JAMCM Journal of Applied Mathematics and Computational Mechanics 2022, 21(4), 52-59
www.amcm.pcz.pl p-ISSN 2299-9965
DOI: 10.17512/jamcm.2022.4.05 e-ISSN 2353-0588

ON ESTIMATION OF PRIORITY VECTORS DERIVED FROM INCONSISTENT PAIRWISE COMPARISON MATRICES

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Received: 2 October 2022; Accepted: 6 November 2022

Abstract. The most critical and purely heuristic assumption about priority vector estimation on the basis of pairwise comparisons is that which states a positive relationship between the consistency of decision makers' judgments and the quality of estimates of their priorities. As this issue constitutes the area of interest of the Multi-Criteria Decision Making theory in relation to AHP, it's examined in this paper via Monte Carlo simulations from the perspective of a new measure of PCM consistency i.e. Index of Square Logarithm Deviations. It needs to be emphasized that such problems of applied mathematics have been already studied via computer simulations as the only way of this phenomenon examination.

MSC 2010: *68T37, 62C86, 90C59, 90C70*
Keywords: *approximate reasoning, pairwise comparisons, consistency, priorities, AHP*

1. Introduction

The hierarchical analysis method known as the Analytic Hierarchy Process (AHP) was defined in the late seventies of the previous century [1]. At that time a complete solution for multiple criteria decision making problems was proposed, including a ranking calculation algorithm, a data quality determination, i.e. the inconsistency index, and a hierarchical model that allows the decision maker (DM) to deal with multiple criteria problems [2]. However, various studies of pairwise comparison methods have led to many definitions of scales applied during comparisons – not examined herein due to limited capacity of the article, many algorithms yielding priorities – for brevity not discussed in this paper, and various approaches of inconsistency measurement – also not scrutinized herein due to consistency of the article. Nevertheless, selected references can be recommended for a reader who would like to familiarize with this topic, see e.g. [3-7]. Moreover, the applicability and popularity of AHP cannot be underestimated as its applications can be found in many areas including engineering and industrial problems. Its recent successful applications in this field encompass among others such papers as e.g. [8-12].

The traditional prioritization methodology (PM) in AHP is based on Saaty's comparison scale and the explicitly defined mathematical structure of pairwise comparison matrices that are consistent and their related Principal Right Eigenvector's (REV) function to produce from them accurate or estimated weights, derived as priority vectors (PV). It has been proven that, if $\boldsymbol{A}(w) = (w_{ij})$, $w_{ij} > 0$, where $i, j = 1, ..., n,$ then $\boldsymbol{A}(w)$, called the Pairwise Comparison Matrix (PCM) has a simple positive eigenvalue EV = $\lambda_{\max}$ called the Principal Eigenvalue (PEV) of $\boldsymbol{A}(w)$ and for the remaining EVs of $\boldsymbol{A}(w)$ the relation $\lambda_{\max} > |\lambda_k|$ is true. The REV denoted as $\boldsymbol{w} = [w_1, ..., w_n]^T$ that is a solution of $\boldsymbol{A}(w)\boldsymbol{w} = \lambda_{\max}\boldsymbol{w}$ has $w_i > 0$, $i = 1, ..., n$, and when $||\boldsymbol{w}|| = \boldsymbol{e}^T\boldsymbol{w}$ where $e = [1, 1, ..., 1]^T$ then $\boldsymbol{w}$ can be normalized by dividing it by its norm. Hereafter, only normalized forms of PV are considered.

Definition 1: If the elements of a matrix $\boldsymbol{A}(w)$ satisfy the condition $w_{ij} = 1/w_{ji}$ for all $i, j = 1, ..., n$, then the matrix $\boldsymbol{A}(w)$ is called *reciprocal*.

Definition 2: If the elements of a matrix $\boldsymbol{A}(w)$ satisfy the condition $w_{ik}w_{kj} = w_{ij}$ for all $i, j, k = 1, ..., n$, and the matrix is *reciprocal*, then it is called *consistent* or *cardinal transitive*. In these circumstances, the relation $\boldsymbol{A}(w)\boldsymbol{w} = \boldsymbol{nw}$ is also true.

In the AHP applications $\boldsymbol{A}(w)$ is unknown, only its estimate $\boldsymbol{A}(a)$ is known, which contains intuitive pairwise comparisons, also known as human judgments which in the assumption are relatively close to $\boldsymbol{A}(w)$. Hence, the relation between elements of $\boldsymbol{A}(w)$ and $\boldsymbol{A}(a)$ can be expressed as below (Formula (1)):

$$a_{ij} = e_{ij}w_{ij} \tag{1}$$

where e_{ij} denotes a randomly selected perturbation element.

