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Controllability of consensus heterogeneous multi-agent networks over continuous time scale*

by

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Abstract: The research, presented in this paper, concernes the controllability of a multi-agent network with a directed, unweighted, cooperative, and time-invariant communication topology. The network's agents follow linear and heterogeneous dynamics, encompassing first-order, second-order, and third-order differential equations over continuous time. Two classes of neighbour-based linear distributed control protocols are considered: the first one utilises average feedback from relative velocities/relative accelerations, and the second one utilises feedback from absolute velocities/absolute accelerations. Under both protocols, the network's agents achieve consensus in their states asymptotically. We observe that both of the considered dynamical rules exploit the random-walk normalised Laplacian matrix of the network's graph. By categorising the agents of the network into leaders and followers, with leaders serving as exogenous control inputs, we analyse the controllability of followers within their state space through the influence of leaders. Specifically, matrix-rank conditions are established to evaluate the leader–follower controllability of the network under both control protocols. These matrix-rank conditions are further refined in terms of the system matrices' eigenvalues and eigenvectors. The inference diagrams presented in this work provide deeper insights into how leader–follower interactions impact the network controllability. The efficacy of the theoretical findings is validated through numerical examples.

Keywords: multi-agent systems, controllability, heterogeneous agent models

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1. Introduction

Multi-agent networked systems have garnered significant attention in recent years due to their numerous advantages over conventional control systems. These advantages include cost reduction, improved system efficiency, enhanced flexibility, reliability, and the ability to provide new capabilities. Multi-agent systems are used to model a wide range of real-world phenomena, from biological agent flocking to multiple mobile robots, spacecrafts, unmanned vehicles, wireless sensor networks, smart grids, social networks, machine learning, economics, and manufacturing, among others (see, for instance, Chen and Ren, 2019, or Mesbahi and Egerstedt, 2010).

Controllability is a fundamental concept in the analysis and synthesis of multi-agent systems. It addresses the question whether the states of the agents can be driven from any initial positions to any desired final positions within a finite time frame using a set of external admissible controls. Controllability is often achieved through a leader–follower strategy, where agents are divided into leaders and followers. The goal is to transfer the followers from their initial configurations to desired configurations within their state-space, while allowing the leaders to move freely within their state-space (see Liu et al, 2008; Lozano et al, 2008; Tanner, 2004; Yazicioğlu and Egerstedt, 2013).

Multi-agent networks are involved in various cooperative tasks, with consensus problems receiving significant attention within the control and network communities. The consensus problem involves driving the states of the network's agents to a common desired value through appropriate agreement protocols. Over the past few decades, consensus problems for multi-agent systems have been extensively studied, particularly within the context of leader–follower architectures. For example, Saber and Murray (2004) discuss consensus problems for multi-agent networks of dynamic agents with fixed and switching topologies. In particular, authors referred to introduced two consensus protocols for networks with and without time-delays and performed convergence analysis. Similar kind of problem is considered in Ren and Beard (2005), where it is shown that the consensus of the agents can be achieved if the union of the directed interaction graphs have a spanning tree frequently enough as the system evolves. Authors in Defoort et al (2015) focused on the design of a consensus model with unknown inherent nonlinear dynamics. In Zhang, Sun and Yang (2021), a new consensus protocol and an event-triggered communication strategy, based on a closed-loop state estimator have been designed.

Notably, the aforementioned results primarily address consensus for multi-agent networks consisting of first-order integrator agents. In the real world, many physical systems have complex inner dynamics. To extend the applicability of multi-agent systems, it is essential to investigate consensus problems for higher-order multi-agent systems. The consensus models with second-order dynamics are considered in Ren (2007), where it is shown that the leader–follower strategy can be unified in the general framework of consensus seeking, and is demonstrated by considering a multiple micro-air-vehicle formation flying. The event-triggered control strategy for the second-order consensus is analysed in Xie et al. (2013) and in Yu et al. (2013) distributed control gains for consensus in multi-agent systems with second-order nonlinear dynamics are discussed. There are also applications for third-order multi-agent networks, such as modelling deflection of curved beams, controlling flying objects in cosmic space (Greguš, 1987), and analysing entry-flow phenomena in fluid dynamics (Jayaraman, Padmanabhan and Mehrotra, 1986; Padhi and Pati, 2014; Reynolds, 1989). Research has focused on consensus in third-order multi-agent networks, for example in Liu, An and Wu (2018).

Most of research has concentrated on homogeneous multi-agent networks, where all agents possess the same-order dynamics. However, in reality, as in population dynamics, epidemic models, economic systems, and social sciences, agent dynamics may differ due to various constraints, making heterogeneity an important parameter to study. Subsequently, researchers have begun to address heterogeneous multi-agent systems under various dynamical rules. For example, in Liu, Xie and Wang (2012), Zheng and Wang (2012a,b), Zheng, Zhu and Wang (2011), authors especially considered the heterogeneous network of agents encompassing first- and second-order integrator dynamics and established their consensus properties under different protocols. A recent study by Geng et al. (2022) investigates the consensus problem in heterogeneous multi-agent systems with directed topology, comprising three classes of agents described by first-order, second-order, and third-order integrator dynamics. In all these works, a common fact is that the underlying dynamical rule exploits the Laplacian matrix of the network's graph.

In the parallel line of research, there has been an investigation on the leader–follower controllability of heterogeneous networks under different kinds of consensus protocols. For example, Guan et al. (2016) examine controllability issues for continuous- and discrete-time consensus in linear, heterogeneous multi-agent systems composed of first- and second-order integrator agents, under directed and weighted communication topologies. The results of Guan et al. (2016) have been extended to the case of third-order integrator agents in Muni et al. (2023). Again in both of these works, the underlying dynamical rule exploits the Laplacian matrix of the network's graph. We would like to mention that in applications such as modelling Brownian motion, epidemic information diffusion, random sampling, computing aggregate functions on complex sets, and the illustration of many real-world stochastic processes, the dynamical model involves the random-walk normalised Laplacian matrix instead of Laplacian (Aldous, 1991; Coppersmith et al., 1993; Leleux et al., 2022).

The first time investigation on controllability in consensus of multi-agent networks under the random-walk normalised Laplacian has been carried out in Muni et al. (2022). However, there, the network is homogeneous, involving only first-order integrator dynamics. There is no work reported on the controllability of consensus of heterogeneous networks, where the dynamics exploits the random-walk normalised Laplacian matrix of the network's graph. The present work addresses this gap. By employing two different kinds of consensus protocols, which are linear and distributed, we analyse the controllability of the network in a leader–follower framework. Specifically, by utilising PBH rank criterion, easily verifiable necessary and sufficient matrix-rank conditions for checking leader–follower controllability of the network are obtained, which are subsequently applied to several networks to evaluate their controllability. We also obtain some necessary controllability conditions, which involve the system matrices' eigenvalues and eigenvectors. The inference diagrams presented in this work provide deeper insights into how leader–follower interactions impact the network controllability.

The rest of the paper is organised as follows: In Section 2, we provide background information on graphs and matrices. Section 3 outlines the problem's modelling. In Section 4, we perform a controllability analysis and derive various mathematical conditions for verifying the controllability of the system. Section 5 includes numerical examples to illustrate our results. Finally, the paper concludes with Section 6.

2. Preliminaries

2.1. Matrix preliminaries

The notations used are standard: $\mathbb{W}$, $\mathbb{Z}^+$, and $\mathbb{R}$ ($\mathbb{C}$) refer to the set of whole numbers, positive integers, and real (complex) numbers, respectively. Let $\mathcal{I}_k$ denote the set of the first k positive integers. For any fixed positive integers n and m, $\mathbb{F}^{n\times m}$ is the vector space of all $(n\times m)$ matrices with entries from the field $\mathbb{F}$. When $m = 1$, this space is denoted as $\mathbb{F}^n$, representing the n-dimensional vector space of $(n \times 1)$ column vectors over the field $\mathbb{F}$.

Let $\mathbf{I}_n$ denote the identity matrix of size $(n\times n)$ and $\mathbf{O}$ ($\mathbf{0}$) represent the zero matrix (zero vector) of the appropriate size. The diagonal matrix of size $(n\times n)$ with principal diagonal entries a_i, $i = 1, \ldots, n$ is designated as $\text{diag}(a_1, \ldots, a_n)$. The transpose of $\mathbf{A} \in \mathbb{F}^{n\times m}$ is denoted as $\mathbf{A}^\intercal \in \mathbb{F}^{m\times n}$. Note that $\mathbb{R}^{n\times n}$ is a proper subset of $\mathbb{C}^{n\times n}$, allowing real $(n \times n)$ matrices to have complex eigenvalues and complex eigenvectors. A nonzero $\boldsymbol{v} \in \mathbb{C}^n$ is called an eigenvector of $\mathbf{A} \in \mathbb{R}^{n\times n}$ if $\mathbf{A}\boldsymbol{v} = \mu\boldsymbol{v}$ for some $\mu \in \mathbb{C}$; here μ is called the eigenvalue of $\mathbf{A}$. The symbol $\sigma(\mathbf{A})$ denotes the set containing all eigenvalues of $\mathbf{A}$. It is important to note that $\sigma(\mathbf{A}) = \sigma(\mathbf{A}^\intercal)$. The nullspace of $\mathbf{A} \in \mathbb{R}^{n\times m}$ is a subspace of $\mathbb{C}^m$, given by $\mathcal{N}(\mathbf{A}) := \{\boldsymbol{v} \,|\, \boldsymbol{v} \in \mathbb{C}^m,\ \mathbf{A}\boldsymbol{v} = \mathbf{0}\}$. The eigenspace of $\mathbf{A} \in \mathbb{R}^{n\times n}$ with respect to $\lambda \in \sigma(\mathbf{A})$ is $\mathbb{E}_\lambda(\mathbf{A}) := \mathcal{N}(\lambda\mathbf{I}_n - \mathbf{A})$. Notice that all nonzero elements of $\mathbb{E}_\lambda(\mathbf{A})$ are the eigenvectors of $\mathbf{A}$ corresponding to the eigenvalue λ. For a more in-depth exploration of matrix theory and its properties, we recommend referring to a monograph by Friedberg, Insel and Spence (1989).

2.2. Graph preliminaries

A directed simple graph $\mathcal{G}$ is an ordered pair $(\mathcal{V}, \mathcal{E})$ consisting of a nonempty vertex set $\mathcal{V}$ (with its elements referred to as vertices of $\mathcal{G}$) and an edge set $\mathcal{E}$ (with its elements known as edges of $\mathcal{G}$), together with an incidence function that associates with each edge 'e' of $\mathcal{G}$ an ordered pair (x, y) of distinct vertices x and y, such that no two edges of $\mathcal{G}$ are associated with a common ordered pair of distinct vertices. Here, e is said to join x to y, with x being called the tail and y being called the head of e. A finite directed simple graph has a finite vertex set.

A directed simple graph is weakly connected if, in its undirected version, there is a path between any pair of distinct vertices. In this paper, the used notion of a graph refers to a finite directed simple graph that is weakly connected and has a nonempty edge set.

The neighbor set of $i \in \mathcal{V}$ is $\mathcal{N}_i := \{j : j \in \mathcal{V},\ (j, i) \in \mathcal{E}\}$. The adjacency matrix of $\mathcal{G}$, $\mathcal{A}(\mathcal{G}) := \left[a_{ij}\right]$, is a square matrix of size $|\mathcal{V}| \times |\mathcal{V}|$ with a_{ij} being the weight of edge (j, i). If there is no edge (j, i) in $\mathcal{G}$, then the a_{ij} value is set to zero. When $\mathcal{N}_i \neq \emptyset$ for all $i \in \mathcal{V}$, then the matrix $\mathcal{L}^{rw}(\mathcal{G})$, defined as $\mathcal{L}^{rw}(\mathcal{G}) := \mathbf{I}_{|\mathcal{V}|} - \Delta^{-1}(\mathcal{G})\mathcal{A}(\mathcal{G})$, is known as the random-walk normalised Laplacian of $\mathcal{G}$, this notion playing a central role in the controllability analysis of our multi-agent network; here $\Delta(\mathcal{G}) := \operatorname{diag}(\sum_{j\in\mathcal{N}_i} a_{ij})$ is the degree matrix of $\mathcal{G}$. Notice that the entries of $\mathcal{L}^{rw}(\mathcal{G})$ are given by

$$\left[\mathcal{L}^{rw}(\mathcal{G})\right]_{ij} = \begin{cases} 1 & \text{for } i = j, \\ -\dfrac{a_{ij}}{\sum_{j\in\mathcal{N}_i} a_{ij}} & \text{for } i \neq j. \end{cases}$$

For a more comprehensive exploration of matrix-theoretic and algebraic approaches to graph theory, we recommend referring to sources like Bondy and Murty (2008); Kurras (2016).

2.3. Interconnection graph of the multi-agent network

In this study, we consider a multi-agent network, characterised by directed and cooperative communications, where the communication weights are unity, and the network's topology remains constant over time. Additionally, we ensure that no agent communicates with itself, there are no multiple interactions between any two agents, each agent is influenced by at least one other agent, and the undirected variant of the network maintains connectivity. This network can be effectively represented by a graph, with its vertices corresponding to the agents and edges representing the interactions between agents. This graph is commonly referred to as the *interconnection graph* of the network.

