ADAPTIVE FILTER FOR INERTIAL SYSTEMS

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Abstract

The stochastic model of the disturbances handled by Kalman filters and the necessity of accurate identification of the dynamic model of the controlled system or process bring about a significant limitation of use of Kalman filters in practice. The paper presents filter designed for inertial systems whose models and control signals are not well known or are beyond description. The assumptions leading to a significant simplification of Kalman algorithm are described. On this basis, the algorithm with an experimentally matched parameter for the filter properties modification is introduced. An example of effective adaptive filtration is presented also.

Key words: digotal signal processing, adaptive filtration, Kalman filter, inertial systems, thermal systems

1 Introduction

Filtration is an important tool in digital signal processing. The adaptive filtration of signals describing the state of the systems and processes in the presence of noise and measurement inaccuracies has been used for a long time. Many algorithms are employed for this purpose (e.g. [1]), including the Kalman filter algorithm. Unfortunately, the stochastic model of the disturbances handled by Kalman filters and the necessity of accurate identification of dynamic model of the controlled system or process bring about a significant limitation of use of Kalman filters in practice. The paper presents an attempt to get around these limitations by using a modified Kalman filter algorithm for adaptive filtration.

This filter is designed for inertial systems whose models and control signals are not well known or are beyond description. An example of such a system can be a thermal system with accidental batch. Another example is the non-explosive combustion of a sample of unknown composition. The filter described in the article has been, among others, applied for the measurement of the SO2 and CO2 contents in combustion gases produced in such a process [2].

2. Kalman filter

The adaptive properties of Kalman filter [1] are associated with the optimal linear quadratic estimation of the system state, based on the knowledge of the system model, on the control vector and on the parameters of stochastic disturbances of state and also on the measurement of state variables. The algorithm is based on a set of equations describing the dynamics in the state space in the presence of disturbances. Its discrete form is

$$\begin{aligned} x(k) &= Ax(k-1) + B(k)u(k) + v(k) \\ y(k) &= Cx(k) + w(k) \end{aligned} \tag{1}$$

where A – state matrix, B – input matrix, C – output matrix, x – state vector, y – output vector, u – control vector, v – state disturbances vector with the covariance matrix Q, and w – measurement disturbance vector with the covariance matrix R. For a better legibility we shall assume later on that the state vector is fully observed, which means that C is a unitary matrix. It does not cause any limitation, because on the basis of the full form of the Kalman filter description the necessary modifications can be easily made.

The Kalman filter algorithm of the estimation of state vector value in the k-th step, made before the measurement y(k), is based on its estimation in the previous step $\hat{x}(k-1 \mid k-1)$, as described by the equation

$$\hat{x}(k \mid k-1) = A\hat{x}(k-1 \mid k-1) + Bu(k-1) \tag{2}$$

The estimation error is defined as

$$\tilde{x}(k-1 \mid k) = x(k) - \hat{x}(k \mid k-1) \tag{3}$$

It has the covariance

$$P(k \mid k-1) = AP(k-1 \mid k-1)A^T + Q(k-1) \tag{4}$$

When the measurement y(k) is completed, the additional information is obtained, on the basis of which the state vector estimator in the k-th step gets the form

$$\hat{x}(k \mid k) = \hat{x}(k \mid k-1) + K(k)[y(k) - \hat{x}(k \mid k-1)] \tag{5}$$

where K(k) is the Kalman gain matrix, which can be calculated from the equation

$$K(k) = P(k \mid k-1)[P(k \mid k-1) + R(k)]^{-1} \tag{6}$$

The covariance of the state estimation error for k step is related to the covariance (4) as follows

$$P(k \mid k) = [1 - K(k)]P(k \mid k-1) \tag{7}$$

3. The assumptions for adaptive filter

It was assumed in general that the described adaptive filter is intended for systems of inertial type. For many inertial systems in which the adaptive filtration can be used the process can be split into three phases of the filtered signal change: rising, stabilization and dropping. A typical example for this is the temperature in thermal systems: they are heated up, kept at a constant temperature and then cooled down. The presented adaptive filter is particularly destined for such a kind of work.

Another assumption was that the filtered state vector consists of only one state variable and its step-to-step changes are small over the applied measurement repetition period τm, especially in comparison with the full range variation that is physically permissible in the system.