Formula (1) is very useful for imitation of imperfect human pairwise comparisons via computer simulations. In such cases, a selected probability distribution (PD) is applied for e_{ij} e.g. *gamma, log-normal, truncated-normal, triangular*, Laplace, *beta*, Cauchy PDs, *uniform* as well Fisher-Snedecor PD [13-15].

The most critical and purely heuristic assumption about PV estimation on the basis of pairwise comparisons is that which states a positive relationship between a consistency of decision makers' judgments and the quality of estimates of their priorities i.e. belief in the truth of the statement: "better consistency of PCM leads to better PV estimates". However, it turns out that such an assumption is not entirely true for every known measure of PCM consistency [5, 14]. Hence, the relationship between variability of different measures of PCM inconsistency, i.e. various Consistency Indices (CI), and estimation errors of PVs is of particular interest for some authors [4, 5, 16]. The issue also constitutes the area of interest of the Multi-Criteria Decision Making (MCDM) theory in relation to AHP, why it's examined in this paper from the perspective of a new measure of PCM consistency that is proposed herein. Noticeable, computer simulations are the only way for this phenomenon examination. It needs to be emphasized that similar problems of applied mathematics have already been studied and their examinations let discover interesting relations between selected CI and PVs errors, see e.g. [13, 14, 16].

The following selected consistency indices can be listed as studied from the presented perspective. The Saaty's consistency index CI_{REV} – Formula (2), the geometric consistency index CI_{LLSM} by Aguaron and Moreno-Jimenez [17] – Formula (3), the Koczkodaj's [18] consistency index $K(A)$ – Formula (4), the inconsistency index $ATI(A)$ – Formula (5), proposed by Grzybowski [13], and Kazibudzki's [5, 16] indices of inconsistency i.e. $ALTI_1(A)$, $ALTI_2(A)$ and $TSL(A)$ – Formulae (6)-(8).

$$CI_{REV} = (\lambda_{max} - n)/(n-1) \tag{2}$$

$$CI_{LLSM} = \frac{2}{(n-1)(n-2)} \sum_{i<j} log^2 \left(\frac{a_{ij} w_j}{w_i}\right) \tag{3}$$

$$K(A) = \max_{i<j<k} \left\{ TI_p \left(min \left\{ \left| 1 - \frac{a_{ik}}{a_{ij} a_{jk}} \right|, \left| 1 - \frac{a_{ij} a_{jk}}{a_{ik}} \right| \right\} \right) \right\}_{p=1,\dots,\binom{n}{3}} \tag{4}$$

$$ATI(A) = \frac{1}{\binom{n}{3}} \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^{n} min \left\{ \left| 1 - \frac{a_{ik}}{a_{ij} a_{jk}} \right|, \left| 1 - \frac{a_{ij} a_{jk}}{a_{ik}} \right| \right\} \tag{5}$$

$$ALTI_1(A) = \frac{1}{\binom{n}{3}} \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^{n} \left| ln \frac{a_{ij} a_{jk}}{a_{ik}} \right| \tag{6}$$

$$ALTI_2(A) = \frac{1}{\binom{n}{3}} \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^{n} ln^2 \frac{a_{ij} a_{jk}}{a_{ik}} \tag{7}$$

$$TSL(A) = \frac{\sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^{n} ln^2 \frac{a_{ik}}{a_{ij} a_{jk}}}{\binom{n}{3} \left(1 + \max_{i<j<k} \left\{ ln_p^2 \frac{a_{ik}}{a_{ij} a_{jk}} \right\}_{p=1,\dots,\binom{n}{3}} \right)} \tag{8}$$

In this research a new consistency measure is proposed, i.e. Index of Square Logarithm Deviations (ISLD) – Formula (9), and it is examined via computer simulations from the earlier discussed perspective.

$$ISLD = Median \left\{ ln^2 \sum_{j=1}^{n} \left(a_{ij} w_j / n w_i \right) \right\}_{i=1,\dots,n} \tag{9}$$

2. Research methodology

Taking into account the objective of this research and assumptions which have been already described and applied for similar analysis, e.g. [5, 13, 14, 16], the following computer simulation algorithm (Fig. 1) is adapted, performed, and its output examined. Generally, computer simulations such as Monte Carlo simulations are widely regarded as a valid and reliable source of scientific information [14].

