Opuscula Math. 40, no. 6 (2020), 737–750
https://doi.org/10.7494/OpMath.2020.40.6.737 Opuscula Mathematica

In memory of Mark Kac

ECHOES AND GLIMPSES OF A DISTANT DRUM

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Communicated by Jean Mawhin

Abstract. To what extent does the spectrum of the Laplacian operator on a domain D with prescribed boundary conditions determine its shape? This paper first retraces the history of this problem, then Kac's approach in terms of a diffusion process with absorbing boundary conditions. It is shown how the restriction to a polygonal boundary for D in this method, which required taking the limit of an infinite number of sides to obtain a smooth one, can be avoided by using the Duhamel method.

Keywords: Kac drum problem, inverse methods, diffusion process.

Mathematics Subject Classification: 01A70, 35B30, 35J05, 35P20.

1. A BRIEF HISTORICAL INTRODUCTION

In 1966 Mark Kac published a paper [5] with the arresting title "Can one hear the shape of a drum?". The problem was to determine if knowing all the eigenvalues of the Laplacian operator Δ on a two-dimensional domain D with Dirichlet boundary conditions was sufficient to reconstruct the shape of D.

Apparently Kac heard of the problem from Salomon Bochner, and later, in a conversation with Lipman Bers, the latter summed it up in the picturesque way which became the title of the paper.

The question can in fact be traced further back. In 1882 the spectroscopist Arthur Schuster (later Sir Arthur), in his report to the British Association for the Advancement of Science, said: "It would baffle the most skillful mathematicians to solve the inverse problem and to find out the shape of a bell by means of the sounds which it is capable of sending out".

A first step followed a seminar *Alte und neue Fragen der Physik* given by H.A. Lorentz in 1910 at Göttingen. He ended with the remark that it should be possible to show that the number of eigenvalues $N(\lambda)$ less than λ for the Laplacian operator on a three-dimensional domain Ω with Dirichlet conditions was approximately $\frac{|\Omega|}{6\pi^2}\lambda^{3/2}$, and within two years Herman Weyl had proved the result.

As stated, the answer to the original question is in the negative. It was already known at the time that some different 16-dimensional tori were isospectral, but the problem restricted to the more realistic two or three-dimensional world remained unsolved for some considerable time.

Much water has flowed under the bridge since the question was first asked.

Clearly the skeptics would have the shorter, but not necessarily easier, task of finding a single counterexample, and a first pair of planar isospectral domains was found in 1992 [3], followed by others of simpler shapes [2]. Each pair has the property of sharing same area, same perimeter and same corners (but not in the same order), but above all of being non-convex.

However, it is now known [12] that the answer can be "yes", provided for example that D be convex with an analytic boundary ∂D and possess certain symmetries.

The problem has also been extended to a variety of cases, notably to the spectrum of the Laplace–Beltrami operator on an n-dimensional Riemannian manifold (M,g) [1].

It is convenient to introduce a kind of "partition function" or Dirichlet series for the spectrum

$$f(t) = \sum_n e^{-\lambda_n t},$$

and to consider its asymptotic behaviour as $t \downarrow 0$.

A result due to Minakshisundaram and Pleijel [7] proves the existence of an asymptotic expansion

$$\sum_i e^{-\lambda_i t} \overset{t\downarrow 0}{\sim} (4\pi t)^{-n/2}(a_0 + a_1 t + a_2 t^2 + \ldots)$$

and Pleijel [9] showed that if D is a two-dimensional simply connected domain, with a smooth boundary, then an asymptotic formula such as

$$\sum e^{-\lambda_n t} \overset{t\downarrow 0}{\sim} \frac{A}{4\pi t} - \frac{L}{8\sqrt{\pi t}} + C + o(1)$$

should hold with A the area of D (this is Weyl's result), L its perimeter, and the constant $C = 1/6$.

The determination of the coefficients of the asymptotic expansion has been studied by McKean and Singer [6], as well as by Smith [10] in the case of a smooth boundary and by van den Berg and Srisatkunarajah [11] in the polygonal case.