The next assumption for the presented filter is a limited knowledge of the system model and its input control signals, which on basis of eq. (2) results in an inaccuracy of prediction of the state vector in the consecutive step. Additionally, in such a case the control signals should be treated as a part of disturbance signal. In such a situation the disturbance signal v(k) cannot be described as stochastic with normal distribution, as it is assumed in Kalman filters. However, for the filter in question, similarly like for a usual Kalman filter, the measurement disturbances will be modelled as the stochastic signal w(k) of normal distribution and known variance R.

It was possible to make use of Kalman filter in the described case thanks to some simplification of the model of inertial system and the adoption of some analogy between the signals influencing the state of the system in both filters. In the consequence, the described filter loses the optimality defined for Kalman filter. As a result of such a modification the additional weight coefficients have been applied to the disturbances. This enables us to experimentally adjust the filter properties to requirements in individual cases.

As mentioned, the presented adaptive filter has been designed for the single variable filtration only. As a result, all matrices and vectors in Kalman filter eq. (1), (7) become scalars and all covariance matrices become variances. This much simplifies the applied algorithm.

To adapt the Kalman filter algorithm to our filtration case the interdependences that would allow the heuristic equivalence of the process signals filtered in both filters should be determined.

It was assumed in the proposed solution that the step-to-step changes Δx of the state variable are small. Basing on this it can be also assumed that when the input signal u(k) and measurement signal y(k) are unknown, the most probable value of state vector in the step k is equal to its value filtered in the step k-1. The appropriate estimator can be written as

$$\hat{x}(k \mid k-1) = \hat{x}(k-1 \mid k-1) \tag{8}$$

Substituting equation (8) for (2) is synonymous with setting the system parameters as: A=1 and B=0. It had been assumed before that C=1, so the eq. (1) can be replaced by

$$\begin{aligned} x(k) &= x(k-1) + v(k) \\ y(k) &= x(k) + w(k) \end{aligned} \tag{9}$$

It is worth to stress that value A=1 describes the system of integrating character, whose pole z=1 (in continues notation s=0). For the time increments Δt small enough (a few times measurement period τm) such a model in many cases well enough approximates the dynamics of inertial systems of the time lag T>> Δt.

To interrelate the respective equations of both filters we assume that the signal v(k) in the Kalman filter equations, now signed as vn(k), corresponds to v(k) in the real system filtered by the presented adaptive filter, upon certain conditions.

The stochastic signal vn(k) can be described by its variance , which for N samples can be calculated as

$$\sigma^2(k) = \frac{1}{N-1} \sum_{i=k-N+1}^{k} \left[v_n(i) - \bar{v} \right]^2 \tag{10}$$

where $\bar{v}$ is the expected value, which for e.g. for white noise is equal zero. As mentioned, v(k) in the real system shows some time trends of unknown rate, so its parameters cannot by characterized using eq. (10). For this it was assumed that the criterion of the signals vn(k) and v(k) equivalence is equality of averaged square values over the step-to-step increments.

During Kalman filtration, the measurements of y(k) values and the estimation of state variables values are carried out. In the result, for the determination of v(k) and $v_n(k)$ equivalence we can accept the equality of parameter described as

$$S(k)=\frac{1}{1-N}\sum_{i=k-N+1}^{k}\left[y(i)-\hat{x}(i-1\mid i-1)\right]^2 \tag{11}$$

Variance Q(k) for vn(k) and u(k)=0 can be easily calculated on the basis of S(k) using the Kalman filter equations. Taking into account the assumption of equivalence of v(k) and vn(k), the value Q(k) calculated similarly on the basis of v(k) can be treated likewise in proposed filter equations. The finally accepted method of calculation for S(k) and that for the substitute value Q(k) in adaptive filter is described in the next part of the paper.

The last essential assumption for the adaptive filter under consideration is the possibility of its properties modification depending on the process needs. Among others, it concerns the compromise between the requirement of low delays during fast changes of system state and that of the effective noise signal rejection in the steady state. To achieve this goal the diversification of weights assigned to the state and measurement disturbances are allowed.

