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The directional error curve as booth elliptical lemniscate

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Abstract

There are two aspects of position coordinates accuracy in navigation: global accuracy and local accuracy – relative to navigational dangers. The latter case refers to the assessment of position (distance) accuracy in respect to the nearest navigational danger, a hazard to navigation. The directional error is a relevant measure for such assessment. The directional error curve is analyzed in this article as a particular case of the Booth elliptical lemniscate. The curve graph illustrates the confidence interval of point position errors in a given direction (about 68%).

Introduction

One essential aspect of navigation and shipping safety is the assessment of accuracy and reliability of navigational data and other information. Various measures of accuracy assessment are used for this purpose. Position coordinates are evaluated with a few different measures. Some of them have practical applications due to the ease of plotting or calculating and simple geometrical interpretation. We have to bear in mind, however, that such measures are more to give an idea than to be exact. For this reason we use advanced methods for estimating navigational parameters of automated and integrated navigational systems. Consequently, the appropriate assessment of their accuracy is made. On the other hand, in systems and equipment for navigational information / situation display we can use and present a variety of accuracy assessment measures for position and other navigational parameters.

The most common measures of position accuracy assessment used in navigational practice are as follows:

- parallelogram of position errors;
- mean error ellipse;
- mean directional error;
- mean position error;
- probable position error;
- 95% error;
- maximum error;
- position covariance matrix;
- ellipsoid and hyperellipsoid of position errors;
- geometric coefficients of a position determination system or parametric navigation.

Note that a different probability corresponds to each of these measures.

In most cases maritime navigation deals with navigational parameters: scalars or two-dimensional vectors. We generally obtain these parameters as direct or indirect measurements of physical or geometric quantities. In either case the accuracy of ultimate results needs to be assessed. Accuracy assessment consists in examining the distributions of scalar random variables or two-dimensional random vectors.

In many theoretical considerations, as well as practical applications a need arises to evaluate the accuracy of linear objects (scalars) based on information on the accuracy of individual points. From known (measured) random vectors we calculate scalar parameters, components of the vectors. One such example is the calculation of a distance between two points, whose coordinates (two-dimensional or more-dimensional) are defined from measurements. A similar case is, when from measured position coordinates we determine a spatial position of linear objects – depth contours, wharf or fairway limits, area boundaries, distance to a danger etc. In such situations we should use the directional error, not the mean circular error [1, 2]. These authors analyze the directional error and its relations with the Booth's elliptical lemniscate.

Two-dimensional normal distribution

In maritime navigation we often adopt a two-dimensional space, in which ship's position coordinates are determined. Therefore, a ship's position can be regarded as a two-dimensional random vector, whose distribution is determined by its probability density function. In our case this function has the following form [3, 4]:

$$f(x,y)=\frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho_{xy}^2}}\exp\left\{-\frac{1}{2\left(1-\rho_{xy}^2\right)}\cdot\right.$$
$$\left.\cdot\left[\frac{(x-m_x)^2}{\sigma_x^2}-2\rho\frac{(x-m_x)(y-m_y)}{\sigma_x\sigma_y}+\frac{(y-m_y)^2}{\sigma_y^2}\right]\right\} \tag{1}$$

where:

- $\bar{x},\bar{y}$ – mean values of random variables X, Y;
- σ_x – standard deviation of random variable X;
- σ_y – standard deviation of random variable Y;
- ρ_{xy} – correlation coefficient of random variables X and Y.

In navigational interpretation σ_x is the mean error of the coordinate X, that is the directional error along the axis X. A similar remark applies to the standard deviation of the variable Y. Besides, as we bear in mind, the correlation coefficient of random variables is defined by this relation:

$$\rho_{xy}=\frac{\sigma_{xy}}{\sigma_x\sigma_y} \tag{2}$$

where: σ_{xy} is a covariance of random variables X, Y.

Let us note that the value of probability density function at its maximum does not change due to the translation (shift), or rotation of the coordinate system, which can be formulated as follows.

Conclusion 1.

The probability density function value for a two-dimensional random vector in the maximum is an invariant of shifts and rotations.