2. THE 1966 PAPER

Kac's idea can be summarized as follows: the solution to the equation

$$\frac{\partial u(\mathbf{r},t)}{\partial t} = \Delta u(\mathbf{r},t), \quad u(\mathbf{r},t)\big|_{\partial D} = 0, \quad u(\mathbf{r},0) = f(\mathbf{r}),$$

where D is some domain in $\mathbb{R}_2$ and $f(\mathbf{r})$ a prescribed initial condition, can be expressed in terms of the eigenvalues and eigenfunctions of the Laplacian on D with the same boundary conditions:

$$\Delta u_n(\mathbf{r}) = \lambda_n u_n(\mathbf{r}), \quad u_n(\mathbf{r})\big|_{\partial D} = 0.$$

Assuming the set of eigenfunctions to form a complete orthonormal basis, the solution can be written as

$$u(\mathbf{r}, t) = \sum_n c_n u_n(\mathbf{r}) e^{-\lambda_n t},$$

where the $\{c_n\}$ are the coefficients of the expansion of the initial conditon $f(\mathbf{r})$ in this basis. In particular if $f(\mathbf{r}) = \delta(\mathbf{r} - \boldsymbol{\xi})$, where $\boldsymbol{\xi}$ is a point *inside* D, then

$$u(\mathbf{r}, t) = \sum_n u_n(\boldsymbol{\xi})\, u_n(\mathbf{r}) e^{-\lambda_n t}$$

and as a consequence

$$\sum_n e^{-\lambda_n t} = \iint\limits_D u(\boldsymbol{\xi}, t)\, \mathrm{d}\boldsymbol{\xi}.$$

In the present context it is the behaviour of the sum when $t \downarrow 0$ which is required, and in particular how it is related to the characteristic features of the domain D.

Disregarding at first the boundary condition, the solution to the equation is simply

$$u(\mathbf{r}, t) = \frac{1}{4\pi t}\, e^{-|\mathbf{r}-\boldsymbol{\xi}|^2/4t},$$

and so

$$\sum_n e^{-\lambda_n t} \overset{t\downarrow 0}{\sim} \iint\limits_D \frac{1}{4\pi t}\, \mathrm{d}\boldsymbol{\xi} \overset{t\downarrow 0}{\sim} \frac{A(D)}{4\pi t}.$$

By the method of images, or rather its Sommerfeld–Carslaw extension, Kac then showed that if the domain were polygonal with an area A, perimeter L and a number of corners of internal angles $\theta_i > \pi/2$, then

$$\sum_n e^{-\lambda_n t} \overset{t\downarrow 0}{\sim} \frac{A}{4\pi t} - \frac{L}{8\sqrt{\pi t}} + C,$$

where

$$C = -\frac{1}{8\pi} \sum_i \sin\left(\frac{\pi^2}{\theta_i}\right) \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}y}{(1+\cosh y)\left[\cosh\left(\frac{\pi y}{\theta_i}\right) - \cos\left(\frac{\pi^2}{\theta_i}\right)\right]}.$$

The integral is not evaluated in the paper, since what was required was the limit when the number of corners goes to infinity and each angle $\theta_i \to \pi$, thus leading to a smooth boundary, for which

$$C_\infty = \frac{1}{8\pi} \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}y}{(1+\cosh y)^2} = \frac{1}{6},$$

thus recovering the third term of the Weyl–Pleijel formula.

The integral can in fact be evaluated quite easily by noticing that if $s, t \in \mathbb{R}$,

$$\frac{\sin s}{\cosh t - \cos s} = -\Re \cot \frac{s + it}{2},$$

then the constant term can be written as

$$-\frac{1}{8\pi} \Re \sum_i \int_{-\infty}^{\infty} \frac{1}{1+\cosh y} \cot \frac{\pi}{2\theta_i}(\pi + iy)\, \mathrm{d}y = -\frac{1}{8\pi} \Im \sum_i \int_{-\infty}^{\infty} \frac{\coth \frac{\pi}{2\theta_i}(y - i\pi)}{1+\cosh y}\, \mathrm{d}y.$$

Consider then the integral

$$I = \int_\Gamma \frac{\coth \beta z}{1 - \cosh z}\, \mathrm{d}z \quad \left(\beta = \frac{\pi}{2\theta}\right)$$

where Γ runs from $\Re z = -\infty$ to $\Re z = \infty$ along the line $\Im z = -i\pi$ and returns along $\Im z = i\pi$.