Assuming there are $N \geqq 2$ agents in the network, the adjacency matrix $\mathcal{A}$ of its interconnection graph is an $N \times N$ binary matrix, given by

$$[\mathcal{A}]_{ij} = \begin{cases} 1 & \text{if there is communication from agent } j \text{ to agent } i, \\ 0 & \text{otherwise.} \end{cases}$$

For our convenience, we refer to the network as $\mathfrak{N}$ and its interconnection graph as $\mathcal{G}_{\mathfrak{N}}$.

3. Problem formulation

Let the agents in $\mathfrak{N}$ update their states over the time interval $\tau := [t_{\mathfrak{i}}, t_{\mathfrak{f}}]$, where $0 \leqq t_{\mathfrak{i}} < t_{\mathfrak{f}} < +\infty$, according to a linear and heterogeneous dynamical framework that encompasses first-order, second-order, and third-order differential equations. Let $m \in \mathbb{W}$ be the number of first-order integrator agents, where $0 \leqq m \leqq N$, $s \in \mathbb{W}$ be the number of second-order integrator agents, where $0 \leqq s \leqq N$, $0 \leqq m+s \leqq N$, and the remaining agents, totalling $N-m-s \in \mathbb{W}$, be the third-order integrator agents, where $0 \leqq N-m-s \leqq N$.

Explicitly, the dynamical model governing the state updates of the network's agents is provided below:

$$\begin{cases} \dot{x}_i(t) = u_i(t) & \text{for } i \in \mathcal{I}_m, \\ \dot{x}_i(t) = v_i(t),\ \dot{v}_i(t) = u_i(t) & \text{for } i \in \mathcal{I}_{m+s} \setminus \mathcal{I}_m, \\ \dot{x}_i(t) = v_i(t),\ \dot{v}_i(t) = a_i(t),\ \dot{a}_i(t) = u_i(t) & \text{for } i \in \mathcal{I}_N \setminus \mathcal{I}_{m+s}. \end{cases} \tag{1}$$

In this model, $x_i(t) \in \mathbb{R}$, $v_i(t) \in \mathbb{R}$, $a_i(t) \in \mathbb{R}$, and $u_i(t) \in \mathbb{R}$ represent the position-like, velocity-like, acceleration-like, and control input, respectively, of agent i at $t \in \tau$. The state updates among agents are realised through two kinds of neighbour-based control protocols, as described below:

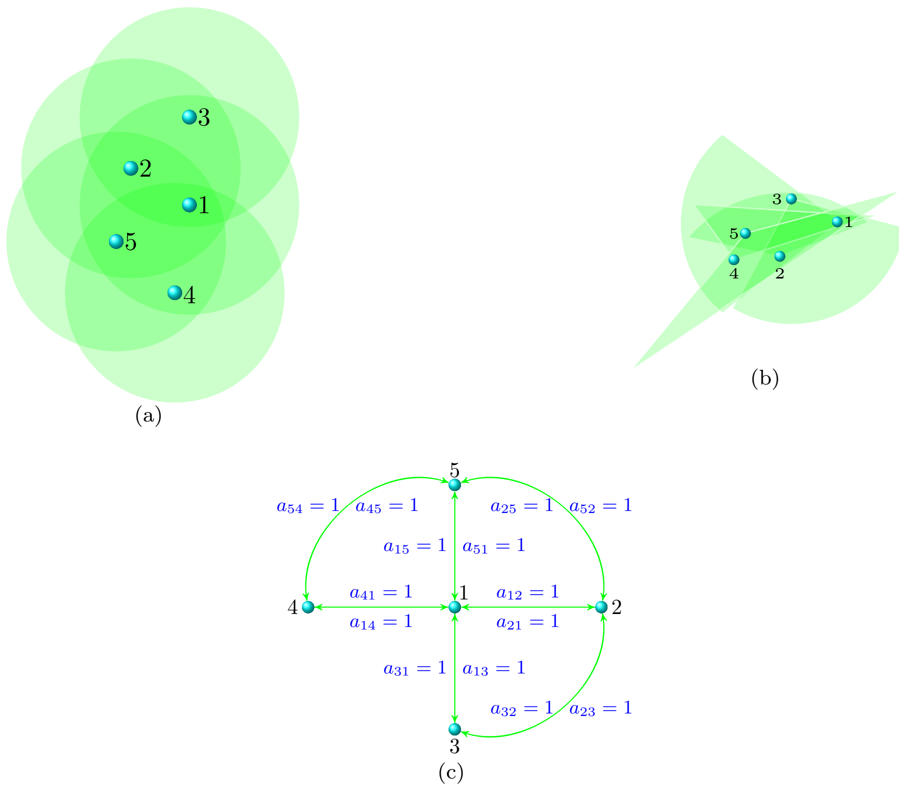


Figure 1. A typical multi-agent network $\mathfrak{N}$: (a) The communication region is circular, resembling visual communication. (b) The communication region forms a $\vee$ shape, resembling verbal communication. (c) The corresponding interconnection graph $\mathcal{G}_{\mathfrak{N}}$

$$u_i(t) = \begin{cases} \dfrac{1}{|\mathcal{N}_i|} \displaystyle\sum_{j \in \mathcal{N}_i} \big(x_j(t) - x_i(t)\big) \text{ for } i \in \mathcal{I}_m, \\ \dfrac{1}{|\mathcal{N}_i|} \displaystyle\sum_{j \in \mathcal{N}_i} \big(x_j(t) - x_i(t)\big) + \dfrac{1}{|\mathcal{N}_i|} \displaystyle\sum_{j \in \mathcal{N}_i} \big(v_j(t) - v_i(t)\big) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } i \in \mathcal{I}_{m+s} \setminus \mathcal{I}_m, \\ \dfrac{1}{|\mathcal{N}_i|} \displaystyle\sum_{j \in \mathcal{N}_i} \big(x_j(t) - x_i(t)\big) + \dfrac{1}{|\mathcal{N}_i|} \displaystyle\sum_{j \in \mathcal{N}_i} \big(v_j(t) - v_i(t)\big) \\ \qquad\qquad\qquad + \dfrac{1}{|\mathcal{N}_i|} \displaystyle\sum_{j \in \mathcal{N}_i} \big(a_j(t) - a_i(t)\big) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } i \in \mathcal{I}_N \setminus \mathcal{I}_{m+s}, \end{cases} \tag{2}$$

or

$$u_i(t) = \begin{cases} \frac{1}{|\mathcal{N}_i|} \sum_{j \in \mathcal{N}_i} \big(x_j(t) - x_i(t)\big) \text{ for } i \in \mathcal{I}_m, \\ \frac{1}{|\mathcal{N}_i|} \sum_{j \in \mathcal{N}_i} \big(x_j(t) - x_i(t)\big) + v_i(t) \text{ for } i \in \mathcal{I}_{m+s} \setminus \mathcal{I}_m, \\ \frac{1}{|\mathcal{N}_i|} \sum_{j \in \mathcal{N}_i} \big(x_j(t) - x_i(t)\big) + v_i(t) + a_i(t) \text{ for } i \in \mathcal{I}_N \setminus \mathcal{I}_{m+s}. \end{cases} \tag{3}$$

Note that both of these protocols are linear and distributed. In the control protocol (2), the neighbour-based law for second-order agents involves averaging feedback from relative velocities, while the neighbour-based law for third-order agents incorporates the average feedback from relative velocities and relative accelerations. In control protocol (3), the neighbour-based law for second-order agents uses feedback from absolute velocities, and for third-order agents, it utilises feedback from absolute velocities and absolute accelerations.

It is worth noting that similar protocols have been employed in Muni et al. (2022) to address the controllability problem of consensus in networks under linear homogeneous dynamics, involving only first-order integrator agents. Interestingly, under both control protocols, (2) and (3), the agents within $\mathfrak{N}$ achieve asymptotic consensus in their positions, velocities, and accelerations (as $t_{\mathfrak{f}} \to +\infty$).

We can present the compact forms of the dynamical system (1) under both control protocols, (2) and (3). For this purpose, we introduce the following notations: Denoting the stack vector of the states of all first-order agents as

$\boldsymbol{x}^{(1)}(t) := \begin{bmatrix} x_1(t) & \dots & x_m(t) \end{bmatrix}^\intercal \in \mathbb{R}^m$,

the stack vector of the states of all second-order agents as

$\boldsymbol{x}^{(2)}(t) := \begin{bmatrix} x_{m+1}(t) & \dots & x_{m+s}(t) \end{bmatrix}^\intercal \in \mathbb{R}^s$

&

$\boldsymbol{v}^{(2)}(t) := \begin{bmatrix} v_{m+1}(t) & \dots & v_{m+s}(t) \end{bmatrix}^\intercal \in \mathbb{R}^s$,

and the stack vector of the states of all third-order agents as

$\boldsymbol{x}^{(3)}(t) := \begin{bmatrix} x_{m+s+1}(t) & \dots & x_N(t) \end{bmatrix}^\intercal \in \mathbb{R}^{N-m-s}$,

$\boldsymbol{v}^{(3)}(t) := \begin{bmatrix} v_{m+s+1}(t) & \dots & v_N(t) \end{bmatrix}^\intercal \in \mathbb{R}^{N-m-s}$,

&

$\boldsymbol{a}^{(3)}(t) := \begin{bmatrix} a_{m+s+1}(t) & \dots & a_N(t) \end{bmatrix}^\intercal \in \mathbb{R}^{N-m-s}$.

With these, the aggregated stack vector of all states of N agents is

$$\begin{aligned}&\boldsymbol{x}(t)\\ &:= \left[\boldsymbol{x}^{(1)^\intercal}(t) \quad \boldsymbol{x}^{(2)^\intercal}(t) \quad \boldsymbol{v}^{(2)^\intercal}(t) \quad \boldsymbol{x}^{(3)^\intercal}(t) \quad \boldsymbol{v}^{(3)^\intercal}(t) \quad \boldsymbol{a}^{(3)^\intercal}(t)\right]^\intercal \in \mathbb{R}^{3N-2m-s}.\end{aligned}$$

Moreover, the random-walk normalised Laplacian matrix of $\mathcal{G}_{\mathfrak{N}}$ can be partitioned as follows:

$$\mathcal{L}^{rw}(\mathcal{G}_{\mathfrak{N}}) = \left[\begin{array}{c|c|c} \mathbf{P}_{11} & \mathbf{P}_{12} & \mathbf{P}_{13} \\ \hline \mathbf{P}_{21} & \mathbf{P}_{22} & \mathbf{P}_{23} \\ \hline \mathbf{P}_{31} & \mathbf{P}_{32} & \mathbf{P}_{33} \end{array}\right]. \tag{4}$$

It is important to note that this is an $N \times N$ matrix comprising only real entries, and it could be symmetric or asymmetric. Each $\mathbf{P}_{ij} \in \mathbb{R}^{p_i \times q_j}$ represents the submatrix of $\mathcal{L}^{rw}(\mathcal{G}_{\mathfrak{N}})$ corresponding to the rows of i^{th} order agents and the columns of j^{th} order agents, where $i = 1, 2, 3,\ j = 1, 2, 3,\ p_1 = q_1 = m$, $p_2 = q_2 = s$, and $p_3 = q_3 = N - m - s$. (If there are no j'^{th} order agents for some $j' = 1, 2$, or 3, then $\mathbf{P}_{ij'}$ and $\mathbf{P}_{j'i}$ do not exist for all $i = 1, 2, 3$.)