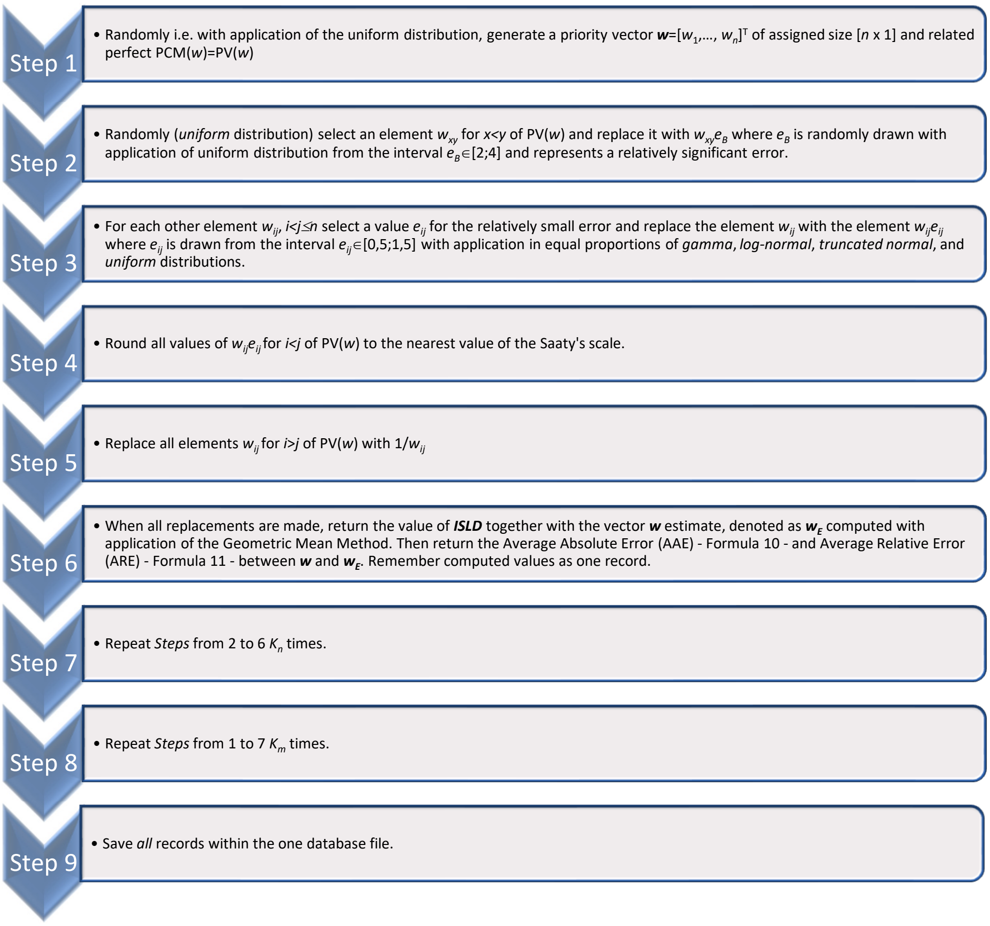

Fig. 1. Simulation algorithm applied for the research

$$AAE = \frac{1}{n}\sum_{i=1}^{n}|w_i - w_{Ei}| \tag{10}$$

$$ARE = \frac{1}{n}\sum_{i=1}^{n}\frac{|w_i - w_{Ei}|}{w_i} \tag{11}$$

In the simulation algorithm (Fig. 1), parameters of implemented PDs are set in the way that $\mathrm{EV}(e_{ij}) = 1$, where EV denotes the expected value.

3. Examination results

The research results conducted on the basis of presented earlier simulation scenario are presented in Tables 1 and 2, and Figures 2-5. Plots within Figures 2-3 and Figures 4-5 present relations – among mean ISLD and p-quantiles of AAE/ARE, between mean ISLD and mean AAE/ARE, and values of Spearman rank correlation coefficients (SRC) for those relations.