3. The adaptive filtration algorithm

At the beginning let us temporarily assume that signals v(k) and w(k0 are white noise, thus they have normal distribution. For the sake of simplified form of model (9), the equations of Kalman filter for such a system also undergo simplification. They get the form

$$\begin{aligned} K(k)&=\frac{P(k-1\mid k-1)+Q(k)}{P(k-1\mid k-1)+Q(k)+R}\\ P(k\mid k)&=(1-K(k))(P(k-1\mid k-1)+Q(k))\\ \hat{x}(k\mid k)&=K(k)y(k)+(1-K(k))\hat{x}(k-1\mid k-1)\end{aligned} \tag{12}$$

The values of y(k) sequence are obtained from measurements. Taking into consideration (9) they are equal to

$$y(k)=x(k-1)+v(k)+w(k) \tag{13}$$

After filtration (12), we obtain the sequence of estimated values of state $\hat{x}(k \mid k)$ whose estimation error referred to the previous step is equal

$$\tilde{x}(k-1 \mid k-1) = x(k-1) - \hat{x}(k-1 \mid k-1) \tag{14}$$

and whose variance is P(k-1|k-1)

The sequences of signal v(k) and w(k) are not correlated and have zero value expected values, due to the assumption, they are white noise. Any sequences of either of them resulting from shifting by arbitrary number of steps are not correlated either. Taking into account eq, (12) and eq, (3) the equivalence parameter S(k) described by eq. (11) can be written as

$$\begin{aligned} S(k) &= \frac{1}{N-1}\left[\sum_{i=k-N+1}^{k}\left(y(i) - \hat{x}(i-1 \mid i-1)\right)^2\right] \\ &= \frac{1}{N-1}\left[\sum_{i=k-N+1}^{k}\left(v(i) + w(i) + \tilde{x}(i-1 \mid i-1)\right)^2\right] \\ &= Q(k) + R + P(k-1 \mid k-1) + Cov(k, \tilde{x}, v, w) \end{aligned} \tag{15}$$

Covariance of error $\tilde{x}(k \mid k)$ referred to signals v(k) and w(k) signed as $Cov(k, \tilde{x}, v, w)$ is for Q<R negligibly small in the total balance of estimation errors. Variance Q(k) assuming that $Cov(k, \tilde{x}, v, w)$ is equal to zero can be determined from the equation

$$Q(k) \cong S - R - P(k \mid k) = \frac{1}{N-1}\left[\sum_{i=k-N+1}^{k}\left(y(i) - \hat{x}(i-1 \mid i-1)\right)^2\right] - R - P(k-1 \mid k-1) \tag{16}$$

However, one should be aware that relationship (16) in some cases can lead to overestimation.

During Kalman filtration (12) recursive calculations are executed, among others the calculations of error variance P(k|k).

To adapt the calculations of the actual value of Q(k) variance to this method of calculation it is convenient to substitute a modified equation (17) for (16) that corrects step by step the value of Q(k). This equation is:

$$Q(k) = (1-\alpha)Q(k-1) + \alpha\left[\left(y(k) - \hat{x}(k-1)\right)^2 - R - P(k-1 \mid k-1)\right] \tag{17}$$

where α is weighting coefficient of value e.g. 0.25. Eq. (17) has the form of a low-pass IIR filter moderating the accidental fluctuations of the calculations results.

It is worthwhile to take into account the effects of Q(k) estimation with reference to R variance. For the described assumptions the Kalman gain K applied in eq. (5) has values shown in Figure 1.