We have on the one hand

$$I = \int_{-\infty}^{\infty} \frac{1}{1+\cosh y} \left[\coth \beta(y + i\pi) - \coth \beta(y - i\pi)\right] \mathrm{d}y = 2i\, \Im \int_{-\infty}^{\infty} \frac{\coth \beta(y - i\pi)}{1+\cosh y}\, \mathrm{d}y,$$

and on the other hand, as $\beta < 1$, if $\theta_i > \frac{\pi}{2}$, the only singularity inside Γ is the triple pole at the origin, where the integrand behaves as

$$\frac{\coth \beta z}{1 - \cosh z} \sim -\frac{2}{\beta z^3} + \left(\frac{1}{6\beta} - \frac{2\beta}{3}\right)\frac{1}{z} + \mathrm{O}(z).$$

Substituting $\beta = \frac{\pi}{2\theta_i}$ gives

$$C_i = \frac{1}{24}\left(\frac{\pi}{\theta_i} - \frac{\theta_i}{\pi}\right).$$

Suppose now the (convex) polygon to have n sides, with each angle $\theta_i = \pi - \epsilon_i$, and take the limit $n \to \infty$, $\epsilon_i \to 0$:

$$C = \lim_{n\to\infty} \frac{1}{24} \sum_i \left(\frac{\pi}{\pi - \epsilon_i} - \frac{\pi - \epsilon_i}{\pi}\right) = \lim_{n\to\infty} \frac{1}{12\pi} \sum_i \epsilon_i = \frac{1}{6},$$

since in this case $\sum_i \theta_i = (n-2)\pi$.

It can be shown that this result is in fact valid for any polygon, even with acute angles, but the analysis along the above lines is involved.

Finally Kac showed that for a domain D with a smooth boundary ∂D and a number G of holes, each one with a smooth boundary, then the constant becomes $\frac{1-G}{6}$, and L is to be understood as the *total* perimeter of the domain.

It will be convenient to rewrite the Weyl–Pleijel–Kac formula in a way that isolates the contributions coming from the possible sharp corners of the boundary: for a domain D with a piecewise smooth, simple border, a number G of holes (with smooth boundaries) and a finite number of corners of angle θ_i,

$$\sum_n e^{-\lambda_n t} \overset{t\downarrow 0}{\sim} \frac{A}{4\pi t} - \frac{L}{8\sqrt{\pi t}} + \frac{1-G}{6} + \frac{1}{24}\sum_i \left(\frac{\theta_i}{\pi} + \frac{\pi}{\theta_i} - 2\right).$$

Thus for a quarter circle (quadrant)

$$C = \frac{1}{6} + 3 \cdot \frac{1}{24}\left(2 + \frac{1}{2} - 2\right) = \frac{11}{48}.$$

3. A DIFFERENT APPROACH

An important part of Kac's paper is devoted to making the "diffusion process method" result rigorous, and justifying the approximation of a smooth shape by a polygon of increasingly large number of sides. What we shall endeavour to achieve in this paper is to show that the diffusion method can in fact be applied directly to the case where D has a smooth, rectifiable, simple boundary ∂D.

Let $A(D)$ and $L(D)$ be its area and perimeter. At each point P of the boundary draw a segment of fixed length a along the inward normal to ∂D at P. Assume that the locus of P' is again simple, and can be considered as the boundary $\partial D'$ of a new domain D'. Then the area $A(D')$ is equal to

$$A(D') = A(D) - L(D)\, a + \pi a^2.$$

For as P' approaches P, the normals n and n' intersect at C, with $|PC| = \mathcal{R}$, the radius of curvature at P (see Figure 1).

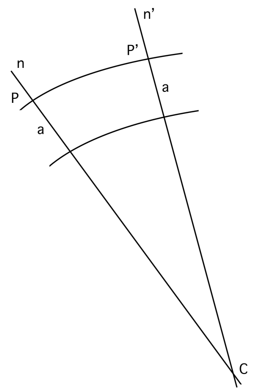

Fig. 1. Merging of two normals to the boundary

So

$$A(D) - A(D') = \frac{1}{2}\int \left[\mathcal{R}^2 - (\mathcal{R}-a)^2\right] \mathrm{d}\alpha$$
$$= a\int \mathrm{d}s - a^2\!\int \mathrm{d}\alpha = La - \pi a^2.$$

The boundary condition $u|_{\partial D} = 0$ can now be taken *partly* into account by noticing that if $\boldsymbol{\xi}$ lies close to the boundary, then by the method of images and neglecting the curvature, i.e. replacing the boundary condition by $u|_{\Delta} = 0$,

$$u(\mathbf{r},t) = \frac{1}{4\pi t}\, e^{-|\mathbf{r}-\boldsymbol{\xi}|^2/4t} - \frac{1}{4\pi t}\, e^{-|\mathbf{r}-\boldsymbol{\xi'}|^2/4t},$$

where $\boldsymbol{\xi'}$ is the image of $\boldsymbol{\xi}$ on the normal to ∂D through $\boldsymbol{\xi}$ (Figure 2).