Table 1. ISLD performance in relation to AAE distribution

MEAN ISLD	p-quantiles of AAE(LLSM)			MEAN AAE
	$p = 0.1$	$p = 0.5$	$p = 0.9$	
0.0022694	0.00596	0.01316	0.03414	0.0183870
0.0202402	0.00676	0.01399	0.03755	0.0195361
0.0452134	0.00966	0.01949	0.04972	0.0258283
0.0725819	0.01235	0.02572	0.06184	0.0323953
0.1002700	0.01506	0.03052	0.06673	0.0366781
0.1283270	0.01649	0.03222	0.06803	0.0383714
0.1562870	0.01741	0.03334	0.06848	0.0393119
0.1837120	0.01854	0.03490	0.07064	0.0409059
0.2112180	0.01970	0.03693	0.07479	0.0432768
0.2391760	0.02072	0.03881	0.07733	0.0450696
0.2668150	0.02195	0.04080	0.08229	0.0475152
0.2941620	0.02301	0.04286	0.08563	0.0498912
0.3222390	0.02420	0.04489	0.08948	0.0518571
0.3498430	0.02507	0.04647	0.09350	0.0540846
0.3772250	0.02591	0.04820	0.09870	0.0565986
0.4795540	0.02923	0.05660	0.12274	0.0675113

Note: results based on 75 000 random reciprocal PCMs for $K_n = 50$, $K_m = 250$, and for $n \in \{4, 5, \ldots, 9\}$

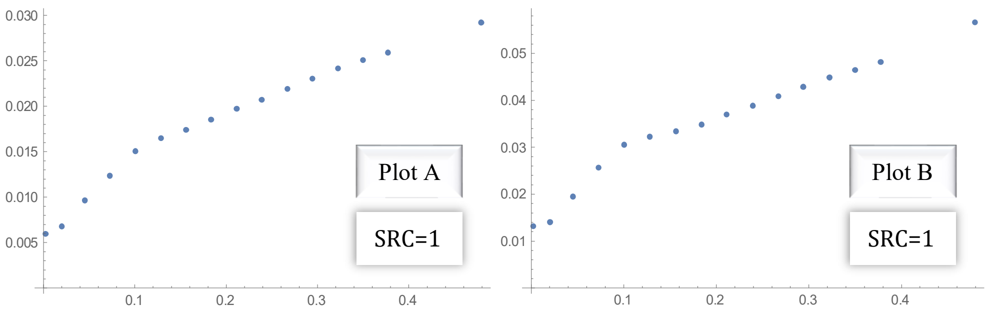

Fig. 2. ISLD performance. Plots of correlation among average values of ISLD and: AAE quantiles of order $p = 0.1$ (Plot A) and $p = 0.5$ (Plot B)

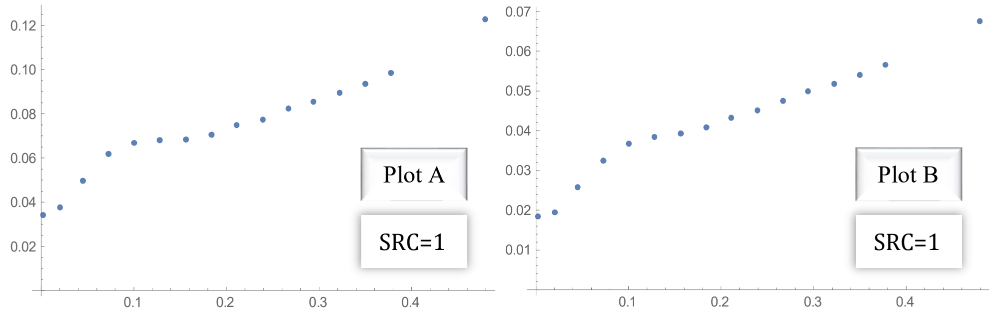

Fig. 3. ISLD performance. Plots of correlation among average values of ISLD and: AAE quantiles of order p = 0.9 (Plot A) and the mean AAE (Plot B)

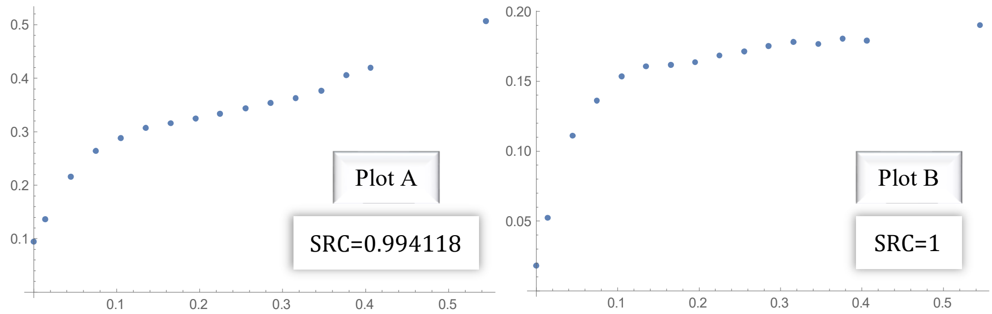

Fig. 4. ISLD performance. Plots of correlation among average values of ISLD and: ARE quantiles of order p = 0.1 (Plot A) (Plot A) and p = 0.5 (Plot B)