With all these considerations, we can represent the compact form of the system (1) under protocol (2) as follows:

$$\dot{\boldsymbol{x}}(t) = \begin{array}{c} \\ \boldsymbol{x}^{(1)} \\ \boldsymbol{x}^{(2)} \\ \boldsymbol{v}^{(2)} \\ \boldsymbol{x}^{(3)} \\ \boldsymbol{v}^{(3)} \\ \boldsymbol{a}^{(3)} \end{array} \!\!\begin{array}{c} \begin{array}{cccccc} \boldsymbol{x}^{(1)^\intercal} & \boldsymbol{x}^{(2)^\intercal} & \boldsymbol{v}^{(2)^\intercal} & \boldsymbol{x}^{(3)^\intercal} & \boldsymbol{v}^{(3)^\intercal} & \boldsymbol{a}^{(3)^\intercal} \end{array} \\ \begin{bmatrix} -\mathbf{P}_{11} & -\mathbf{P}_{12} & \mathbf{O} & -\mathbf{P}_{13} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{I}_s & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ -\mathbf{P}_{21} & -\mathbf{P}_{22} & -\mathbf{P}_{22} & -\mathbf{P}_{23} & -\mathbf{P}_{23} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I}_{N-m-s} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I}_{N-m-s} \\ -\mathbf{P}_{31} & -\mathbf{P}_{32} & -\mathbf{P}_{32} & -\mathbf{P}_{33} & -\mathbf{P}_{33} & -\mathbf{P}_{33} \end{bmatrix} \end{array} \boldsymbol{x}(t),\ t \in \tau. \tag{5}$$

Similarly, under protocol (3), we have:

$$\dot{\boldsymbol{x}}(t) = \begin{array}{c} \\ \boldsymbol{x}^{(1)} \\ \boldsymbol{x}^{(2)} \\ \boldsymbol{v}^{(2)} \\ \boldsymbol{x}^{(3)} \\ \boldsymbol{v}^{(3)} \\ \boldsymbol{a}^{(3)} \end{array} \!\!\begin{array}{c} \begin{array}{cccccc} \boldsymbol{x}^{(1)^\intercal} & \boldsymbol{x}^{(2)^\intercal} & \boldsymbol{v}^{(2)^\intercal} & \boldsymbol{x}^{(3)^\intercal} & \boldsymbol{v}^{(3)^\intercal} & \boldsymbol{a}^{(3)^\intercal} \end{array} \\ \begin{bmatrix} -\mathbf{P}_{11} & -\mathbf{P}_{12} & \mathbf{O} & -\mathbf{P}_{13} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{I}_s & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ -\mathbf{P}_{21} & -\mathbf{P}_{22} & \mathbf{I}_s & -\mathbf{P}_{23} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I}_{N-m-s} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I}_{N-m-s} \\ -\mathbf{P}_{31} & -\mathbf{P}_{32} & \mathbf{O} & -\mathbf{P}_{33} & \mathbf{I}_{N-m-s} & \mathbf{I}_{N-m-s} \end{bmatrix} \end{array} \boldsymbol{x}(t),\ t \in \tau. \tag{6}$$

It can be shown that, under both of the above dynamical rules, the positions, velocities, and accelerations of all agents in $\mathfrak{N}$ will achieve asymptotic consensus. That is, they will satisfy the following conditions:

$$\lim_{t\to+\infty} |x_j(t) - x_i(t)| = 0 \quad \text{for all } i,\, j \in \mathcal{I}_N,$$
$$\lim_{t\to+\infty} |v_j(t) - v_i(t)| = 0 \quad \text{for all } i,\, j \in \mathcal{I}_N \setminus \mathcal{I}_m,$$

and

$$\lim_{t\to+\infty} |a_j(t) - a_i(t)| = 0 \quad \text{for all } i,\, j \in \mathcal{I}_N \setminus \mathcal{I}_{m+s}.$$

Example 3.1 *Consider the network* $\mathfrak{N}$ *depicted in Fig. 1. Let agents* 1 *and* 2 *be the first-order integrators, agent* 3 *be the second-order integrator, while agents* 4 *and* 5 *are third-order integrators. This results in*

$$\mathcal{L}^{rw}(\mathcal{G}_{\mathfrak{N}}) = \left[\begin{array}{cc|c|cc} 1 & -1/4 & -1/4 & -1/4 & -1/4 \\ -1/3 & 1 & -1/3 & 0 & -1/3 \\ \hline -1/2 & -1/2 & 1 & 0 & 0 \\ \hline -1/2 & 0 & 0 & 1 & -1/2 \\ -1/3 & -1/3 & 0 & -1/3 & 1 \end{array}\right],$$

so that the compact forms of dynamical system (1) *under protocols* (2) *and* (3) *here becomes*

$$\dot{\boldsymbol{x}}(t) =$$

$$\begin{array}{rc}
 & \begin{array}{cccccc} \boldsymbol{x}^{(1)\mathsf{T}} & \boldsymbol{x}^{(2)\mathsf{T}} & \boldsymbol{v}^{(2)\mathsf{T}} & \boldsymbol{x}^{(3)\mathsf{T}} & \boldsymbol{v}^{(3)\mathsf{T}} & \boldsymbol{a}^{(3)\mathsf{T}} \end{array} \\
\begin{array}{r} \boldsymbol{x}^{(1)} \\ \\ \boldsymbol{x}^{(2)} \\ \boldsymbol{v}^{(2)} \\ \boldsymbol{x}^{(3)} \\ \boldsymbol{v}^{(3)} \\ \boldsymbol{a}^{(3)} \\ \\ \end{array} &
\left[\begin{array}{cc|c|c|cc|cc|cc}
-1 & 1/4 & 1/4 & \mathbf{O} & 1/4 & 1/4 & \multicolumn{2}{c|}{\mathbf{O}} & \multicolumn{2}{c}{\mathbf{O}} \\
1/3 & -1 & 1/3 & & 0 & 1/3 & \multicolumn{2}{c|}{} & \multicolumn{2}{c}{} \\ \hline
\multicolumn{2}{c|}{\mathbf{O}} & \mathbf{O} & \mathbf{I}_1 & \multicolumn{2}{c|}{\mathbf{O}} & \multicolumn{2}{c|}{\mathbf{O}} & \multicolumn{2}{c}{\mathbf{O}} \\ \hline
1/2 & 1/2 & -1 & -1 & \multicolumn{2}{c|}{\mathbf{O}} & \multicolumn{2}{c|}{\mathbf{O}} & \multicolumn{2}{c}{\mathbf{O}} \\ \hline
\multicolumn{2}{c|}{\mathbf{O}} & \mathbf{O} & \mathbf{O} & \multicolumn{2}{c|}{\mathbf{O}} & \multicolumn{2}{c|}{\mathbf{I}_2} & \multicolumn{2}{c}{\mathbf{O}} \\ \hline
\multicolumn{2}{c|}{\mathbf{O}} & \mathbf{O} & \mathbf{O} & \multicolumn{2}{c|}{\mathbf{O}} & \multicolumn{2}{c|}{\mathbf{O}} & \multicolumn{2}{c}{\mathbf{I}_2} \\ \hline
1/2 & 0 & \mathbf{O} & \mathbf{O} & -1 & 1/2 & -1 & 1/2 & -1 & 1/2 \\
1/3 & 1/3 & & & 1/3 & -1 & 1/3 & -1 & 1/3 & -1
\end{array}\right]
\end{array}$$

$$\boldsymbol{x}(t),\ t \in \tau \tag{7}$$

and

$$\dot{\boldsymbol{x}}(t) = \begin{array}{cc}
 & \begin{array}{cccccc} \boldsymbol{x}^{(1)\top} & \boldsymbol{x}^{(2)\top} & \boldsymbol{v}^{(2)\top} & \boldsymbol{x}^{(3)\top} & \boldsymbol{v}^{(3)\top} & \boldsymbol{a}^{(3)\top} \end{array} \\
\begin{array}{c} \boldsymbol{x}^{(1)} \\ \\ \boldsymbol{x}^{(2)} \\ \boldsymbol{v}^{(2)} \\ \boldsymbol{x}^{(3)} \\ \boldsymbol{v}^{(3)} \\ \boldsymbol{a}^{(3)} \\ \\ \end{array} & \left[\begin{array}{cc|c|c|cc|c|c}
-1 & 1/4 & 1/4 & \mathbf{O} & 1/4 & 1/4 & \mathbf{O} & \mathbf{O} \\
1/3 & -1 & 1/3 & & 0 & 1/3 & & \\ \hline
\multicolumn{2}{c|}{\mathbf{O}} & \mathbf{O} & \mathbf{I}_1 & \multicolumn{2}{c|}{\mathbf{O}} & \mathbf{O} & \mathbf{O} \\ \hline
1/2 & 1/2 & -1 & \mathbf{I}_1 & \multicolumn{2}{c|}{\mathbf{O}} & \mathbf{O} & \mathbf{O} \\ \hline
\multicolumn{2}{c|}{\mathbf{O}} & \mathbf{O} & \mathbf{O} & \multicolumn{2}{c|}{\mathbf{O}} & \mathbf{I}_2 & \mathbf{O} \\ \hline
\multicolumn{2}{c|}{\mathbf{O}} & \mathbf{O} & \mathbf{O} & \multicolumn{2}{c|}{\mathbf{O}} & \mathbf{O} & \mathbf{I}_2 \\ \hline
1/2 & 0 & \mathbf{O} & \mathbf{O} & -1 & 1/2 & \mathbf{I}_2 & \mathbf{I}_2 \\
1/3 & 1/3 & & & 1/3 & -1 & &
\end{array}\right]
\end{array} \boldsymbol{x}(t),$$

$$t \in \tau, \tag{8}$$

respectively. The corresponding inference diagrams are displayed in Fig. 2.

Now, we classify the agents in $\mathfrak{N}$ into leaders and followers. Let l $(1 \leqq l < N)$ be the number of leaders and remaining $N - l$ be the number of followers. Specifically, we decompose l as $l_1 + l_2 + l_3$, where l_1 is the number of first-order integrator leaders $(0 \leqq l_1 \leqq m)$, l_2 is the number of second-order integrator leaders $(0 \leqq l_2 \leqq s)$, and l_3 is the number of third-order integrator leaders $(0 \leqq l_3 \leqq N - m - s)$.

To simplify the analysis, we rename the positions of l_1 leaders (if $l_1 \neq 0$) as $z_1(t), \ldots, z_{l_1}(t)$ in the increasing order of the original first-order agents' labels, and their stack vector is defined as $\mathbf{z}^{(1)}(t) := \begin{bmatrix} z_1(t) & \ldots & z_{l_1}(t) \end{bmatrix}^\top \in \mathbb{R}^{l_1}$. Similarly, for the $(m - l_1)$ first-order followers (if $m - l_1 \neq 0$), we rename their positions as $y_1(t), \ldots, y_{m-l_1}(t)$ in the increasing order of the original first-order agents' labels and create their stack vector as $\mathbf{y}^{(1)}(t) := \begin{bmatrix} y_1(t) & \ldots & y_{m-l_1}(t) \end{bmatrix}^\top \in \mathbb{R}^{m-l_1}$. We follow a similar relabelling process for second-order integrator leaders, where we rename their positions and velocities

$$\text{(if } l_2 \neq 0\text{) as } z_{m+1}(t), \ldots, z_{m+l_2}(t) \text{ and } \beta_{m+1}(t), \ldots, \beta_{m+l_2}(t),$$

respectively.

The corresponding stack vectors are

$$\mathbf{z}^{(2)}(t) := \begin{bmatrix} z_{m+1}(t) & \ldots & z_{m+l_2}(t) \end{bmatrix}^\top \in \mathbb{R}^{l_2}$$

and

$$\mathbf{w}^{(2)}(t) := \begin{bmatrix} \beta_{m+1}(t) & \ldots & \beta_{m+l_2}(t) \end{bmatrix}^\top \in \mathbb{R}^{l_2},$$

respectively.

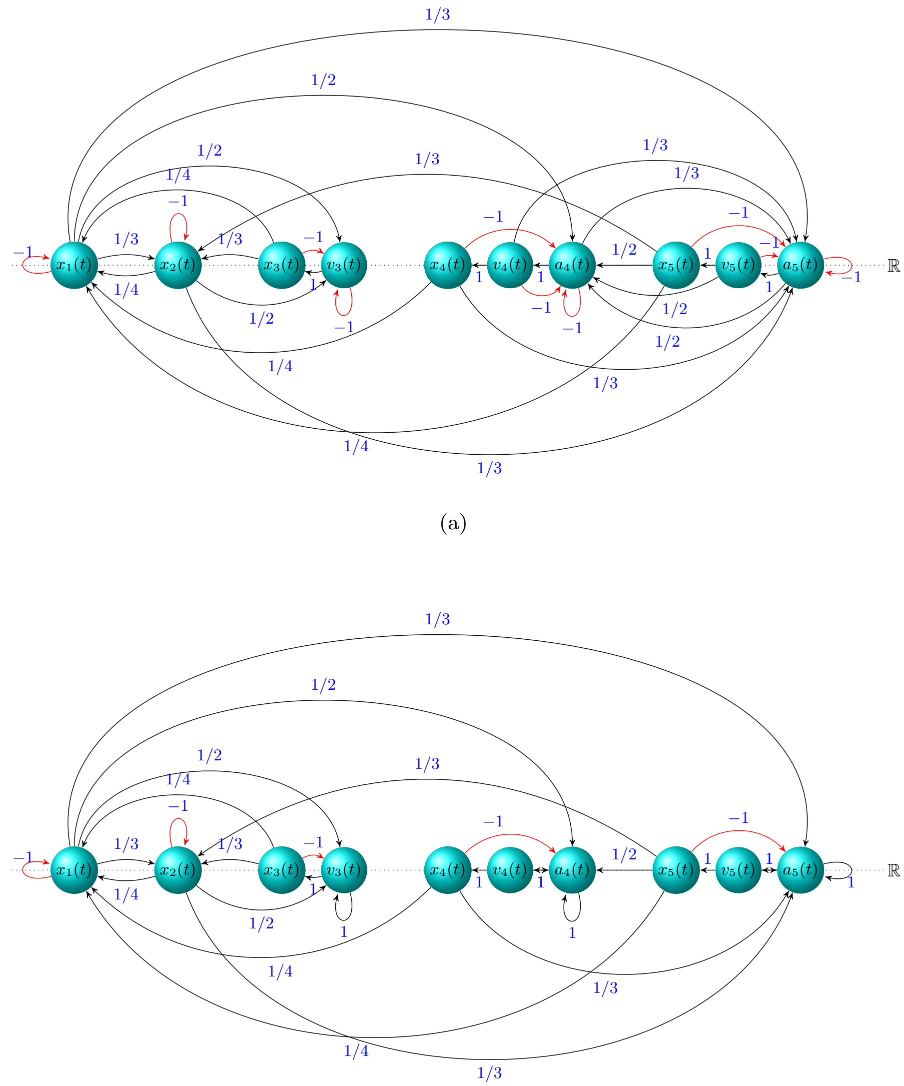


Figure 2. Inference diagrams for systems (3.1) and (3.1). Directed arrows connecting two state components, accompanied by the associated factor 'a', indicate the possibility of gathering information about one state by monitoring another. The concept of creating inference diagrams is further elaborated in Liu and Barabási (2016)

For the $(s-l_2)$ second-order followers (if $s-l_2 \neq 0$), we rename their positions and velocities as

$$y_{m+1}(t), \ldots, y_{m+s-l_2}(t) \text{ and } \alpha_{m+1}(t), \ldots, \alpha_{m+s-l_2}(t),$$

respectively, with corresponding stack vectors

$$\mathbf{y}^{(2)}(t) := \begin{bmatrix} y_{m+1}(t) & \ldots & y_{m+s-l_2}(t) \end{bmatrix}^{\mathsf{T}} \in \mathbb{R}^{s-l_2}$$

and

$$\mathbf{v}^{(2)}(t) := \begin{bmatrix} \alpha_{m+1}(t) & \ldots & \alpha_{m+s-l_2}(t) \end{bmatrix}^{\mathsf{T}} \in \mathbb{R}^{s-l_2},$$

respectively.