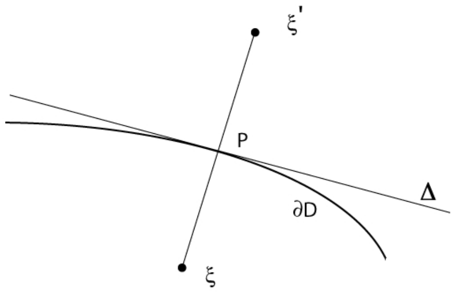

Fig. 2. Approximating the boundary by its tangent

It then follows that

$$u(\boldsymbol{\xi},t) = \frac{1}{4\pi t}\left[1 - e^{-|\boldsymbol{\xi}-\boldsymbol{\xi'}|^2/4t}\right]$$

and

$$\sum_n e^{-\lambda_n t} = \iint\limits_D u(\boldsymbol{\xi},t)\,\mathrm{d}\boldsymbol{\xi} \overset{t\downarrow 0}{\sim} \frac{1}{4\pi t}\iint\limits_D \left[1 - e^{-|\boldsymbol{\xi}-\boldsymbol{\xi'}|^2/4t}\right] \mathrm{d}\boldsymbol{\xi}$$
$$\overset{t\downarrow 0}{\sim} \frac{A(D)}{4\pi t} - \frac{1}{4\pi t}\iint\limits_D e^{-|\boldsymbol{\xi}-\boldsymbol{\xi'}|^2/4t}\mathrm{d}\boldsymbol{\xi}.$$

The integral is exponentially small unless $|\boldsymbol{\xi}-\boldsymbol{\xi'}| \sim O(\sqrt{t})$. Setting therefore $|\boldsymbol{\xi}-\boldsymbol{\xi'}| = 2p\sqrt{t}$ shows that it behaves when $t \downarrow 0$ as

$$\int\limits_0^\infty e^{-p^2}\left[\,A[\,(p+\mathrm{d}p)\sqrt{t}\,] - A[\,p\sqrt{t}\,]\,\right],$$

$A[p\sqrt{t}]$ being the area of the domain where the boundary has been "shrunk" by an amount $p\sqrt{t}$.

In view of the previous result, this becomes

$$\int_0^\infty e^{-p^2}\left[-L\,(p+\mathrm{d}p)\sqrt{t}\,+L\,p\sqrt{t}\,+\pi(p+\mathrm{d}p)^2t-\pi p^2 t\right]=L\,\frac{\sqrt{\pi t}}{2}+\pi t,$$

L being the length of the domain D.

To summarize one finds

$$\sum_n e^{-\lambda_n\,t}\overset{t\downarrow 0}{\sim}\frac{A(D)}{4\pi t}-\frac{L(D)}{8\sqrt{\pi t}}+\frac{1}{4}+o(1).$$

The same argument can also be applied to the case where the domain has a number G of holes. Assuming again their boundaries to be smooth and simple, and in view of the orientation of the curvilinear integrals the result would be

$$\sum_n e^{-\lambda_n\,t}\overset{t\downarrow 0}{\sim}\frac{A(D)}{4\pi t}-\frac{L(D)}{8\sqrt{\pi t}}+\frac{1-G}{4}+o(1).$$

where $L(D)$ is the the total length (including the holes) of the domain.

