha-1} f(s)^\alpha ds \geq \left(⨍_0^1 f(s) ds \right)^\alpha, \quad \alpha \geq 2. \tag{64}$$

The proof starts with the aforementioned fuzzy Chebyshev inequality and utilises a number of techniques from fuzzy measure theory. We omit it because it is out of the scope of our review. Note that the above is valid for $\alpha \geq 2$ and, surprisingly, the inequality is reversed, in contrast with the result for Lebesgue integrals. However, as was shown recently by D. Hong in 2020, the above formulation of the inequality is not optimal [17]. Instead, with the above assumptions, we have the following

$$\left(⨍_0^1 s^{\alpha-1} ds \right)^{-1} ⨍_0^1 s^{\alpha-1} f(s)^\alpha ds \geq \left(⨍_0^1 f(s) ds \right)^\alpha, \quad \alpha \geq 1, \tag{65}$$

where now we allow for the whole range of α. Notice the constant in parentheses above. For the Lebesgue integral, it would equal α. However, as noted in [17], for the Sugeno case, it is always smaller or equal to it (for example, when $\alpha = 3$ it equals 2.618). Hong also gives some useful estimates of this prefactor. Notice a completely different behaviour of the Sugeno integral compared with the Labesgue case. For a literature concerning different inequalities for Sugeno integrals, the reader is referred to [34].

Apart from Sugeno integrals, various generalisations of the concept of integration have been proposed, analysed, and applied. Reviewing these would take us too far from the main topic of our short exposition. Some of these generalisations posses their own Bushell-Okrasiński type inequalities. For instance, pseudo-integrals for which, loosely speaking, instead of the field of real numbers one considers a semi-ring defined on a real interval, exhibit a version of (14) with redefined multiplication and addition [12]. Notice how different various properties of these integrals might be from the Lebesgue case (like the loss of linearity). But nevertheless, Bushell-Okrasiński inequality (or its variants) remains valid. This strengthens its universal character.

5. Conclusion The Bushell-Okrasiński inequality is a little mathematical gem discovered when studying nonlinear integral equations. The wide array of different possible extensions and generalisations indicates that it is a fundamental relation in mathematical analysis. It waited to be found until almost the end of the twentieth century, but now sits comfortably within the collection of its older siblings - Chebyshev, Hölder, and Jensen inequalities.

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Nierówność Bushella-Okrasińskiego

Ł. Płociniczak

Streszczenie W niniejszej pracy omawiamy nierówność Bushella-Okrasiego: jej historię, motywacje za nią stojące oraz kilka uogólnień. Ta nierówność pierwotnie pojawiła się w badaniach nieliniowych równań Volterry, ale bardzo szybko zdobyła zainteresowanie wielu matematyków. Podstawowy wynik został szybko uogólniony i rozszerzony w różnych kierunkach. Między innymi inni autorzy wzmocnili główną tezę, uogólnili jądro oraz nieliniowość, wyznaczyli optymalną stałą multiplikatywną, znaleźli warunki, przy których występuje równość oraz sformułowali liczne warianty ważne dla całek innych niż Lebesgue'a. Dokonujemy przeglądu wszystkich tych aspektów.

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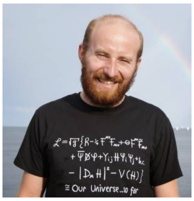

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