Finally, we relabel the positions, velocities, and accelerations of l_3 leaders

$$(\text{if } l_3 \neq 0) \text{ as } z_{m+s+1}(t), \ldots, z_{m+s+l_3}(t),\ \beta_{m+s+1}(t), \ldots, \beta_{m+s+l_3}(t),$$

and

$$\delta_{m+s+1}(t), \ldots, \delta_{m+s+l_3}(t),$$

respectively. Their stack vectors are defined as

$$\mathbf{z}^{(3)}(t) := [z_{m+s+1}(t) \ldots z_{m+s+l_3}](t)^{\mathsf{T}} \in \mathbb{R}^{l_3},$$

$$\mathbf{w}^{(3)}(t) := [\beta_{m+s+1}(t) \ldots \beta_{m+s+l_3}](t)^{\mathsf{T}} \in \mathbb{R}^{l_3},$$

and

$$\mathbf{b}^{(3)}(t) := [\delta_{m+s+1}(t) \ \ldots \delta_{m+s+l_3}](t)^{\mathsf{T}} \in \mathbb{R}^{l_3},$$

respectively.

For the $(N-m-s-l_3)$ third-order followers (if $N-m-s-l_3 \neq 0$), we relabel their positions, velocities, accelerations as

$$y_{m+s+1}(t), \ldots, y_{N-l_3}(t),\ \alpha_{m+s+1}(t), \ldots, \alpha_{N-l_3}(t), \text{ and } \gamma_{m+s+1}(t), \ldots \gamma_{N-l_3}(t),$$

respectively, and their stack vectors are

$$\mathbf{y}^{(3)}(t) := \begin{bmatrix} y_{m+s+1}(t) & \ldots & y_{N-l_3}(t) \end{bmatrix}^{\mathsf{T}} \in \mathbb{R}^{N-m-s-l_3},$$

$$\mathbf{v}^{(3)}(t) := \begin{bmatrix} \alpha_{m+s+1}(t) & \ldots & \alpha_{N-l_3}(t) \end{bmatrix}^{\mathsf{T}} \in \mathbb{R}^{N-m-s-l_3},$$

and

$$\mathbf{a}^{(3)}(t) := \begin{bmatrix} \gamma_{m+s+1}(t) & \ldots & \gamma_{N-l_3}(t) \end{bmatrix}^{\mathsf{T}} \in \mathbb{R}^{N-m-s-l_3}, \qquad \text{respectively.}$$

This systematic relabelling allows us to rephrase the compact forms (5) and (6) in leader–follower frameworks. These are respectively written as this is shown in formulae (9) and (10), which are shown, due to their specificity, on the following separate pages.

Here, $\mathbf{F}_{ij} \in \mathbb{R}^{(p_i - l_i)\times(q_j - l_j)}$ represents the submatrix of $\mathbf{P}_{ij}$, obtained by omitting the rows corresponding to leaders agreeing with the i^{th} order dynamics and columns corresponding to leaders, agreeing with the j^{th} order dynamics. Similarly, $\mathbf{D}_{ij} \in \mathbb{R}^{(p_i - l_i)\times(l_j)}$ is the submatrix of $\mathbf{P}_{ij}$, obtained by excluding the rows corresponding to leaders agreeing with the i^{th} order dynamics and columns corresponding to followers agreeing with the j^{th} order dynamics. Here, i and j take values from 1 to 3, and we have $p_1 = q_1 = m$, $p_2 = q_2 = s$, and $p_3 = q_3 = N - m - s$.

Furthermore, $\boldsymbol{f}^{(i)}$ (and also $\mathfrak{f}^{(i)}$) $\in \mathbb{R}^{l_i}$ is a vector, whose components represent right-hand sides of (1) corresponding to the leaders' positions agreeing on the ith order dynamics. Similarly, $\boldsymbol{g}^{(i)}$ (also $\mathfrak{g}^{(i)}$) $\in \mathbb{R}^{l_i}$ is a vector whose components represent the right-hand sides of (1) corresponding to the leaders' velocities agreeing on the ith order dynamics, and $\boldsymbol{h}^{(i)}$ (also $\mathfrak{h}^{(i)}$) $\in \mathbb{R}^{l_i}$ is a vector whose components represent the right-hand sides of (1) corresponding to the leaders' accelerations agreeing on the ith order dynamics. Specifically, $\boldsymbol{f}^{(i)}$, $\boldsymbol{g}^{(i)}$, and $\boldsymbol{h}^{(i)}$ are obtained from (2), while $\mathfrak{f}^{(i)}$, $\mathfrak{g}^{(i)}$, and $\mathfrak{h}^{(i)}$ are obtained from (3).

We now assume that the states of the leaders (positions, velocities, and accelerations) are not influenced by $\boldsymbol{f}^{(i)}$, $\boldsymbol{g}^{(i)}$, and $\boldsymbol{h}^{(i)}$ (or $\mathfrak{f}^{(i)}$, $\mathfrak{g}^{(i)}$, and $\mathfrak{h}^{(i)}$), allowing us to independently regulate their states. In other words, leaders act as control inputs to the followers' dynamics. Specifically, we manage the aggregated leader function in a manner that places it within $\mathfrak{L}^2(\tau : \mathbb{R}^q)$, where $q := l_1 + 2l_2 + 3l_3$ represents the total state components of l leaders. The selection of this permissible space is motivated by the fact that as long as the aggregated leader function belongs to $\mathfrak{L}^2(\tau : \mathbb{R}^q)$, its energy remains finite, ensuring that the effort required to control the followers' states is bounded and manageable.

$$
\left[\begin{array}{c}
\dot{\mathbf{y}}^{(1)}\\ \dot{\mathbf{y}}^{(2)}\\ \dot{\mathbf{v}}^{(2)}\\ \dot{\mathbf{y}}^{(3)}\\ \dot{\mathbf{v}}^{(3)}\\ \dot{\mathbf{a}}^{(3)}\\ \hline
\dot{\mathbf{z}}^{(1)}\\ \dot{\mathbf{z}}^{(2)}\\ \dot{\mathbf{w}}^{(2)}\\ \dot{\mathbf{z}}^{(3)}\\ \dot{\mathbf{w}}^{(3)}\\ \dot{\mathbf{b}}^{(3)}
\end{array}\right] =
$$

$$
= \left[\begin{array}{cccccc|cccccc}
-\mathbf{F}_{11} & -\mathbf{F}_{12} & \mathbf{O} & -\mathbf{F}_{13} & \mathbf{O} & \mathbf{O} & -\mathbf{D}_{11} & -\mathbf{D}_{12} & \mathbf{O} & -\mathbf{D}_{13} & \mathbf{O} & \mathbf{O}\\
\mathbf{O} & \mathbf{O} & \mathbf{I}_{s-l_2} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O}\\
-\mathbf{F}_{21} & -\mathbf{F}_{22} & -\mathbf{F}_{22} & -\mathbf{F}_{23} & -\mathbf{F}_{23} & \mathbf{O} & -\mathbf{D}_{21} & -\mathbf{D}_{22} & -\mathbf{D}_{22} & -\mathbf{D}_{23} & -\mathbf{D}_{23} & \mathbf{O}\\
\mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I}_{N-m-s-l_3} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O}\\
\mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I}_{N-m-s-l_3} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O}\\
-\mathbf{F}_{31} & -\mathbf{F}_{32} & -\mathbf{F}_{32} & -\mathbf{F}_{33} & -\mathbf{F}_{33} & -\mathbf{F}_{33} & -\mathbf{D}_{31} & -\mathbf{D}_{32} & -\mathbf{D}_{32} & -\mathbf{D}_{33} & -\mathbf{D}_{33} & -\mathbf{D}_{33}\\ \hline
 & & \mathbf{O} & & & & & & \mathbf{O} & & &
\end{array}\right]
\left[\begin{array}{c}
\mathbf{y}^{(1)}\\ \mathbf{y}^{(2)}\\ \mathbf{v}^{(2)}\\ \mathbf{y}^{(3)}\\ \mathbf{v}^{(3)}\\ \mathbf{a}^{(3)}\\ \hline
\mathbf{z}^{(1)}\\ \mathbf{z}^{(2)}\\ \mathbf{w}^{(2)}\\ \mathbf{z}^{(3)}\\ \mathbf{w}^{(3)}\\ \mathbf{b}^{(3)}
\end{array}\right]
$$

$$
+ \left[\begin{array}{c}
\mathbf{0}\\ \hline
\boldsymbol{f}^{(1)}\\ \boldsymbol{f}^{(2)}\\ \boldsymbol{g}^{(2)}\\ \boldsymbol{f}^{(3)}\\ \boldsymbol{g}^{(3)}\\ \boldsymbol{h}^{(3)}
\end{array}\right] \tag{9}
$$

and

$$
\left[\begin{array}{c}
\dot{\mathbf{y}}^{(1)} \\ \dot{\mathbf{y}}^{(2)} \\ \dot{\mathbf{v}}^{(2)} \\ \dot{\mathbf{y}}^{(3)} \\ \dot{\mathbf{v}}^{(3)} \\ \dot{\mathbf{a}}^{(3)} \\ \hline
\dot{\mathbf{z}}^{(1)} \\ \dot{\mathbf{z}}^{(2)} \\ \dot{\mathbf{w}}^{(2)} \\ \dot{\mathbf{z}}^{(3)} \\ \dot{\mathbf{w}}^{(3)} \\ \dot{\mathbf{b}}^{(3)}
\end{array}\right] =
$$

$$
= \left[\begin{array}{cccccc|cccccc}
-\mathbf{F}_{11} & -\mathbf{F}_{12} & \mathbf{O} & -\mathbf{F}_{13} & \mathbf{O} & \mathbf{O} & -\mathbf{D}_{11} & -\mathbf{D}_{12} & \mathbf{O} & -\mathbf{D}_{13} & \mathbf{O} & \mathbf{O} \\
\mathbf{O} & \mathbf{O} & \mathbf{I}_{s-l_2} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\
-\mathbf{F}_{21} & -\mathbf{F}_{22} & \mathbf{I}_{s-l_2} & -\mathbf{F}_{23} & \mathbf{O} & \mathbf{O} & -\mathbf{D}_{21} & -\mathbf{D}_{22} & \mathbf{O} & -\mathbf{D}_{23} & \mathbf{O} & \mathbf{O} \\
\mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I}_{N-m-s-l_3} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\
\mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I}_{N-m-s-l_3} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\
-\mathbf{F}_{31} & -\mathbf{F}_{32} & \mathbf{O} & -\mathbf{F}_{33} & \mathbf{I}_{N-m-s-l_3} & \mathbf{I}_{N-m-s-l_3} & -\mathbf{D}_{31} & -\mathbf{D}_{32} & \mathbf{O} & -\mathbf{D}_{33} & \mathbf{O} & \mathbf{O} \\ \hline
 & & \mathbf{O} & & & & & & \mathbf{O} & & &
\end{array}\right]
\left[\begin{array}{c}
\mathbf{y}^{(1)} \\ \mathbf{y}^{(2)} \\ \mathbf{v}^{(2)} \\ \mathbf{y}^{(3)} \\ \mathbf{v}^{(3)} \\ \mathbf{a}^{(3)} \\ \hline
\mathbf{z}^{(1)} \\ \mathbf{z}^{(2)} \\ \mathbf{w}^{(2)} \\ \mathbf{z}^{(3)} \\ \mathbf{w}^{(3)} \\ \mathbf{b}^{(3)}
\end{array}\right]
$$

$$
+ \left[\begin{array}{c}
\mathbf{o} \\ \hline
\mathfrak{f}^{(1)} \\ \mathfrak{f}^{(2)} \\ \mathfrak{g}^{(2)} \\ \mathfrak{f}^{(3)} \\ \mathfrak{g}^{(3)} \\ \mathfrak{h}^{(3)}
\end{array}\right]. \tag{10}
$$