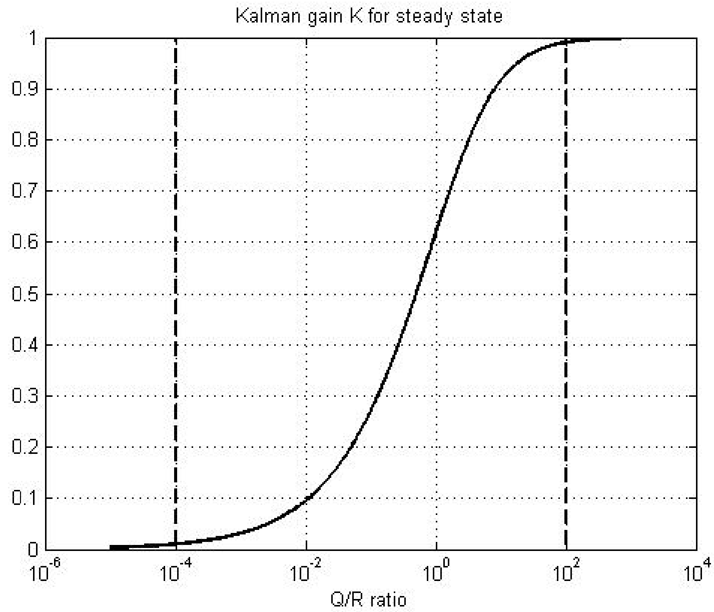

Figure 1. Kalman filter gain vs. Q/R ratio

Accidental fluctuation of Q(k), calculated from eq. (17) on the basis of disturbed measurement values, can lead to even negative values. So it is necessary to limit the lowest value of Q(k)/R to e.g. 1/10000 (K=0.01). According to Figure 1 the value of gain K is very small for this ratio. Similarly the upper limit e.g. Q(k)/R=100 (K=0.99) is also reasonable. So, we have the following limitation

$$0.0001R \leq Q(k) \leq 100R \tag{18}$$

So far, the model of the system dynamics has been considered in a simplified form (9). The state disturbances taken into consideration were treated as white noise. Additionally, the value of Q(k) derived from eq. (16) can be overestimated.

As a result, some corrections in the application of the presented equations should be made in order to obtain effective adaptive filtering in the real system. Figure 1 shows that the filter gain K depends on Q/R ratio. Let us add a new weight coefficient β in the equation describing Q(k). Then the modified value of state disturbances designated as Qm(k) can be written as

$$Q_m(k) = (1-\alpha)Q_m(k-1) + \alpha\left[\beta\left(y(k) - \hat{x}(k-1)\right)^2 - R - P(k-1\,|\,k-1)\right] \tag{19}$$

The weight coefficient β can reduce the effects of Q(k) overestimation and that of the poor evaluation of the measurement disturbances variance R. If β drops down, it reduces the influence of measurement disturbances on the filtered variable in the steady state, but on the other hand, leads to the time delays increase during the periods of fast changes.

The right value of β should be a trade-off based on experimental evaluation.

After the modification associated with introducing the coefficient β and substituting Qm(k) for Q(k) variance in the Kalman filter, the final equations of the presented adaptive filter take the form

$$\begin{aligned} K(k) &= \frac{P(k-1\,|\,k-1) + Q_m(k)}{P(k-1\,|\,k-1) + Q_m(k) + R} \\ P(k\,|\,k) &= \left(1 - K(k)\right)\left(P(k-1\,|\,k-1) + Q_m(k)\right) \\ \hat{x}(k\,|\,k) &= K(k)y(k) + \left(1 - K(k)\right)\hat{x}(k-1\,|\,k-1) \end{aligned} \tag{20}$$

The operation of the filter described by (18)-(19) is shown in Figure 1 to 3, using a test signal that was deterministic by nature, being a combination of single steps and exponents. This signal was jammed with a measuring noise of variance R. The test signal proper simulates the state changes going on at various rates and its character is much different from that of the noise with normal distribution, which Kalman filtration concerns.

Figure 2 shows simultaneously the test signal and the filtration result obtained with a LP filter having large time-constant. It allows for good filtration in the quasi-steady state, but brings in considerable delays, particularly in the phase of initial rise. Reduction of the filter time-constant much reduces the delays, but the filtration becomes practically ineffective in the quasi-steady state (Figure 3). Figure 4 demonstrates the operation of the discussed filter. We can easily notice that the adaptive filter is free of the above-described faults of the filters with an invariable time-constant. The value of β=0.5 in equation (12) was assumed for this filter.