For a given probability distribution, the quantities σ_x, σ_x, ρ_{xy} and $\bar{x},\bar{y}$ are constants. The function (1) reaches a maximum when:

$$x=\bar{x},\;\; y=\bar{y} \tag{3}$$

After the substitution of conditions (3) into the relation (1) we obtain:

$$e^0 = 1.$$

Hence at the maximum, the value of probability density function will be:

$$f(x,y)_{\max}=\frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho_{xy}^2}} \tag{4}$$

After incorporating the correlation coefficient (2) and doing some transformations, we bring the term $\sigma_x\sigma_y\sqrt{1-\rho_{xy}^2}$ from the denominator of equation (4) to this formula:

$$\sigma_x\sigma_y\sqrt{1-\rho_{xy}^2}=\sqrt{\sigma_x^2\sigma_y^2-\sigma_{xy}^2} \tag{5}$$

As mentioned before, according to Conclusion 1, the equation below holds:

$$f(x,y)_{\max}=f(x',y')_{\max} \tag{6}$$

where x', y' are new coordinates after a rotation or translation of the coordinate system, also the following equation will hold:

$$\sigma_x^2\sigma_y^2-\sigma_{xy}^2=\sigma_{x'}^2\sigma_{y'}^2-\sigma_{x'y'}^2 \tag{7}$$

The matrix of two-dimensional covariance of probability density function (1) has this form:

$$\mathbf{P}=\begin{bmatrix}\sigma_x^2 & \sigma_{xy}\\ \sigma_{xy} & \sigma_y^2\end{bmatrix}$$

The left-hand and right-hand side of equation (7) represent the determinant of that matrix, from which another conclusion is implied.

Conclusion 2.

The determinant of the covariance matrix of a two-dimensional random vector is an invariant of rotations and translations.

Let us take a probability density function for a random vector, whose mean vector is a zero vector. This will simplify algebraic transformations and maintain the general nature of considerations. It simply means a shift of the coordinate system by a vector [$\bar{x},\bar{y}$].

In this case, we will obtain the following form of the probability density function:

$$f(x,y)=\frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho_{xy}^2}}\cdot$$
$$\cdot\exp\left\{-\frac{1}{2\left(1-\rho_{xy}^2\right)}\left[\frac{x^2}{\sigma_x^2}-2\rho\frac{xy}{\sigma_x\sigma_y}+\frac{y^2}{\sigma_y^2}\right]\right\} \tag{8}$$

In our further analysis we will utilize the following theorem [4, 5, 6].

Theorem

If n-dimensional random vector $\mathbf{x}$ has a multidimensional normal distribution with expected values vector $\bar{\mathbf{x}}$ and covariance matrix $\mathbf{P}$ of n order, then for any real $m \times n$ of matrix $\mathbf{A}$ of the $m \le n$ order, the random vector $\mathbf{y} = \mathbf{Ax}$ has a multidimensional normal distribution with an expected value $E(\mathbf{y}) = \mathbf{A}\bar{\mathbf{x}}$ and covariance matrix $\mathbf{APA}^T$.

Conclusion 3.

In particular for $m = 1$ the random variable $\mathbf{w}^T\mathbf{x}$ has a normal distribution with expected value $\mathbf{w}^T\bar{\mathbf{x}}$ and variance $\mathbf{w}^T\mathbf{P}\mathbf{w}$.

We can use the above theorem for determining the random variable distribution after a change of the coordinate system. Then [7] matrix $\mathbf{A}$ will be a Jacobian matrix for the transformation of the old coordinate system into the new one. This theorem is also a generalization of the law of mean error propagation [8].

Matrix $\mathbf{A}$ can be a matrix of rotations – a change of coordinates due to a rotation of the system. We can calculate directional errors from the newly obtained covariance matrix, and for this purpose the conclusion drawn from the theorem can be used. In this case vector $\mathbf{w}$ will be a directional vector.

Let us now rotate the coordinate system by an angle α. The old coordinates will be expressed by the new ones and the rotation angle in this formula:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \cdot \begin{bmatrix} x' \\ y' \end{bmatrix} \tag{9}$$