Denote by $p := (m - l_1) + 2(s - l_2) + 3(N - m - s - l_3)$ the total state components of the $N - l$ followers. From (9) and (10), the dynamics of these p follower states are extracted as follows:

$$\begin{bmatrix} \dot{\mathbf{y}}^{(1)}(t) \\ \dot{\mathbf{y}}^{(2)}(t) \\ \dot{\mathbf{v}}^{(2)}(t) \\ \dot{\mathbf{y}}^{(3)}(t) \\ \dot{\mathbf{v}}^{(3)}(t) \\ \dot{\mathbf{a}}^{(3)}(t) \end{bmatrix} =$$

$$\begin{bmatrix} -\mathbf{F}_{11} & -\mathbf{F}_{12} & \mathbf{O} & -\mathbf{F}_{13} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{I}_{s-l_2} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ -\mathbf{F}_{21} & -\mathbf{F}_{22} & -\mathbf{F}_{22} & -\mathbf{F}_{23} & -\mathbf{F}_{23} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I}_{N-m-s-l_3} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I}_{N-m-s-l_3} \\ -\mathbf{F}_{31} & -\mathbf{F}_{32} & -\mathbf{F}_{32} & -\mathbf{F}_{33} & -\mathbf{F}_{33} & -\mathbf{F}_{33} \end{bmatrix} \begin{bmatrix} \mathbf{y}^{(1)}(t) \\ \mathbf{y}^{(2)}(t) \\ \mathbf{v}^{(2)}(t) \\ \mathbf{y}^{(3)}(t) \\ \mathbf{v}^{(3)}(t) \\ \mathbf{a}^{(3)}(t) \end{bmatrix}$$

$$+ \begin{bmatrix} -\mathbf{D}_{11} & -\mathbf{D}_{12} & \mathbf{O} & -\mathbf{D}_{13} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ -\mathbf{D}_{21} & -\mathbf{D}_{22} & -\mathbf{D}_{22} & -\mathbf{D}_{23} & -\mathbf{D}_{23} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ -\mathbf{D}_{31} & -\mathbf{D}_{32} & -\mathbf{D}_{32} & -\mathbf{D}_{33} & -\mathbf{D}_{33} & -\mathbf{D}_{33} \end{bmatrix} \begin{bmatrix} \mathbf{z}^{(1)}(t) \\ \mathbf{z}^{(2)}(t) \\ \mathbf{w}^{(2)}(t) \\ \mathbf{z}^{(3)}(t) \\ \mathbf{w}^{(3)}(t) \\ \mathbf{b}^{(3)}(t) \end{bmatrix}, \ t \in \tau, \tag{11}$$

and

$$\begin{bmatrix} \dot{\mathbf{y}}^{(1)}(t) \\ \dot{\mathbf{y}}^{(2)}(t) \\ \dot{\mathbf{v}}^{(2)}(t) \\ \dot{\mathbf{y}}^{(3)}(t) \\ \dot{\mathbf{v}}^{(3)}(t) \\ \dot{\mathbf{a}}^{(3)}(t) \end{bmatrix} =$$

$$\begin{bmatrix} -\mathbf{F}_{11} & -\mathbf{F}_{12} & \mathbf{O} & -\mathbf{F}_{13} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{I}_{s-l_2} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ -\mathbf{F}_{21} & -\mathbf{F}_{22} & \mathbf{I}_{s-l_2} & -\mathbf{F}_{23} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I}_{N-m-s-l_3} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I}_{N-m-s-l_3} \\ -\mathbf{F}_{31} & -\mathbf{F}_{32} & \mathbf{O} & -\mathbf{F}_{33} & \mathbf{I}_{N-m-s-l_3} & \mathbf{I}_{N-m-s-l_3} \end{bmatrix} \begin{bmatrix} \mathbf{y}^{(1)}(t) \\ \mathbf{y}^{(2)}(t) \\ \mathbf{v}^{(2)}(t) \\ \mathbf{y}^{(3)}(t) \\ \mathbf{v}^{(3)}(t) \\ \mathbf{a}^{(3)}(t) \end{bmatrix}$$

$$+\begin{bmatrix} -\mathbf{D}_{11} & -\mathbf{D}_{12} & \mathbf{O} & -\mathbf{D}_{13} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ -\mathbf{D}_{21} & -\mathbf{D}_{22} & \mathbf{O} & -\mathbf{D}_{23} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ -\mathbf{D}_{31} & -\mathbf{D}_{32} & \mathbf{O} & -\mathbf{D}_{33} & \mathbf{O} & \mathbf{O} \end{bmatrix} \begin{bmatrix} \mathbf{z}^{(1)}(t) \\ \mathbf{z}^{(2)}(t) \\ \mathbf{w}^{(2)}(t) \\ \mathbf{z}^{(3)}(t) \\ \mathbf{w}^{(3)}(t) \\ \mathbf{b}^{(3)}(t) \end{bmatrix}, \ t \in \tau, \tag{12}$$

respectively.

Definition 1 (Controllability) *Multi-agent network $\mathfrak{N}$ is considered leader–follower controllable in $\mathbb{R}^p$ over τ if effective regulation of all q states of leaders in $\mathbb{R}$, such that the aggregated leader function is contained within $\mathfrak{L}^2(\tau : \mathbb{R}^q)$, enables all p states of followers to transition from any initial values to any desired final values within $\mathbb{R}$ over τ, while ensuring that the aggregated follower function resides within $\mathfrak{L}^2(\tau : \mathbb{R}^p)$ and adheres to the dynamics presented in (11) (or (12)).*

It is evident that the leader–follower controllability of $\mathfrak{N}$ is equivalent to the controllability of the system (11) (or (12)).

4. Controllability analysis

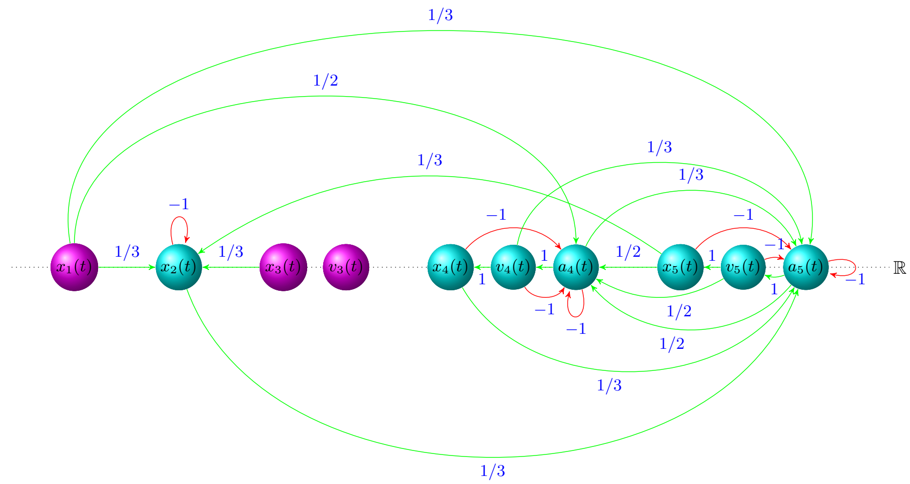
In this section, we analyse the conditions, under which system (11) (or (12)) is controllable. Since the system is linear time-invariant, we employ Popov-Belevitch-Hautus rank test (Terrell, 2009), which states that system (11) is controllable if and only if

$$\operatorname{rank}\begin{bmatrix}\mu\mathbf{I}_p - \mathbf{A} & \mathbf{B}\end{bmatrix} = p \tag{13}$$

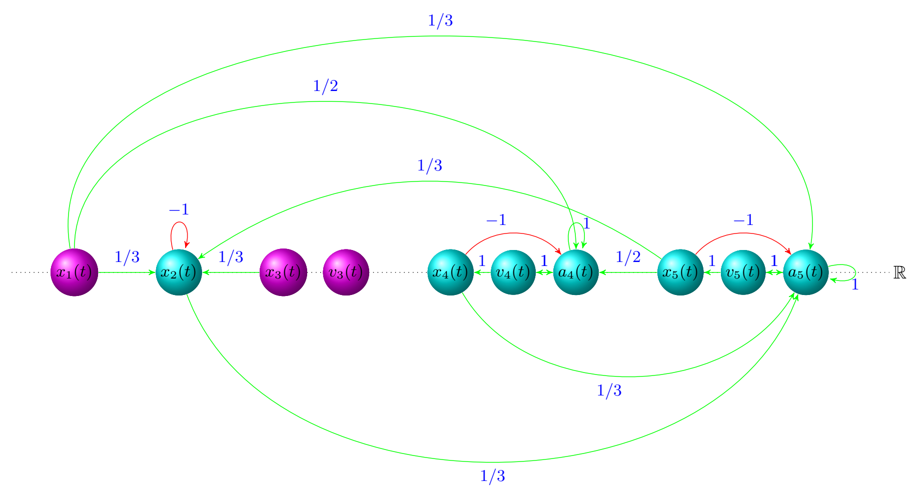
holds good for every $\mu \in \mathbb{C}$, where

$$\mathbf{A} = \begin{bmatrix} -\mathbf{F}_{11} & -\mathbf{F}_{12} & \mathbf{O} & -\mathbf{F}_{13} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{I}_{s-l_2} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ -\mathbf{F}_{21} & -\mathbf{F}_{22} & -\mathbf{F}_{22} & -\mathbf{F}_{23} & -\mathbf{F}_{23} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I}_{N-m-s-l_3} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I}_{N-m-s-l_3} \\ -\mathbf{F}_{31} & -\mathbf{F}_{32} & -\mathbf{F}_{32} & -\mathbf{F}_{33} & -\mathbf{F}_{33} & -\mathbf{F}_{33} \end{bmatrix}_{p\times p} \quad \text{and}$$

$$\mathbf{B} = \begin{bmatrix} -\mathbf{D}_{11} & -\mathbf{D}_{12} & \mathbf{O} & -\mathbf{D}_{13} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ -\mathbf{D}_{21} & -\mathbf{D}_{22} & -\mathbf{D}_{22} & -\mathbf{D}_{23} & -\mathbf{D}_{23} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ -\mathbf{D}_{31} & -\mathbf{D}_{32} & -\mathbf{D}_{32} & -\mathbf{D}_{33} & -\mathbf{D}_{33} & -\mathbf{D}_{33} \end{bmatrix}_{p\times q}.$$



(a)



(b)

Figure 3. In the context of Example 3.1, by assigning the agents 1 and 3 as leaders, the inference diagrams for the corresponding systems (11) and (12) are displayed here. These diagrams offer insights into the fundamental observation that, generally, once we partition the agents in $\mathfrak{N}$ into leaders and followers, the communications from the followers' states to the leaders' states will be lost. Consequently, the states of followers no longer have an influence on the states of leaders. Furthermore, there is no intercommunication among the states of leaders, and the state components of leaders do not self-communicate

The necessary and sufficient condition for the controllability of the system (12) can be written in a similar manner. By performing elementary row and column operations on $\begin{bmatrix}\mu\mathbf{I}_p - \mathbf{A} & \mathbf{B}\end{bmatrix}$, we can transform it into the form

$$\left[\begin{array}{c|c} * & \mathbf{O} \\ \hline \mathbf{O} & \star \end{array}\right],$$

which allows us to determine its rank as the sum of ranks of the diagonal blocks $*$ and $\star$. This transformation is demonstrated in the next two pages.