However, this result has been obtained by neglecting the curvature of the boundary. It will now be shown that if this is taken into account, the formula becomes

$$\sum_n e^{-\lambda_n\,t}\overset{t\downarrow 0}{\sim}\frac{A(D)}{4\pi t}-\frac{L(D)}{8\sqrt{\pi t}}+\frac{1-G}{6}+o(1).$$

4. THE DUHAMEL METHOD

To solve the one-dimensional equation

$$\frac{\partial u(x,t)}{\partial t}=\frac{\partial^2 u(x,t)}{\partial x^2},\quad u(0,t)=f(t),\ u(x,0)=0,$$

let $u_0(x,t)$ be the solution in the case $f(t)=1$ $(t>0)$, *viz.*

$$u_0(x,t)=\frac{2}{\sqrt{\pi}}\int_{\frac{x}{2\sqrt{t}}}^{\infty}e^{-s^2}\mathrm{d}s.$$

Then by the Duhamel method [4],

$$\begin{aligned}u(x,t)&=\int_0^t f(s)\,\partial_t u_0(x,t-s)\,\mathrm{d}s\\&=\frac{x}{2\sqrt{\pi}}\int_0^t f(s)\;e^{-\frac{x^2}{4(t-s)}}\,(t-s)^{-3/2}\,\mathrm{d}s.\end{aligned}$$

Likewise for the two-dimensional case

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \quad u(0,y,t) = g(y,t),\ u(x,y,0) = 0,$$

consider first the special case $u_0(0,y,t) = g(y)$, for which

$$u_0(x,y,t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x g(\eta)}{x^2 + (\eta - y)^2} \ e^{-\frac{x^2+(\eta-y)^2}{4t}} \mathrm{d}\eta.$$

Then again by the Duhamel method,

$$\begin{aligned} u(x,y,t) &= \int_0^t f(s)\ \partial_t u_0(x,y,t-s)\ \mathrm{d}s \\ &= \frac{1}{\pi} \int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \frac{x g(\eta,s)}{x^2 + (\eta - y)^2}\ \partial_t\ e^{-\frac{x^2+(\eta-y)^2}{4(t-s)}}\ \mathrm{d}\eta \\ &= \frac{1}{4\pi} \int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \frac{x g(\eta,s)}{(t-s)^2}\ e^{-\frac{x^2+(\eta-y)^2}{4(t-s)}}\ \mathrm{d}\eta. \end{aligned}$$

To solve therefore

$$\frac{\partial u}{\partial t} = \Delta u, \quad u(\mathbf{r},0) = f(\mathbf{r}), \quad u\big|_{\Gamma} = 0$$

where $\Gamma \equiv y^2 + 2\,\mathcal{R}_P\, x = 0$ is the osculating parabola to ∂D at P (see Figure 3), let u_0 be the solution of

$$\frac{\partial u_0}{\partial t} = \Delta u_0, \quad u_0(\mathbf{r},0) = f(\mathbf{r}), \quad u_0\big|_{x=0} = 0.$$

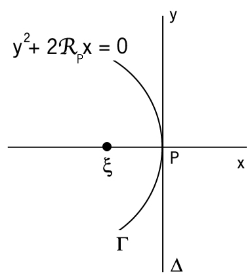

Fig. 3. Osculating parabola to boundary

Then $v = u - u_0$ obeys the equation

$$\frac{\partial v}{\partial t} = \Delta v, \quad v(\mathbf{r}, 0) = 0, \quad v\big|_\Gamma = -u_0\big|_\Gamma,$$

and in the vicinity of P, the last condition can be replaced by the approximation

$$v\big|_{x=0} = -u_0\big|_\Gamma.$$

Now

$$\begin{aligned} u_0\big|_\Gamma &= \frac{1}{4\pi t}\left[e^{-\frac{(x-\xi)^2+y^2}{4t}} - e^{-\frac{(x+\xi)^2+y^2}{4t}}\right]_\Gamma \\ &= \frac{1}{4\pi t}\left[e^{-\frac{\left(\frac{y^2}{2\mathcal{R}}+\xi\right)^2+y^2}{4t}} - e^{-\frac{\left(\frac{y^2}{2\mathcal{R}}-\xi\right)^2+y^2}{4t}}\right] \\ &= \frac{1}{2\pi t}e^{-\frac{y^2+\xi^2}{4t}} \sinh\frac{\xi y^2}{4\mathcal{R}t}e^{-\frac{y^4}{16\mathcal{R}^2 t}}, \end{aligned}$$

and this quantity is exponentially small as $t \downarrow 0$ unless both y and ξ are $\mathrm{O}(\sqrt{t})$, in which case

$$u_0\big|_\Gamma \sim \frac{1}{8\pi\mathcal{R}t^2}\,\xi y^2\, e^{-\frac{y^2+\xi^2}{4t}}.$$