Table 2. ISLD performance in relation to ARE distribution

MEAN ISLD	p-quantiles of ARE(LLSM)			MEAN ARE
	p = 0.1	p = 0.5	p = 0.9	
0.0000472	0.01832	0.09433	0.57914	0.2284600
0.0138869	0.05217	0.13668	0.49012	0.2673050
0.0443866	0.11117	0.21633	0.52771	0.3505270
0.0745758	0.13594	0.26390	0.60734	0.4332360
0.1050080	0.15359	0.28847	0.67207	0.5049580
0.1348460	0.16051	0.30718	0.75553	0.5699490
0.1649670	0.16187	0.31613	0.79601	0.6141030
0.1951390	0.16344	0.32496	0.86659	0.6443390
0.2249220	0.16851	0.33375	0.89710	0.6771590
0.2551780	0.17154	0.34344	0.95147	0.7154860
0.2851730	0.17509	0.35392	0.99331	0.7093450
0.3156630	0.17811	0.36318	1.01989	0.7453040
0.3462380	0.17665	0.37642	1.11668	0.7510230
0.3763400	0.18069	0.40547	1.23410	0.8654410
0.4061370	0.17917	0.41991	1.43266	0.9596730
0.5451050	0.19016	0.50745	2.66733	1.4335300

Note: results based on 50 000 random reciprocal PCMs for K_n = 50, K_m = 250, and for $n \in \{3, 4,\ldots, 6\}$

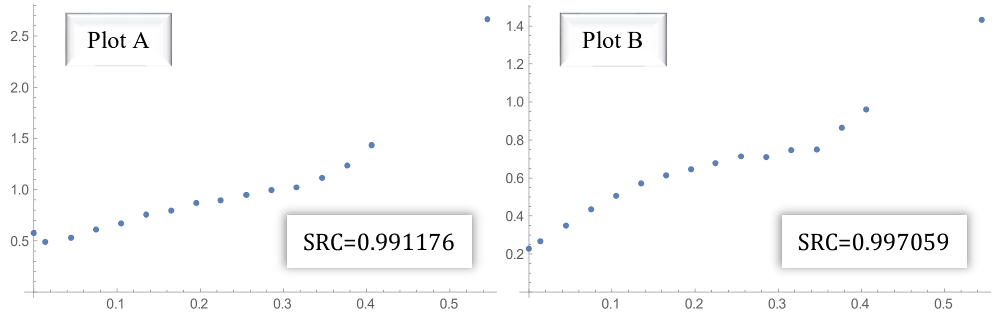

Fig. 5. ISLD performance. Plots of correlation among average values of ISLD and: ARE quantiles of order p = 0.9 (Plot A) and the mean ARE (Plot B)

4. Conclusions

As is known, there are three fundamental reasons of priority vectors estimation imperfection, i.e. the condition of PCM reciprocity, the necessity of scale application for preference expression during pairwise comparisons what generates rounding errors, and inconsistency of human judgments expressed in various measures of PCM consistency. The examination and discussion about the two first elements of the above list was beyond the scope of this research, however the third reason of priority vectors estimation imperfection fall into this research focus.

That is why the examination was conducted on relations among the new PCM consistency measure introduced herein, and possible estimation errors of priority vectors generated on the basis of variously distorted PCM. Such examinations give the foundation of established preferences trustworthiness and as before are conducted only by few authors e.g. [4, 5, 14, 16].

This paper proposes a new measure of PCM consistency which presents very attractive features in relation to its association with possible estimation errors of PV (both absolute and relative). The research outcome provides an added value for other examinations focusing on ways of PV credibility verification [19] within the AHP.

References

[1] Saaty, T.L. (1977). A scaling method for priorities in hierarchical structures. *Journal of Mathematical Psychology*, *15*(3), 234-281. https://doi.org/10.1016/0022-2496(77)90033-5.

[2] Saaty, T.L. (2008). Decision making with the analytic hierarchy process. *International Journal of Services Sciences*, *1*(1), 83. https://doi.org/10.1504/IJSSCI.2008.017590.

[3] Kou, G., Ergu, D., Chen, Y., & Lin, C. (2016). Pairwise comparison matrix in multiple criteria decision making. *Technological and Economic Development of Economy*, *22*(5), 738-765. https://doi.org/10.3846/20294913.2016.1210694.