From this last step, we conclude that (13) holds true if and only if for every $\mu \in \mathbb{C}$, the following condition is satisfied:

$$\begin{aligned}&\text{rank}\begin{bmatrix} \mu\mathbf{I}_{m-l_1}+\mathbf{F}_{11} & \mathbf{F}_{12} & \mathbf{F}_{13} & \mathbf{D}_{11} & \mathbf{D}_{12} & \mathbf{O} & \mathbf{D}_{13} & \mathbf{O} & \mathbf{O} \\ \mathbf{F}_{21} & \mu^2\mathbf{I}_{s-l_2}+(\mu+1)\mathbf{F}_{22} & (\mu+1)\mathbf{F}_{23} & \mathbf{D}_{21} & \mathbf{O} & \mathbf{D}_{22} & \mathbf{O} & \mathbf{D}_{23} & \mathbf{O} \\ \mathbf{F}_{31} & (\mu+1)\mathbf{F}_{32} & \mu^3\mathbf{I}_{N-m-s-l_3}+(\mu^2+\mu+1)\mathbf{F}_{33} & \mathbf{D}_{31} & \mathbf{O} & \mathbf{D}_{32} & \mathbf{O} & \mathbf{O} & \mathbf{D}_{33} \end{bmatrix}\\ &= N-l. \end{aligned} \tag{14}$$

Similarly, the necessary and sufficient condition for the controllability of the system (12) is as follows:

$$\begin{aligned}&\text{rank}\begin{bmatrix} \mu\mathbf{I}_{m-l_1}+\mathbf{F}_{11} & \mathbf{F}_{12} & \mathbf{F}_{13} & \mathbf{D}_{11} & \mathbf{D}_{12} & \mathbf{D}_{13} \\ \mathbf{F}_{21} & (\mu^2-\mu)\mathbf{I}_{s-l_2}+\mathbf{F}_{22} & \mathbf{F}_{23} & \mathbf{D}_{21} & \mathbf{D}_{22} & \mathbf{D}_{23} \\ \mathbf{F}_{31} & \mathbf{F}_{32} & (\mu^3-\mu^2-\mu)\mathbf{I}_{N-m-s-l_3}+\mathbf{F}_{33} & \mathbf{D}_{31} & \mathbf{D}_{32} & \mathbf{D}_{33} \end{bmatrix}\\ &= N-l \ \ \forall\, \mu \in \mathbb{C}. \end{aligned} \tag{15}$$

Theorem 4.1 (Necessary conditions) *Suppose that the system* (11) (*or* (12)) *is controllable. Then the following conditions hold true for every* $\lambda \in \mathbb{C}$*:*

$$\mathit{rank}\begin{bmatrix}\lambda\mathbf{I}_{m-l_1}+\mathbf{F}_{11} & \mathbf{F}_{12} & \mathbf{F}_{13} & \mathbf{D}_{11} & \mathbf{D}_{12} & \mathbf{D}_{13}\end{bmatrix} = m-l_1. \tag{16}$$

$$\mathit{rank}\begin{bmatrix}\mathbf{F}_{21} & \lambda\mathbf{I}_{s-l_2}+\mathbf{F}_{22} & \mathbf{F}_{23} & \mathbf{D}_{21} & \mathbf{D}_{22} & \mathbf{D}_{23}\end{bmatrix} = s-l_2. \tag{17}$$

$$\mathit{rank}\begin{bmatrix}\mathbf{F}_{31} & \mathbf{F}_{32} & \lambda\mathbf{I}_{N-m-s-l_3}+\mathbf{F}_{33} & \mathbf{D}_{31} & \mathbf{D}_{32} & \mathbf{D}_{33}\end{bmatrix} = N-m-s-l_3. \tag{18}$$

$$
\begin{aligned}
&\operatorname{rank}\left[\mu \mathbf{I}_p - \mathbf{A}\,\mathbf{B}\right] = \\
&= \operatorname{rank}\left[\begin{array}{cccccc:cccccc}
\mu\mathbf{I}_{m-l_1}+\mathbf{F}_{11} & \mathbf{F}_{12} & \mathbf{O} & \mathbf{F}_{13} & \mathbf{O} & \mathbf{O} & -\mathbf{D}_{11} & -\mathbf{D}_{12} & \mathbf{O} & -\mathbf{D}_{13} & \mathbf{O} & \mathbf{O} \\
\mathbf{O} & \mu\mathbf{I}_{s-l_2} & -\,\mathbf{I}_{s-l_2} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\
\mathbf{F}_{21} & \mathbf{F}_{22} & \mu\mathbf{I}_{s-l_2}+\mathbf{F}_{22} & \mathbf{F}_{23} & \mathbf{F}_{23} & \mathbf{O} & -\mathbf{D}_{21} & -\mathbf{D}_{22} & -\mathbf{D}_{22} & -\mathbf{D}_{23} & -\mathbf{D}_{23} & \mathbf{O} \\
\mathbf{O} & \mathbf{O} & \mathbf{O} & \mu\mathbf{I}_{N-m-s-l_3} & -\,\mathbf{I}_{N-m-s-l_3} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\
\mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mu\mathbf{I}_{N-m-s-l_3} & -\,\mathbf{I}_{N-m-s-l_3} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\
\mathbf{F}_{31} & \mathbf{F}_{32} & \mathbf{F}_{32} & \mathbf{F}_{33} & \mathbf{F}_{33} & \mu\mathbf{I}_{N-m-s-l_3}+\mathbf{F}_{33} & -\mathbf{D}_{31} & -\mathbf{D}_{32} & -\mathbf{D}_{32} & -\mathbf{D}_{33} & -\mathbf{D}_{33} & -\mathbf{D}_{33}
\end{array}\right] \\
&= \operatorname{rank}\left[\begin{array}{cccccc:cccccc}
\mu\mathbf{I}_{m-l_1}+\mathbf{F}_{11} & \mathbf{F}_{12} & \mathbf{F}_{13} & \mathbf{O} & \mathbf{O} & \mathbf{O} & -\mathbf{D}_{11} & -\mathbf{D}_{12} & \mathbf{O} & -\mathbf{D}_{13} & \mathbf{O} & \mathbf{O} \\
\mathbf{F}_{21} & \mu^2\mathbf{I}_{s-l_2}+(\mu+1)\mathbf{F}_{22} & (\mu+1)\mathbf{F}_{23} & \mathbf{O} & \mathbf{O} & \mathbf{O} & -\mathbf{D}_{21} & \mathbf{O} & -\mathbf{D}_{22} & \mathbf{O} & -\mathbf{D}_{23} & \mathbf{O} \\
\mathbf{F}_{31} & (\mu+1)\mathbf{F}_{32} & \mu^3\mathbf{I}_{N-m-s-l_3}+(\mu^2+\mu+1)\mathbf{F}_{33} & \mathbf{O} & \mathbf{O} & \mathbf{O} & -\mathbf{D}_{31} & \mathbf{O} & -\mathbf{D}_{32} & \mathbf{O} & \mathbf{O} & -\mathbf{D}_{33} \\
\mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I}_{s-l_2} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\
\mathbf{O} & \mathbf{O} & (\mu+1)\mathbf{I}_{N-m-s-l_3} & \mathbf{O} & \mathbf{I}_{N-m-s-l_3} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\
\mathbf{O} & \mathbf{O} & \mu^2\mathbf{I}_{N-m-s-l_3} & \mathbf{O} & \mathbf{O} & \mathbf{I}_{N-m-s-l_3} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O}
\end{array}\right]
\end{aligned}
$$

$$
= \operatorname{rank}\left[\begin{array}{ccccccccc:ccc}
\mu\mathbf{I}_{m-l_1}+\mathbf{F}_{11} & \mathbf{F}_{12} & \mathbf{F}_{13} & -\mathbf{D}_{11} & -\mathbf{D}_{12} & \mathbf{O} & -\mathbf{D}_{13} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O}\\
\mathbf{F}_{21} & \mu^2\mathbf{I}_{s-l_2}+(\mu+1)\mathbf{F}_{22} & (\mu+1)\mathbf{F}_{23} & -\mathbf{D}_{21} & \mathbf{O} & -\mathbf{D}_{22} & \mathbf{O} & -\mathbf{D}_{23} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O}\\
\mathbf{F}_{31} & (\mu+1)\mathbf{F}_{32} & \mu^3\mathbf{I}_{N-m-s-l_3}+(\mu^2+\mu+1)\mathbf{F}_{33} & -\mathbf{D}_{31} & \mathbf{O} & -\mathbf{D}_{32} & \mathbf{O} & \mathbf{O} & -\mathbf{D}_{33} & \mathbf{O} & \mathbf{O} & \mathbf{O}\\
\hdashline
\mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I}_{s-l_2} & \mathbf{O} & \mathbf{O}\\
\mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I}_{N-m-s-l_3} & \mathbf{O}\\
\mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I}_{N-m-s-l_3}
\end{array}\right]
$$

$$
= \operatorname{rank}\left[\begin{array}{ccccccccc}
\mu\mathbf{I}_{m-l_1}+\mathbf{F}_{11} & \mathbf{F}_{12} & \mathbf{F}_{13} & -\mathbf{D}_{11} & -\mathbf{D}_{12} & \mathbf{O} & -\mathbf{D}_{13} & \mathbf{O} & \mathbf{O}\\
\mathbf{F}_{21} & \mu^2\mathbf{I}_{s-l_2}+(\mu+1)\mathbf{F}_{22} & (\mu+1)\mathbf{F}_{23} & -\mathbf{D}_{21} & \mathbf{O} & -\mathbf{D}_{22} & \mathbf{O} & -\mathbf{D}_{23} & \mathbf{O}\\
\mathbf{F}_{31} & (\mu+1)\mathbf{F}_{32} & \mu^3\mathbf{I}_{N-m-s-l_3}+(\mu^2+\mu+1)\mathbf{F}_{33} & -\mathbf{D}_{31} & \mathbf{O} & -\mathbf{D}_{32} & \mathbf{O} & \mathbf{O} & -\mathbf{D}_{33}
\end{array}\right]
$$

$$
+(s-l_2)+2(N-m-s-l_3).
$$

PROOF It is sufficient to demonstrate the validity of the above conditions only for $\lambda \in \sigma(-\mathbf{F}_{ii})$, $i = 1, 2, 3$, under the hypothesis that system (11) (or (12)) is controllable. This is because for all other $\lambda \in \mathbb{C}$, the matrix $\lambda\mathbf{I}+\mathbf{F}_{ii}$, $i = 1, 2, 3$, possesses full rank, so that the three conditions naturally hold true, irrespective of whether the system (11) (or (12)) is controllable or uncontrollable. We will now proceed to prove the necessity of these conditions for the controllability of the system (11) through a contrapositive argument. The proof will be similar to that for the system (12).

(i) To demonstrate the necessity of condition (16). Suppose that for a certain $\lambda \in \sigma(-\mathbf{F}_{11})$, let's say $\lambda = \lambda_1$, we have

$$0 \leqq \operatorname{rank}\begin{bmatrix}\lambda_1\mathbf{I}_{m-l_1} + \mathbf{F}_{11} & \mathbf{F}_{12} & \mathbf{F}_{13} & \mathbf{D}_{11} & \mathbf{D}_{12} & \mathbf{D}_{13}\end{bmatrix} < m - l_1.$$

Then the controllability condition (14) is violated for $\mu = \lambda_1$, implying that the system (11) is uncontrollable.

(ii) To demonstrate the necessity of condition (17), suppose for a certain $\lambda \in \sigma(-\mathbf{F}_{22})$, say $\lambda = \lambda_2$, we have

$$0 \leqq \operatorname{rank}\begin{bmatrix}\mathbf{F}_{21} & \lambda_2\mathbf{I}_{s-l_2} + \mathbf{F}_{22} & \mathbf{F}_{23} & \mathbf{D}_{21} & \mathbf{D}_{22} & \mathbf{D}_{23}\end{bmatrix} < s - l_2.$$

Now, if we set $\mu := \frac{\lambda_2 \pm \sqrt{\lambda_2^2 + 4\lambda_2}}{2}$, then $\mu \in \mathbb{C}$, $\mu \neq -1$, and $\lambda_2 = \frac{\mu^2}{\mu+1}$.

Then the above inequality becomes:

$$0 \leqq \operatorname{rank}\begin{bmatrix}\mathbf{F}_{21} & \frac{\mu^2}{\mu+1}\mathbf{I}_{s-l_2} + \mathbf{F}_{22} & \mathbf{F}_{23} & \mathbf{D}_{21} & \mathbf{D}_{22} & \mathbf{D}_{23}\end{bmatrix} < s - l_2,$$

which, through elementary column operations, reduces to:

$$0 \leqq \operatorname{rank}\begin{bmatrix}\mathbf{F}_{21} & \mu^2\mathbf{I}_{s-l_2} + (\mu+1)\mathbf{F}_{22} & (\mu+1)\mathbf{F}_{23} & \mathbf{D}_{21} & \mathbf{D}_{22} & \mathbf{D}_{23}\end{bmatrix} < s - l_2.$$

This is the violation of the controllability condition (14) for $\mu = \frac{\lambda_2 \pm \sqrt{\lambda_2^2 + 4\lambda_2}}{2}$. Therefore, system (11) is uncontrollable.

(iii) To demonstrate the necessity of condition (18). As usual, suppose for a certain $\lambda \in \sigma(-\mathbf{F}_{33})$, say $\lambda = \lambda_3$, we have:

$$0 \leqq \operatorname{rank}\begin{bmatrix}\mathbf{F}_{31} & \mathbf{F}_{32} & \lambda_3\mathbf{I}_{N-m-s-l_3} + \mathbf{F}_{33} & \mathbf{D}_{31} & \mathbf{D}_{32} & \mathbf{D}_{33}\end{bmatrix} < N - m - s - l_3.$$

Consider the polynomial equation in μ: $\mu^3 - (\lambda_3)\mu^2 - (\lambda_3)\mu - \lambda_3 = 0$. Select one of the zeros of this equation that is not equal to -1, let's call it μ_0. The existence of such μ_0 is always guaranteed for any value of $\lambda_3 \in \sigma(-\mathbf{F}_{33})$. (The solution set of this polynomial equation is $\{-1, i, -i\}$ if and only if $\lambda_3 = -1$, and in this case μ_0 can be either i or $-i$. For $\lambda_3(\neq -1) \in \sigma(-\mathbf{F}_{33})$, we observe that none of the zeros of this polynomial equation is equal to -1, and in this case μ_0 can be any one of the three zeros.) Clearly, $\mu_0 \in \mathbb{C}$, $\mu_0 \neq \omega, \omega^2$ (here, $\omega = \frac{-1+i\sqrt{3}}{2}$ is one of the cube roots of unity), and $\lambda_3 = \frac{\mu_0^3}{\mu_0^2+\mu_0+1}$. With this, the above inequality reduces to:

$$0 \leqq \operatorname{rank}\begin{bmatrix}\mathbf{F}_{31} & \mathbf{F}_{32} & \frac{\mu_0^3}{\mu_0^2+\mu_0+1}\mathbf{I}_{N-m-s-l_3} + \mathbf{F}_{33} & \mathbf{D}_{31} & \mathbf{D}_{32} & \mathbf{D}_{33}\end{bmatrix} < N - m - s - l_3.$$

Now, using elementary column operations, we get:

$$0 \leqq \operatorname{rank}\big[\mathbf{F}_{31}\; (\mu_0+1)\mathbf{F}_{32}\; \mu_0^3\,\mathbf{I}_{N-m-s-l_3} + (\mu_0^2+\mu_0+1)\mathbf{F}_{33}\; \mathbf{D}_{31}\; \mathbf{D}_{32}\; \mathbf{D}_{33}\big] < N - m - s - l_3.$$

This shows the violation of the controllability condition (14) for $\mu = \mu_0 \in \mathbb{C}$. Therefore, the system (11) is uncontrollable.