It follows then from the preceding paragraph that

$$v(x,y,t) \sim -\frac{x}{4\pi}\int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \frac{1}{(t-s)^2}e^{-\frac{x^2+(y-\eta)^2}{4(t-s)}}\frac{1}{8\pi\mathcal{R}_P\, s^2}\,\xi\eta^2\, e^{-\frac{\eta^2+\xi^2}{4s}}\,\mathrm{d}\eta,$$

and in particular

$$\begin{aligned} v(\xi,0,t) &= -\frac{\xi}{4\pi}\int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \frac{1}{(t-s)^2}e^{-\frac{\xi^2+\eta^2}{4(t-s)}}\frac{1}{8\pi\mathcal{R}_P\, s^2}\,\xi\eta^2\, e^{-\frac{\eta^2+\xi^2}{4s}}\,\mathrm{d}\eta \\ &= -\frac{\xi^2}{32\pi^2\mathcal{R}_P}\int_0^t \mathrm{d}s\, e^{-\frac{\xi^2 t}{4s(t-s)}}\frac{1}{s^2(t-s)^2}\int_{-\infty}^{\infty}\eta^2\, e^{-\frac{\eta^2 t}{4s(t-s)}}\,\mathrm{d}\eta \\ &= -\frac{\xi^2}{8\pi^2\mathcal{R}_P\, t^{3/2}}\int_0^t \mathrm{d}s\, e^{-\frac{\xi^2 t}{4s(t-s)}}\sqrt{\frac{\pi}{s(t-s)}}. \end{aligned}$$

It now remains to integrate over ξ, and since it lies inside the domain, assumed to be convex, we find that the correction to the asymptotic sum, due to the curvature of the boundary, is

$$
\begin{aligned}
&-\frac{1}{8\pi^2\, t^{3/2}} \int_0^t \mathrm{d}s \sqrt{\frac{\pi}{s(t-s)}} \int_{-\infty}^0 \xi^2\, e^{-\frac{\xi^2 t}{4s(t-s)}}\, \frac{\pi(\mathcal{R}_P+\xi+\mathrm{d}\xi)^2-\pi(\mathcal{R}_P+\xi)^2}{\mathcal{R}_P}\,\mathrm{d}\xi \\
&= -\frac{1}{4\pi\, t^{3/2}} \int_0^t \mathrm{d}s \sqrt{\frac{\pi}{s(t-s)}} \int_{-\infty}^0 \xi^2\, e^{-\frac{\xi^2 t}{4s(t-s)}} \left(1+\frac{\xi}{\mathcal{R}_P}\right)\mathrm{d}\xi \\
&= -\frac{1}{2t^3}\int_0^t \mathrm{d}s\; s(t-s) + O(\sqrt{t}) \overset{t\downarrow 0}{=} -\frac{1}{12}.
\end{aligned}
$$

The same reasoning can be held in the case the domain having holes, and when this correction $-\frac{1-G}{12}$ is taken into account, the W–P–K formula is recovered.

It will have been noticed that this result holds provided the radius of curvature $\mathcal{R}_P \neq 0$ at all points of the border. For instance, the third term of the asymptotic expansion for a square with rounded corners, however small but non-vanishing the radius may be, is different from that for a square.

The method followed by Kac, namely of starting with a polygon and increasing the number of edges to obtain in the limit a smooth boundary, cannot be put into reverse. The Duhamel method is inapplicable if there is no well-defined normal, and consequently tangent, to the boundary at a sharp corner.

However, the previous results suggest that the modifications to the asymptotic expansion due to sharp corners are local effects, that is to say the contribution of each one is independent of the actual shape of the rest of the boundary, at least to the first order. As a consequence it is possible to evaluate this contribution from the known spectrum of the circular sector of angle $\varpi \leq \pi$ (to retain convexity), for which the eigenfunctions are the Bessel functions $J_{m\nu}(j_{m\nu,n}\rho)\sin n\theta$, where $\nu = \pi/\varpi$ and $m, n = 1, 2, \ldots$

To find the asymptotic expansion of

$$f(t) = \sum_{m,n} e^{-j^2_{m\nu,n} t},$$

we consider the Laplace transform $F(s)$ of $tf(t)$ (that of $f(t)$ does not exist):

$$F(s) = \int_0^\infty e^{-st}\, t f(t)\,\mathrm{d}t = \sum_{m,n} \frac{1}{(s+j^2_{m\nu,n})^2}$$

and study its asymptotic behaviour as $s \to \infty$.