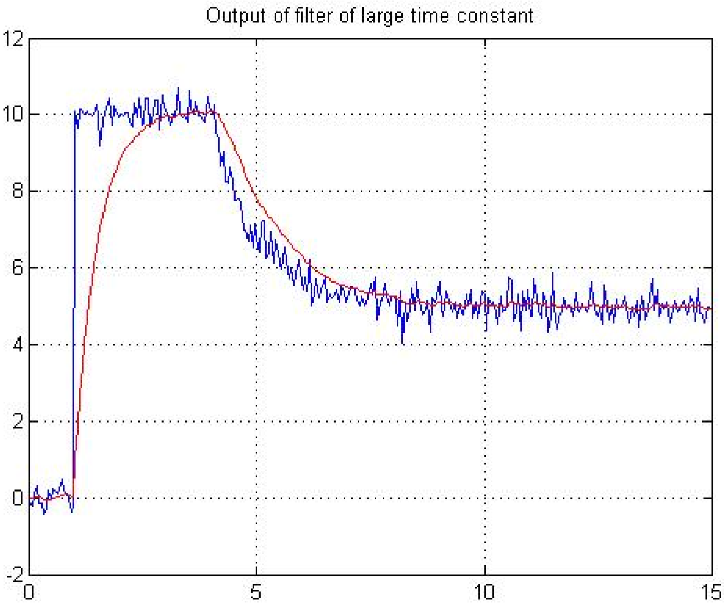

Figure 2. Filtration of LP filter of invariable large time constant

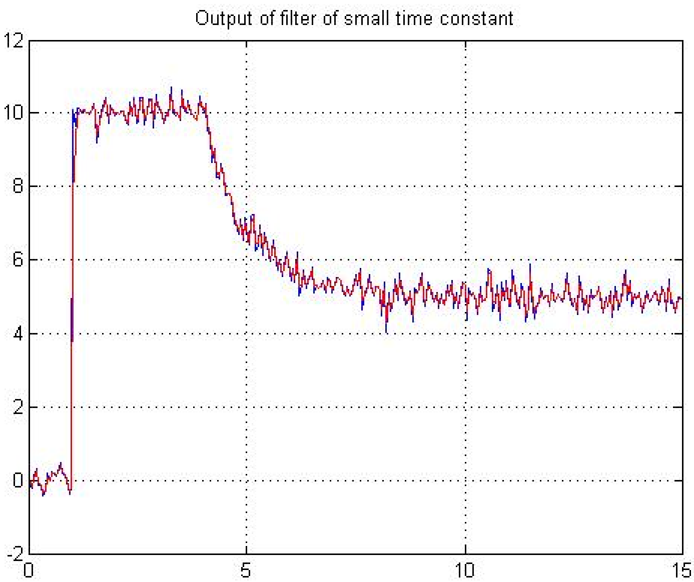

Figure 3. Filtration of LP filter of invariable small time constant

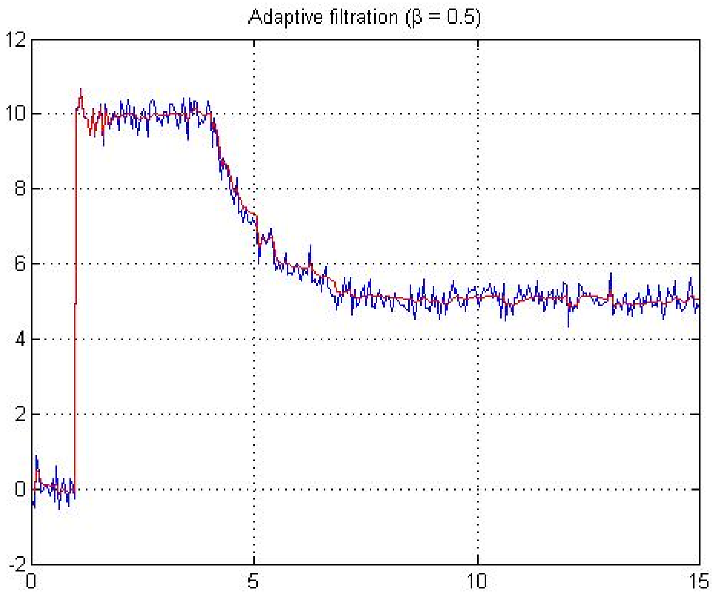

Figure 4. Filtration of adaptive LP filter

4. Conclusions

The paper presents an adaptive filter based on Kalman filter algorithm. This filter doesn't need accurate information either about the system dynamics model or about the control signal on the system input. The filter estimates the values of state variable on the basis of the measurement disturbances variance and the variations of signal on the system output. As it is shown in Figure 4 such a filter can be very effective in various phases of the real process.

The described filter was successfully applied by the author in a number of thermal systems for control (eg. [3]) and measurement purposes.

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