■

Corollary 4.1 (Necessary conditions) *If system (11) (or (12)) is controllable, the following statements hold true:*

(i) No eigenvector of $\mathbf{F}_{11}^{\intercal}$ *is simultaneously orthogonal to all the columns of the matrix* $\begin{bmatrix}\mathbf{F}_{12} & \mathbf{F}_{13} & \mathbf{D}_{11} & \mathbf{D}_{12} & \mathbf{D}_{13}\end{bmatrix}$.

(ii) No eigenvector of $\mathbf{F}_{22}^{\intercal}$ *is simultaneously orthogonal to all the columns of the matrix* $\begin{bmatrix}\mathbf{F}_{21} & \mathbf{F}_{23} & \mathbf{D}_{21} & \mathbf{D}_{22} & \mathbf{D}_{23}\end{bmatrix}$.

(iii) No eigenvector of $\mathbf{F}_{33}^{\intercal}$ *is simultaneously orthogonal to all the columns of the matrix* $\begin{bmatrix}\mathbf{F}_{31} & \mathbf{F}_{32} & \mathbf{D}_{31} & \mathbf{D}_{32} & \mathbf{D}_{33}\end{bmatrix}$.

Proof We will demonstrate that the above statements are equivalent to the conditions (16)–(18) in Theorem 4.1. The proof of this corollary follows automatically.

To show that condition (16) is equivalent to statement (i):

(Necessity): Suppose that condition (16) holds, i.e.,

$$\operatorname{rank}\begin{bmatrix}\lambda\mathbf{I}_{m-l_1} + \mathbf{F}_{11} & \mathbf{F}_{12} & \mathbf{F}_{13} & \mathbf{D}_{11} & \mathbf{D}_{12} & \mathbf{D}_{13}\end{bmatrix} = m - l_1,$$

for every $\lambda \in \mathbb{C}$. This implies that the linear operator,

$$\begin{bmatrix}\lambda\mathbf{I}_{m-l_1} + \mathbf{F}_{11} & \mathbf{F}_{12} & \mathbf{F}_{13} & \mathbf{D}_{11} & \mathbf{D}_{12} & \mathbf{D}_{13}\end{bmatrix} : \mathbb{C}^N \to \mathbb{C}^{m-l_1}$$

possesses full rank $= m - l_1$ for every $\lambda \in \mathbb{C}$. This is possible if and only if the transpose operator,

$$\begin{bmatrix}\lambda \mathbf{I}_{m-l_1} + \mathbf{F}_{11} & \mathbf{F}_{12} & \mathbf{F}_{13} & \mathbf{D}_{11} & \mathbf{D}_{12} & \mathbf{D}_{13}\end{bmatrix}^{\mathsf{T}} : \mathbb{C}^{m-l_1} \to \mathbb{C}^{N}$$

possesses full rank $= m - l_1$ for every $\lambda \in \mathbb{C}$. This is equivalent to writing that the following homogeneous linear system of equations:

$$\begin{bmatrix}\lambda \mathbf{I}_{m-l_1} + \mathbf{F}_{11}^{\mathsf{T}} \\ \mathbf{F}_{12}^{\mathsf{T}} \\ \mathbf{F}_{13}^{\mathsf{T}} \\ \mathbf{D}_{11}^{\mathsf{T}} \\ \mathbf{D}_{12}^{\mathsf{T}} \\ \mathbf{D}_{13}^{\mathsf{T}}\end{bmatrix} \boldsymbol{\theta} = \mathbf{0} \in \mathbb{C}^{N},$$

possesses only the trivial solution $\boldsymbol{\theta} = \mathbf{0} \in \mathbb{C}^{m-l_1}$ for every $\lambda \in \mathbb{C}$. This means that the only (common) solution to the following infinite set of homogeneous equations:

$$\begin{cases} (\lambda \mathbf{I}_{m-l_1} + \mathbf{F}_{11}^{\mathsf{T}})\boldsymbol{\theta} = \mathbf{0} \in \mathbb{C}^{m-l_1}, \ \lambda \in \mathbb{C}, \\ \mathbf{F}_{12}^{\mathsf{T}}\boldsymbol{\theta} = \mathbf{0} \in \mathbb{C}^{s-l_2}, \\ \mathbf{F}_{13}^{\mathsf{T}}\boldsymbol{\theta} = \mathbf{0} \in \mathbb{C}^{N-(m+s)-l_3}, \\ \mathbf{D}_{11}^{\mathsf{T}}\boldsymbol{\theta} = \mathbf{0} \in \mathbb{C}^{l_1}, \\ \mathbf{D}_{12}^{\mathsf{T}}\boldsymbol{\theta} = \mathbf{0} \in \mathbb{C}^{l_2}, \\ \mathbf{D}_{13}^{\mathsf{T}}\boldsymbol{\theta} = \mathbf{0} \in \mathbb{C}^{l_3} \end{cases} \tag{$\dagger$}$$

is $\boldsymbol{\theta} = \mathbf{0}$. This requirement can be restated in terms of the eigenvectors of $\mathbf{F}_{11}^{\mathsf{T}}$. Observe that for all $\lambda \notin \sigma(-\mathbf{F}_{11})$, the zero vector is the only (common) solution to the infinite set of equations in (†). (For in this case, the operator $(\lambda \mathbf{I}_{m-l_1} + \mathbf{F}_{11}^{\mathsf{T}}) : \mathbb{C}^{m-l_1} \to \mathbb{C}^{m-l_1}$ possesses full rank $= m - l_1$, and hence the only solution to $(\mu \mathbf{I}_{m-l_1} + \mathbf{F}_{11}^{\mathsf{T}})\boldsymbol{\theta} = \mathbf{0}$ is $\boldsymbol{\theta} = \mathbf{0}$; this zero solution obviously satisfies the remaining equations of (†), and, accordingly, the only (common) solution to the infinite set of equations in (†) is zero.) Therefore, we will focus only on the case when $\lambda \in \sigma(-\mathbf{F}_{11})$. In this case, the equation $(\lambda \mathbf{I}_{m-l_1} + \mathbf{F}_{11}^{\mathsf{T}})\boldsymbol{\theta} = \mathbf{0} \in \mathbb{C}^{m-l_1}$ means that $\boldsymbol{\theta} \in \mathbb{E}_{-\lambda}(\mathbf{F}_{11})$.

Now, the only (common) solution to the infinite set of equations in (†) will be zero if and only if:

$$\mathbb{E}_{-\lambda}(\mathbf{F}_{11}^{\mathsf{T}}) \cap \mathcal{N}(\mathbf{F}_{12}^{\mathsf{T}}) \cap \mathcal{N}(\mathbf{F}_{13}^{\mathsf{T}}) \cap \mathcal{N}(\mathbf{D}_{11}^{\mathsf{T}}) \cap \mathcal{N}(\mathbf{D}_{12}^{\mathsf{T}}) \cap \mathcal{N}(\mathbf{D}_{13}^{\mathsf{T}}) = \{\mathbf{0}\} \subsetneqq \mathbb{C}^{m-l_1} \tag{19}$$

for every $\lambda \in \sigma(-\mathbf{F}_{11})$. All the nonzero elements of $\mathbb{E}_{-\lambda}(\mathbf{F}_{11}^{\mathsf{T}})$ are the eigenvectors of $\mathbf{F}_{11}^{\mathsf{T}}$ corresponding to its eigenvalue $-\lambda$. So, condition (19) means that no eigenvector of $\mathbf{F}_{11}^{\intercal}$ belongs to $\mathcal{N}(\mathbf{F}_{12}^{\intercal})$, $\mathcal{N}(\mathbf{F}_{13}^{\intercal})$, $\mathcal{N}(\mathbf{D}_{11}^{\intercal})$, $\mathcal{N}(\mathbf{D}_{12}^{\intercal})$, and $\mathcal{N}(\mathbf{D}_{13}^{\intercal})$ simultaneously. In other words, no eigenvector of $\mathbf{F}_{11}^{\intercal}$ is simultaneously orthogonal to all the columns in the matrix $\begin{bmatrix}\mathbf{F}_{12} & \mathbf{F}_{13} & \mathbf{D}_{11} & \mathbf{D}_{12} & \mathbf{D}_{13}\end{bmatrix}$.

(Sufficiency) By contrapositive. Suppose there exists a $\lambda \in \mathbb{C}$, say $\lambda = \lambda'$ such that

$$0 \leqq \operatorname{rank}\begin{bmatrix}\lambda'\mathbf{I}_{m-l_1} + \mathbf{F}_{11} & \mathbf{F}_{12} & \mathbf{F}_{13} & \mathbf{D}_{11} & \mathbf{D}_{12} & \mathbf{D}_{13}\end{bmatrix} < m - l_1.$$

Because a linear operator and its transpose possess the same rank, the operator:

$$\begin{bmatrix}\lambda'\mathbf{I}_{m-l_1} + \mathbf{F}_{11} & \mathbf{F}_{12} & \mathbf{F}_{13} & \mathbf{D}_{11} & \mathbf{D}_{12} & \mathbf{D}_{13}\end{bmatrix}^{\intercal} : \mathbb{C}^{m-l_1} \to \mathbb{C}^{N}$$

also possesses rank $< m - l_1$, and is equivalent to asserting that the homogeneous system:

$$\begin{bmatrix}\lambda'\mathbf{I}_{m-l_1} + \mathbf{F}_{11}^{\intercal} \\ \mathbf{F}_{12}^{\intercal} \\ \mathbf{F}_{13}^{\intercal} \\ \mathbf{D}_{11}^{\intercal} \\ \mathbf{D}_{12}^{\intercal} \\ \mathbf{D}_{13}^{\intercal}\end{bmatrix}\boldsymbol{\theta} = \mathbf{0} \in \mathbb{C}^{N}$$

has at least one nonzero solution (in addition to the zero solution), say $\boldsymbol{\theta} = \boldsymbol{\theta}' \in \mathbb{C}^{m-l_1} \setminus \{\mathbf{0}\}$. This means that the following system of homogeneous equations:

$$\begin{cases} (\lambda'\mathbf{I}_{m-l_1} + \mathbf{F}_{11}^{\intercal})\boldsymbol{\theta} = \mathbf{0} \in \mathbb{C}^{m-l_1}, \\ \mathbf{F}_{12}^{\intercal}\boldsymbol{\theta} = \mathbf{0} \in \mathbb{C}^{s-l_2}, \\ \mathbf{F}_{13}^{\intercal}\boldsymbol{\theta} = \mathbf{0} \in \mathbb{C}^{N-m-s-l_3}, \\ \mathbf{D}_{11}^{\intercal}\boldsymbol{\theta} = \mathbf{0} \in \mathbb{C}^{l_1}, \\ \mathbf{D}_{12}^{\intercal}\boldsymbol{\theta} = \mathbf{0} \in \mathbb{C}^{l_2}, \\ \mathbf{D}_{13}^{\intercal}\boldsymbol{\theta} = \mathbf{0} \in \mathbb{C}^{l_3} \end{cases} \tag{$\ddagger$}$$

possesses a nonzero solution $\boldsymbol{\theta}'$ in addition to the zero solution. In other words,

$$\mathbb{E}_{-\lambda'}(\mathbf{F}_{11}^{\intercal}) \cap \mathcal{N}(\mathbf{F}_{12}^{\intercal}) \cap \mathcal{N}(\mathbf{F}_{13}^{\intercal}) \cap \mathcal{N}(\mathbf{D}_{11}^{\intercal}) \cap \mathcal{N}(\mathbf{D}_{12}^{\intercal}) \cap \mathcal{N}(\mathbf{D}_{13}^{\intercal})$$

contains $\boldsymbol{\theta}'$. This means that $\boldsymbol{\theta}'$ is the eigenvector of $\mathbf{F}_{11}^{\intercal}$ corresponding to its eigenvalue $-\lambda'$ and belonging to $\mathcal{N}(\mathbf{F}_{12}^{\intercal})$, $\mathcal{N}(\mathbf{F}_{13}^{\intercal})$, $\mathcal{N}(\mathbf{D}_{11}^{\intercal})$, $\mathcal{N}(\mathbf{D}_{12}^{\intercal})$, and $\mathcal{N}(\mathbf{D}_{13}^{\intercal})$. In other words, $\boldsymbol{\theta}'$ is simultaneously orthogonal to all the columns of the matrix $\begin{bmatrix}\mathbf{F}_{12} & \mathbf{F}_{13} & \mathbf{D}_{11} & \mathbf{D}_{12} & \mathbf{D}_{13}\end{bmatrix}$.