Since

$$J_\mu(z) = \frac{\left(\frac{z}{2}\right)^\mu}{\Gamma(\mu+1)} \prod_{k=1}^\infty \left(1-\frac{z^2}{j^2_{\mu,k}}\right),$$

it follows that
$$\sum_{\mu,n}^{\infty} \frac{1}{(s+j_{\mu,n}^2)^2} = -\sum_{\mu} \frac{\mathrm{d}^2}{\mathrm{d}s^2}\left[\ln \frac{I_\mu(\sqrt{s})}{s^{\mu/2}}\right].$$

From Olver's uniform (saddle point) asymptotic expansion [8],
$$I_\mu(w) \sim \frac{1}{\sqrt{2\pi}} \frac{e^\eta}{(\mu^2+w^2)^{1/4}} \left[1+\frac{u_1(t)}{\mu}+\ldots\right],$$
where
$$\eta = \sqrt{\mu^2+w^2}+\mu\ln w - \mu\ln(\mu+\sqrt{\mu^2+w^2})$$
$$t = \frac{\mu}{\sqrt{\mu^2+w^2}}, \quad u_1(t) = \frac{3t-5t^3}{24}, \quad \ldots$$
it follows that
$$-\frac{\mathrm{d}^2}{\mathrm{d}s^2}\ln\frac{I_\mu(\sqrt{s})}{s^{\mu/2}} \sim \frac{1}{4(\mu^2+s)^{3/2}} - \frac{\mu}{2s^2} + \frac{\mu^2(2\mu^2+3s)}{4s^2(\mu^2+s)^{3/2}} - \frac{1}{4(\mu^2+s)^2} + \frac{22\mu^2-3s}{32(\mu^2+s)^{7/2}} + \ldots$$

Let $\varpi = \pi/p$, $p>1$ for the sector to be convex, but not necessarily an integer. Then the values for μ are the integer multiples of p, and as
$$\sum_{k=1}^{\infty}(k^2p^2+s)^{-1} = \frac{1}{2s}\left[\frac{\pi\sqrt{s}}{p}\coth\frac{\pi\sqrt{s}}{p}-1\right],$$
consequently,
$$\sum_{k=1}^{\infty}(k^2p^2+s)^{-2} \sim \frac{\pi}{4ps^{3/2}} - \frac{1}{2s^2} + O(s^{-1}e^{-2\pi\sqrt{s}/p}).$$

Similarly
$$\sum_{k=1}^{\infty}(k^2p^2+s)^{-3/2} \overset{s\uparrow\infty}{\sim} \frac{1}{ps} - \frac{1}{2s^{3/2}} + O(s^{-3/2}e^{-2\pi\sqrt{s}/p}),$$
for
$$2\sum_{n=1}^{\infty}(n^2+s)^{-3/2}+s^{-3/2} = \sum_{n=-\infty}^{\infty}(n^2+s)^{-3/2} = \lim_{R\to\infty}\frac{1}{2\pi i}\int_{\Gamma_R}\frac{\pi\cot\pi z}{(z^2+s)^{3/2}}\mathrm{d}z,$$
and this is equal to
$$\int_{\gamma}\frac{\coth\pi y}{(y^2-s)^{3/2}}\,\mathrm{d}y,$$
where Γ_R and γ are as shown in Figure 4.