It is known that rotation affects only the values of variables x, y. It does not affect the determinant of covariance matrix (conclusion 2), so it does not affect the denominator of the equation (4), either. Therefore, we can consider only the function exponent (8). Let us do subsequent transformations, taking into account relation (2). We get:

$$-\frac{1}{2\left(1-\rho_{xy}^2\right)}\left[\frac{x^2}{\sigma_x^2} - 2\rho\frac{xy}{\sigma_x\sigma_y} + \frac{y^2}{\sigma_y^2}\right] = \\ = -\frac{1}{2}\left(\frac{\sigma_y^2 x^2}{\sigma_x^2\sigma_y^2 - \sigma_{xy}^2} - \frac{2\sigma_{xy}xy}{\sigma_x^2\sigma_y^2 - \sigma_{xy}^2} + \frac{\sigma_x^2 y^2}{\sigma_x^2\sigma_y^2 - \sigma_{xy}^2}\right) \tag{10}$$

On the right-hand side of equation (10) is the covariance matrix determinant det$\mathbf{P}$ that can be factored out. Then we obtain a simplified form on the right (10):

$$-\frac{1}{2\det\mathbf{P}}\left(\sigma_y^2 x^2 - 2\sigma_{xy}xy + \sigma_x^2 y^2\right) \tag{11}$$

Now we can exchange the variables (rotation of the coordinate system by an angle α). To do this, we put formulas (9) into (11). After transformations the final form of expression (11) in the new coordinate system will be:

$$-\frac{1}{2\det\mathbf{P}}\Big[\left(\sigma_y^2\cos^2\alpha - \sigma_{xy}\sin 2\alpha + \sigma_x^2\sin^2\alpha\right)x'^2 + \\ -\left(\sigma_y^2\sin 2\alpha + \sigma_{xy}\cos 2\alpha - \sigma_x^2\sin 2\alpha\right)x'y' + \\ +\left(\sigma_y^2\sin^2\alpha - \sigma_{xy}\sin 2\alpha + \sigma_x^2\cos^2\alpha\right)y'^2\Big]$$

Let us write this expression somewhat differently, so that after sorting out particular terms we get:

$$-\frac{1}{2\det\mathbf{P}}\Big[\left(\sigma_x^2\sin^2\alpha - \sigma_{xy}\sin 2\alpha + \sigma_y^2\cos^2\alpha\right)x'^2 + \\ -2\left(-\frac{1}{2}\left(\sigma_x^2 - \sigma_y^2\right)\sin 2\alpha + \sigma_{xy}\cos 2\alpha\right)x'y' + \\ +\left(\sigma_x^2\cos^2\alpha - \sigma_{xy}\sin 2\alpha + \sigma_y^2\sin^2\alpha\right)y'^2\Big] \tag{12}$$

According to formula (7) this equation also holds:

$$\det\mathbf{P} = \sigma_{x'}^2\sigma_{y'}^2 - \sigma_{x'y'}^2$$

After taking into consideration the form of expression (11) and comparison with appropriate terms of expression (12), we come to a situation, where the variances and covariance, after a rotation of the coordinate system, will be written as:

$$\sigma_{x'}^2 = \sigma_x^2\cos^2\alpha - \sigma_{xy}\sin 2\alpha + \sigma_y^2\sin^2\alpha \tag{13}$$

$$\sigma_{y'}^2 = \sigma_x^2\sin^2\alpha - \sigma_{xy}\sin 2\alpha + \sigma_y^2\cos^2\alpha \tag{14}$$

$$\sigma_{x'y'} = -\frac{1}{2}\left(\sigma_x^2 - \sigma_y^2\right)\sin 2\alpha + \sigma_{xy}\cos 2\alpha \tag{15}$$

If in the above formulas we express the covariance by the correlation coefficient and standard deviations (relation (2)), then they will appear in another form:

$$\sigma_{x'}^2 = \sigma_x^2\cos^2\alpha - \rho_{xy}\sigma_x\sigma_y\sin 2\alpha + \sigma_y^2\sin^2\alpha \tag{16}$$

$$\sigma_{y'}^2 = \sigma_x^2\sin^2\alpha - \rho_{xy}\sigma_x\sigma_y\sin 2\alpha + \sigma_y^2\cos^2\alpha \tag{17}$$

$$\sigma_{x'y'} = -\frac{1}{2}\left(\sigma_x^2 - \sigma_y^2\right)\sin 2\alpha + \rho_{xy}\sigma_x\sigma_y\cos 2\alpha \tag{18}$$

Formulas (13)–(15) or (16)–(18) describe the variances and the covariance of the normal distribution after the coordinate system rotation.