[4] Grzybowski, A.Z., & Starczewski, T. (2017). Remarks about inconsistency analysis in the pairwise comparison technique. *2017 IEEE 14th International Scientific Conference on Informatics*, 227-231. https://doi.org/10.1109/INFORMATICS.2017.8327251.

[5] Kazibudzki, P.T. (2019). An examination of ranking quality for simulated pairwise judgments in relation to performance of the selected consistency measure. *Advances in Operations Research*, *2019*, e3574263. https://doi.org/10.1155/2019/3574263.

[6] Franek, J., & Kresta, A. (2014). Judgment scales and consistency measure in AHP. *Procedia Economics and Finance*, *12*, 164-173. https://doi.org/10.1016/S2212-5671(14)00332-3.

[7] Cavallo, B., & Ishizaka, A. (2022). Evaluating scales for pairwise comparisons. *Annals of Operations Research*. https://doi.org/10.1007/s10479-022-04682-8.

[8] Silva Lopez, J.O., Salas Lopez, R., Rojas Briceno, N.B., Gomez Fernandez, D., Terrones Murga, R.E., Iliquin Trigoso, D., Barboza Castillo, E., Oliva Cruz, M., & Barrena Gurbillon, M.A. (2022). Analytic Hierarchy Process (AHP) for a landfill site selection in Chachapoyas and Huancas (NW Peru): Modeling in a GIS-RS Environment. *Advances in Civil Engineering*, *2022*, 9733322. https://doi.org/10.1155/2022/9733322.

[9] Li, L., Liu, Z., & Du, X. (2021). Improvement of analytic hierarchy process based on grey correlation model and its engineering application. *Asce-Asme Journal of Risk and Uncertainty in Engineering Systems Part a-Civil Engineering*, *7*(2), 04021007. https://doi.org/10.1061/AJRUA6.0001126.

[10] Ba, Z., Wang, Y., Fu, J., & Liang, J. (2022). Corrosion risk assessment model of gas pipeline based on improved AHP and its engineering application. *Arabian Journal for Science and Engineering*, *47*(9), 10961-10979. https://doi.org/10.1007/s13369-021-05496-9.

[11] Wang, P., Xue, Y., Su, M., Qiu, D., & Li, G. (2022). A TBM tunnel collapse risk prediction model based on AHP and normal cloud model. *Geomechanics and Engineering*, *30*(5), 413-422. https://doi.org/10.12989/gae.2022.30.5.413.

[12] Zhu, H., Xiang, Q., Luo, B., Du, Y., & Li, M. (2022). Evaluation of failure risk for prestressed anchor cables based on the AHP-ideal point method: An engineering application. *Engineering Failure Analysis*, *138*, 106293. https://doi.org/10.1016/j.engfailanal.2022.106293.

[13] Grzybowski, A.Z. (2016). New results on inconsistency indices and their relationship with the quality of priority vector estimation. *Expert Systems with Applications*, *43*, 197-212. https://doi.org/10.1016/j.eswa.2015.08.049.

[14] Grzybowski, A.Z., & Starczewski, T. (2020). New look at the inconsistency analysis in the pairwise-comparisons-based prioritization problems. *Expert Systems with Applications*, 113549. https://doi.org/10.1016/j.eswa.2020.113549.

[15] Kazibudzki, P.T. (2021). On the statistical discrepancy and affinity of priority vector heuristics in pairwise-comparison-based methods. *Entropy*, *23*(9), Article 9. https://doi.org/10.3390/e23091150.

[16] Kazibudzki, P.T. (2016). Redefinition of triad's inconsistency and its impact on the consistency measurement of pairwise comparison matrix. *Journal of Applied Mathematics and Computational Mechanics*, *15*(1), 71-78. https://doi.org/10.17512/jamcm.2016.1.07.

[17] Aguarón, J., & Moreno-Jiménez, J.M. (2003). The geometric consistency index: Approximated thresholds. *European Journal of Operational Research*, *147*(1), 137-145. https://doi.org/10.1016/S0377-2217(02)00255-2.

[18] Koczkodaj, W.W. (1993). A new definition of consistency of pairwise comparisons. *Mathematical and Computer Modelling*, *18*(7), 79-84. https://doi.org/10.1016/0895-7177(93)90059-8.

[19] Botelho, M. (2022). Analyzing priority vectors: Going beyond inconsistency indexes. *International Journal of the Analytic Hierarchy Process*, *14*(2). https://doi.org/10.13033/ijahp.v14i2.922.