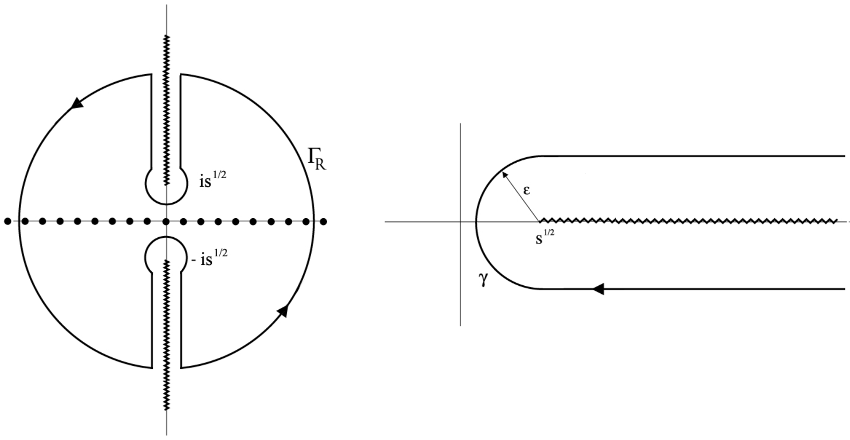

Fig. 4. Integration contours

Integration by parts allows the contour γ to be shrunk ($\epsilon \to 0$) around the cut, and gives

$$\frac{2}{s}\lim_{y\to\infty}\frac{y\ \coth\pi y}{(y^2-s)^{1/2}}+\frac{2}{s}\int_{\sqrt{s}}^{\infty}\frac{\mathrm{d}y}{\sinh^2\pi y\ (y^2-s)^{1/2}}=\frac{2}{s}+O(s^{-3/2}e^{-2\pi\sqrt{s}}),$$

and by scaling the result follows.

Finally, proceeding along similar lines,

$$\sum_{k=1}^{\infty}\left[\frac{kp}{s^2}-\frac{k^2p^2\,(2k^2p^2+3s)}{2s^2(k^2p^2+s)^{3/2}}\right]=\frac{1}{8ps}-\frac{p}{24s^2}+\mathrm{O}(s^{-3}),$$

$$\sum_{k=1}^{\infty}\frac{22k^2p^2-3s}{32(k^2p^2+s)^{7/2}}=\frac{1}{24ps^2}+\mathrm{O}(s^{-5/2}),$$

and all remaining terms are at at most $\mathrm{O}(s^{-5/2})$.

Collecting all terms, we find

$$F(s)=\sum_{k,n}\frac{1}{(s+j^2_{k\pi/\varpi,n})^2}\sim\frac{a_0}{s}+\frac{a_1}{s^{3/2}}+\frac{a_2}{s^2}+\mathrm{O}(s^{-5/2}),$$

and consequently

$$f(t)=\sum_n e^{-\lambda_n t}\overset{t\downarrow 0}{\sim}\frac{a_0}{t}+\frac{2a_1}{\sqrt{\pi t}}+a_2+\mathrm{O}(\sqrt{t})$$

with

$$a_0=\frac{1}{8p},\quad a_1=-\frac{1}{8}-\frac{\pi}{16p},\quad a_2=\frac{1}{8}+\frac{p}{24}+\frac{24}{p}.$$

In terms of the area $A = \varpi/2 = \frac{\pi}{2p}$ and the perimeter $L = 2 + \varpi = 2 + \frac{\pi}{p}$ of the sector, this becomes

$$f(t) = \sum_n e^{-\lambda_n t} \overset{t\downarrow 0}{\sim} \frac{A}{4\pi t} - \frac{L}{8\sqrt{\pi t}} + \frac{1}{8} + \frac{1}{24}\left(\frac{\pi}{\varpi} + \frac{\varpi}{\pi}\right).$$

We now write this result as

$$\sum_n e^{-\lambda_n t} \overset{t\downarrow 0}{\sim} \frac{A}{4\pi t} - \frac{L}{8\sqrt{\pi t}} + \frac{1}{6} + C$$

with

$$C = -\frac{1}{24} + \frac{1}{24}\left(\frac{\pi}{\varpi} + \frac{\varpi}{\pi}\right),$$

in order to evaluate the modification to the smooth boundary formula due to a single corner. Let $\varpi = \pi$, then the domain is a half-circle, and has two corners, both of internal angle $\pi/2$. As

$$C = -\frac{1}{24} + \frac{1}{12} = \frac{1}{24},$$

each right-angle corner contributes $1/48$.

Consequently, a corner of internal angle ϖ must contribute

$$-\frac{1}{24} + \frac{1}{24}\left(\frac{\pi}{\varpi} + \frac{\varpi}{\pi}\right) - \frac{1}{24} = \frac{1}{24}\left(\frac{\pi}{\varpi} + \frac{\varpi}{\pi} - 2\right),$$

and this completes the proof.

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Received: March 3, 2020.
Revised: October 14, 2020.
Accepted: October 16, 2020.