Directional error

Standard deviations (mean errors) are determined as an arithmetic root of the random variable variance. Hence the roots of variances (16) and (17) are directional errors along the new coordinate axes *X*′, *Y*′. Then, the formula defining the directional error (in the direction indicated by an angle, axis *X*′) is as follows:

$$\sigma_\alpha^2 = \sigma_x^2 \cos^2\alpha - \rho_{xy}\sigma_x\sigma_y \sin 2\alpha + \sigma_y^2 \sin^2\alpha \quad (19)$$

or

$$\sigma_\alpha^2 = \sigma_x^2 \cos^2\alpha + \sigma_{xy} \sin 2\alpha + \sigma_y^2 \sin^2\alpha \quad (20)$$

The probability corresponding to the directional error is equal to the probability of mean error. The geometrical interpretation of directional error is shown below (Fig. 1).

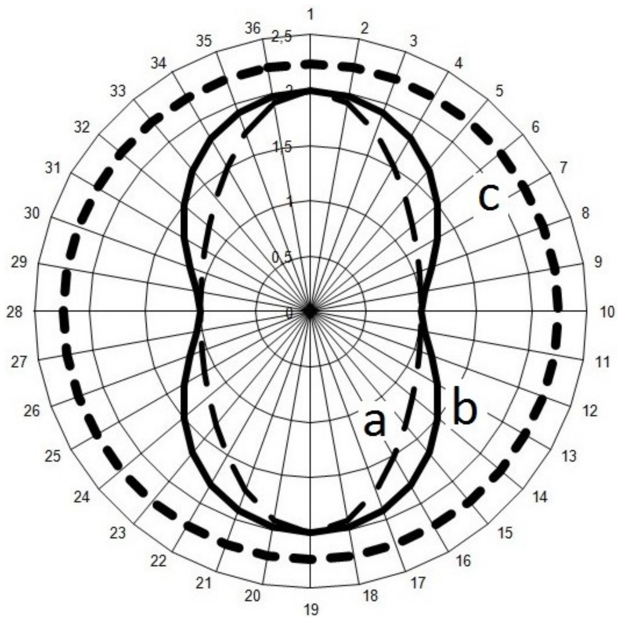

Fig. 1. Comparison of mean error ellipse, mean directional error and directional error (a – confidence ellipse, b – directional error, c – distance root mean square)

In the particular case, when mean errors of coordinates are equal to semi-axes of the mean error ellipse, i.e.

$$a = \sigma_x \quad \text{and} \quad b = \sigma_y$$

the covariance equals zero, that is $\sigma_{xy} = 0$ (consequently, also the correlation coefficient will equal zero). We will then get the following form of distribution variance (1) after a rotation by angle α, the variance being expressed as a function of mean error ellipse parameters:

$$\sigma_{x'}^2 = a^2 \cos^2\alpha + b^2 \sin^2\alpha \quad (21)$$

$$\sigma_{y'}^2 = a^2 \sin^2\alpha + b^2 \cos^2\alpha \quad (22)$$

The directional error, in turn, will be expressed by this relation:

$$\sigma_\alpha^2 = a^2 \cos^2\alpha + b^2 \sin^2\alpha \quad (23)$$

It is the same relation as the one stated in [1, 9].

Let us take a look at the relation between the mean position error and the directional error. We know that the mean position error *M* is expressed by the equivalent formulas [5]:

$$M = \sqrt{trP} = \sqrt{\sigma_x^2 + \sigma_y^2} \quad (24)$$

$$M = \sqrt{a^2 + b^2} \quad (25)$$

A comparison of formulas (23) and (25) implies that the directional error is always smaller than the mean position error:

$$\sigma_\alpha < M$$

This means that it is incorrect to use mean position errors instead of directional errors in accuracy analysis of linear objects. Unfortunately, it often happens in theoretical analyses as well as in practice. The reason is sometimes that during a series of GPS (or DGPS) measurements covariance in not computed, and so the directional error or parameters of the mean error ellipse cannot be determined.

Let us check how a rotation of the coordinate system changes the value of mean position error. To this end, let us sum up variances (16) and (17), and we obtain:

$$\sigma_{x'}^2 + \sigma_{y'}^2 = \sigma_x^2 \cos^2\alpha - \rho_{xy}\sigma_x\sigma_y \sin 2\alpha +$$
$$+ \sigma_y^2 \sin^2\alpha + \sigma_x^2 \sin^2\alpha - \rho_{xy}\sigma_x\sigma_y \sin 2\alpha +$$
$$+ \sigma_y^2 \cos^2\alpha = \sigma_x^2 + \sigma_y^2$$

The above equation leads to the following conclusion.

Conclusion 4.

The mean position error *M* (trace of covariance matrix **P**) is an invariant of shifts and rotations.

We would arrive at the same conclusion by adding variances (21) and (22).

Booth lemniscates

Now, we will demonstrate that the directional error is a Booth elliptical lemniscate. The name of the curve is derived from the fact it is the pedal curve of the ellipse relative to its centre. Its general form is described by the implicit equation [10]:

$$\left(x^2 + y^2\right)^2 - a^2x^2 - b^2y^2 = 0 \quad (26)$$

Let us consider polar coordinates. We substitute:

$$x = r\cos\alpha \quad \text{and} \quad y = r\sin\alpha$$

into equation (26):

$$\left(r^2\cos^2\alpha + r^2\sin^2\alpha\right)^2 - a^2r^2\cos^2\alpha +$$
$$-b^2r^2\sin^2\alpha = 0$$

that is:

$$r^4 = a^2r^2\cos^2\alpha + b^2r^2\sin^2\alpha\ .$$

After dividing both sides by r^2, we obtain:

$$r^2 = a^2\cos^2\alpha + b^2\sin^2\alpha \qquad (27)$$

The directional error σ_α is a distance of its curve to the origin of coordinate system, that is radius r. Therefore, we have obtained an equivalent relation (23). It proves that the directional error curve is a Booth elliptical lemniscate and thus, is characterized by the same properties [10].

Conclusions

The above stated relations determining the directional error are general and can be used regardless of the orientation of the mean error ellipse. This allows us to skip one stage of calculations – those of parameters of the mean error ellipse based on elements of the position covariance matrix [5] or analytical-graphical methods. It is essential in a situation where we obtain positions by various position determination systems or when parameters of the mean error ellipse are time- and space-varying (satellite navigational systems). We can also see that the curve of directional errors is of a more general geometric character as a Booth elliptical lemniscate. That is why properties of that pedal curve result directly from geometric analysis and depend on the related ellipse.

From the analysis herein conducted, we can state that invariants of shifts and rotations are:

– maximum of the probability density function of a two-dimensional random vector;
– determinant of the covariance matrix of a two-dimensional random vector;
– mean position error (trace of the covariance matrix).

In practical terms it means that locally we can make transformations of linear coordinate systems to a form most convenient for accuracy analysis of geometrical parameters of our concern.

References

1. Banachowicz A.: Directional Error. Scientific Bulletin 64. WSM, Szczecin 2001.
2. Banachowicz A., Wolski A.: The Fusion of Point and Linear Object in Navigation. TRANSNAV – International Journal on Marine Navigation and Safety of Sea Transportation 7, 3, September 2013, Gdynia.
3. Gersternkorn T., Śródka T.: Kombinatoryka i rachunek prawdopodobieństwa. PWN, Warszawa 1978.
4. Morrison D.F.: Wielowymiarowa analiza statystyczna. PWN, Warszawa 1990.
5. Banachowicz A.: Geometria liniowego modelu nawigacji parametrycznej. Zeszyty Naukowe Akademii Marynarki Wojennej 109A, Gdynia 1991.
6. Rao R.C.: Modele liniowe statystyki matematycznej. PWN, Warszawa 1982. Linear Statistical Inference and its Applications. John Wiley & Sons, New York-London-Sydney-Toronto 1973.
7. Vaniček P., Krakiwsky E.J.: Geodesy: the concepts. 2nd edition. North Holland, Amsterdam 1986.
8. Banachowicz A., Uriasz J.: Accuracy Assessment of Spatial Coordinates of Ship's Position on the Fairway. International Symposium on "Modern Technologies, Education and Professional Practice in the Globalizing World" 6–7 November 2003, Sofia, Bulgaria.
9. Niczyporowicz E.: Krzywe płaskie. Wybrane zagadnienia z geometrii analitycznej i różniczkowej. PWN, Warszawa 1991.
10. Osada E.: Geodezja. Oficyna Wydawnicza Politechniki Wrocławskiej, Wrocław 2001.

Other

11. Kopacz Z., Morgaś W., Urbański J.: Ocena dokładności pozycji okrętu. AMW, Gdynia 2007.