The proof that condition (17) is equivalent to statement (ii) and condition (18) is equivalent to the statement (iii) follows a similar logic. ∎

Remark 4.1 *Suppose $\mathcal{G}_{\mathfrak{N}}$ is bidirectional such that each vertex has the same in-degree. In that case, $\mathcal{L}^{rw}(\mathcal{G})$ will be symmetric, making all $\mathbf{M}_{ij}$s, and consequently, all $\mathbf{F}_{ij}$s and $\mathbf{D}_{ij}$s symmetric as well. As a result, the necessary conditions in Corollary 4.1 become equivalent to the following:*

(i) No eigenvector of $\mathbf{F}_{11}$ is simultaneously orthogonal to all the columns of the matrix $\begin{bmatrix}\mathbf{F}_{12} & \mathbf{F}_{13} & \mathbf{D}_{11} & \mathbf{D}_{12} & \mathbf{D}_{13}\end{bmatrix}$.
(ii) No eigenvector of $\mathbf{F}_{22}$ is simultaneously orthogonal to all the columns of the matrix $\begin{bmatrix}\mathbf{F}_{21} & \mathbf{F}_{23} & \mathbf{D}_{21} & \mathbf{D}_{22} & \mathbf{D}_{23}\end{bmatrix}$.
(iii) No eigenvector of $\mathbf{F}_{33}$ is simultaneously orthogonal to all the columns of the matrix $\begin{bmatrix}\mathbf{F}_{31} & \mathbf{F}_{32} & \mathbf{D}_{31} & \mathbf{D}_{32} & \mathbf{D}_{33}\end{bmatrix}$.

Remark 4.2 *We refer to $\mathbf{F} := \begin{bmatrix}\mathbf{F}_{11} & \mathbf{F}_{12} & \mathbf{F}_{13}\\ \mathbf{F}_{21} & \mathbf{F}_{22} & \mathbf{F}_{23}\\ \mathbf{F}_{31} & \mathbf{F}_{32} & \mathbf{F}_{33}\end{bmatrix}$ and $\mathbf{D} := \begin{bmatrix}\mathbf{D}_{11} & \mathbf{D}_{12} & \mathbf{D}_{13}\\ \mathbf{D}_{21} & \mathbf{D}_{22} & \mathbf{D}_{23}\\ \mathbf{D}_{31} & \mathbf{D}_{32} & \mathbf{D}_{33}\end{bmatrix}$ as the follower matrix and leader matrix of $\mathfrak{N}$, respectively. We also refer to the matrix pair $(\mathbf{F},\, \mathbf{D})$ as the communication topology of $\mathfrak{N}$.*

5. Illustrative examples

In this section, we demonstrate the effectiveness of the derived theoretical results through illustrative examples.

Example 5.1 *The purpose of this example is to validate Theorem 4.1 (or Corollary 4.1). Figure 4 displays the interconnection graph of a multi-agent network. Let $\{1,\, 2,\, 3,\, 4\}$ be the set of first-order integrator agents, $\{5,\, 6,\, 7\}$ be the set of*

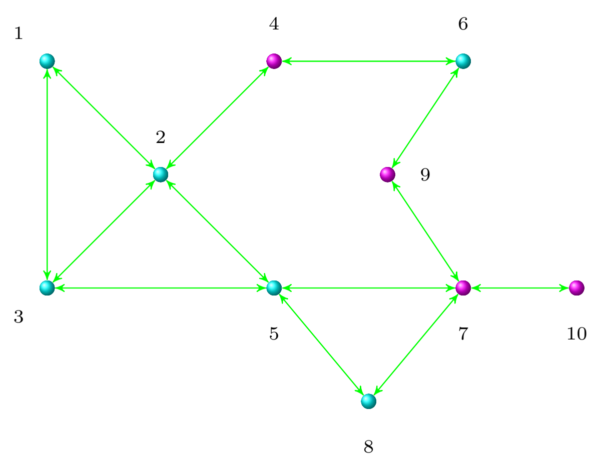


Figure 4. Interconnection graph of the multi-agent network in Example 5.1

second-order integrator agents, and $\{8, 9, 10\}$ *be the set of third-order integrator agents. Let* 4, 7, 9, *and* 10 *be the leaders, while the rest of the agents are followers. The random-walk normalised Laplacian of the interconnection graph is given by*

$$\mathcal{L}^{rw} =$$

$$= \left[\begin{array}{cccc:ccc:ccc} 1 & -1/2 & -1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1/4 & 1 & -1/4 & -1/4 & -1/4 & 0 & 0 & 0 & 0 & 0 \\ -1/3 & -1/3 & 1 & 0 & -1/3 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1/2 & 0 & 1 & 0 & -1/2 & 0 & 0 & 0 & 0 \\ \hdashline 0 & -1/4 & -1/4 & 0 & 1 & 0 & -1/4 & -1/4 & 0 & 0 \\ 0 & 0 & 0 & -1/2 & 0 & 1 & 0 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & 0 & -1/4 & 0 & 1 & -1/4 & -1/4 & -1/4 \\ \hdashline 0 & 0 & 0 & 0 & -1/2 & 0 & -1/2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1/2 & -1/2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right].$$

From this, we obtain $\mathbf{F}_{ij}$ *and* $\mathbf{D}_{ij}$ *as follows:*

$$\mathbf{F}_{11} = \begin{bmatrix} 1 & -1/2 & -1/2 \\ -1/4 & 1 & -1/4 \\ -1/3 & -1/3 & 1 \end{bmatrix}, \quad \mathbf{F}_{12} = \begin{bmatrix} 0 & 0 \\ -1/4 & 0 \\ -1/3 & 0 \end{bmatrix}, \quad \mathbf{F}_{13} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\mathbf{F}_{21} = \begin{bmatrix} 0 & -1/4 & -1/4 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{F}_{22} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{F}_{23} = \begin{bmatrix} -1/4 \\ 0 \end{bmatrix},$$

$$\mathbf{F}_{31} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{F}_{32} = \begin{bmatrix} -1/2 & 0 \end{bmatrix}, \quad \mathbf{F}_{33} = \begin{bmatrix} 1 \end{bmatrix};$$

$$\mathbf{D}_{11} = \begin{bmatrix} 0 \\ -1/4 \\ 0 \end{bmatrix}, \quad \mathbf{D}_{12} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{D}_{13} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\mathbf{D}_{21} = \begin{bmatrix} 0 \\ -1/2 \end{bmatrix}, \quad \mathbf{D}_{22} = \begin{bmatrix} -1/4 \\ 0 \end{bmatrix}, \quad \mathbf{D}_{23} = \begin{bmatrix} 0 & 0 \\ -1/2 & 0 \end{bmatrix},$$

$$\mathbf{D}_{31} = \begin{bmatrix} 0 \end{bmatrix}, \quad \mathbf{D}_{32} = \begin{bmatrix} -1/2 \end{bmatrix},$$

and

$$\mathbf{D}_{33} = \begin{bmatrix} 0 & 0 \end{bmatrix}.$$

The computation of the matrices in (14) and (15) yields the forms that can be seen on the following page, respectively, both of which possess full row rank $= 6 = (10 - 4) = N - l$ *for every* $\mu \in \mathbb{C}$.

This means that the network is leader–follower controllable under both protocols (2) *and* (3).

$$\begin{bmatrix}
(\mu+1) & -1/2 & -1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-1/4 & (\mu+1) & -1/4 & -1/4 & 0 & 0 & -1/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-1/3 & -1/3 & (\mu+1) & -1/3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -1/4 & -1/4 & (\mu^2+\mu+1) & 0 & (-1/4)(\mu+1) & 0 & 0 & -1/4 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & (\mu^2+\mu+1) & 0 & -1/2 & 0 & 0 & 0 & 0 & -1/2 & 0 & 0 & 0\\
0 & 0 & 0 & (-1/2)(\mu+1) & 0 & (\mu^3+\mu^2+\mu+1) & 0 & 0 & -1/2 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

and

$$\begin{bmatrix}
(\mu+1) & -1/2 & -1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-1/4 & (\mu+1) & -1/4 & -1/4 & 0 & 0 & -1/4 & 0 & 0 & 0\\
-1/3 & -1/3 & (\mu+1) & -1/3 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -1/4 & -1/4 & (\mu^2-\mu+1) & 0 & -1/4 & 0 & -1/4 & 0 & 0\\
0 & 0 & 0 & 0 & (\mu^2-\mu+1) & 0 & -1/2 & 0 & -1/2 & 0\\
0 & 0 & 0 & -1/2 & 0 & (\mu^3-\mu^2-\mu+1) & 0 & -1/2 & 0 & 0
\end{bmatrix}.$$

Next, we can see that:

$$\begin{aligned}&\operatorname{rank}\begin{bmatrix}\lambda\mathbf{I}_3+\mathbf{F}_{11} & \mathbf{F}_{12} & \mathbf{F}_{13} & \mathbf{D}_{11} & \mathbf{D}_{12} & \mathbf{D}_{13}\end{bmatrix}\\ &\quad=\operatorname{rank}\begin{bmatrix}\lambda+1 & -1/2 & -1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ -1/4 & \lambda+1 & -1/4 & -1/4 & 0 & 0 & -1/4 & 0 & 0 & 0\\ -1/3 & -1/3 & 1 & -1/3 & 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}=3,\end{aligned}$$

$$\begin{aligned}&\operatorname{rank}\begin{bmatrix}\mathbf{F}_{21} & \lambda\mathbf{I}_2+\mathbf{F}_{22} & \mathbf{F}_{23} & \mathbf{D}_{21} & \mathbf{D}_{22} & \mathbf{D}_{23}\end{bmatrix}\\ &=\operatorname{rank}\begin{bmatrix}0 & -1/4 & -1/4 & \lambda+1 & 0 & -1/4 & 0 & -1/4 & 0 & 0\\ 0 & 0 & 0 & 0 & \lambda+1 & 0 & -1/2 & 0 & -1/2 & 0\end{bmatrix}=2,\end{aligned}$$

$$\begin{aligned}&\operatorname{rank}\begin{bmatrix}\mathbf{F}_{31} & \mathbf{F}_{32} & \lambda\mathbf{I}_1+\mathbf{F}_{33} & \mathbf{D}_{31} & \mathbf{D}_{32} & \mathbf{D}_{33}\end{bmatrix}\\ &\quad=\operatorname{rank}\begin{bmatrix}0 & 0 & 0 & -1/2 & 0 & \lambda+1 & 0 & -1/2 & 0 & 0\end{bmatrix}=1,\end{aligned}$$

for all $\lambda\in\mathbb{C}$, showing the validity of the necessary conditions (16)–(18) *for controllability. Similarly, the statements in Corollary 4.1 can be verified.*

Example 5.2 *This example aims to illustrate that the converse of Theorem 4.1 or Corollary 4.1 is not necessarily true. The example demonstrates that system (11) is uncontrollable, despite the conditions in Theorem 4.1 and Corollary 4.1 being valid. A similar example can be constructed for system* (12). *Figure 5 depicts the interconnection graph of a multi-agent network.*

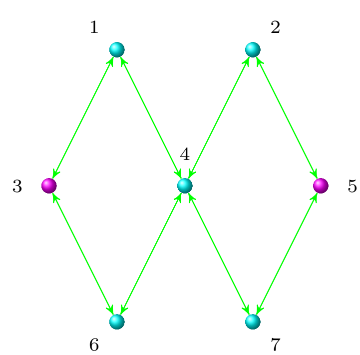


Figure 5. Interconnection graph of the multi-agent network in Example 5.2

Let $\{1, 2, 3\}$ *be the set of first-order integrator agents,* $\{4, 5\}$ *be the set of second-order integrator agents, and* $\{6, 7\}$ *be the set of third-order integrator agents. Let* 3 *and* 5 *be the leaders, while the rest of the agents are followers.*

The random-walk normalised Laplacian of the interconnection graph is given by:

$$\mathcal{L}^{rw} = \left[\begin{array}{ccc|cc|cc} 1 & 0 & -1/2 & -1/2 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1/2 & -1/2 & 0 & 0 \\ -1/2 & 0 & 1 & 0 & 0 & -1/2 & 0 \\ \hline -1/4 & -1/4 & 0 & 1 & 0 & -1/4 & -1/4 \\ 0 & -1/2 & 0 & 0 & 1 & 0 & -1/2 \\ \hline 0 & 0 & -1/2 & -1/2 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1/2 & -1/2 & 0 & 1 \end{array}\right].$$

From this, we obtain $\mathbf{F}_{ij}$ *and* $\mathbf{D}_{ij}$ *as follows:*

$$\mathbf{F}_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \mathbf{F}_{12} = \begin{bmatrix} -1/2 \\ -1/2 \end{bmatrix}, \mathbf{F}_{13} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\mathbf{F}_{21} = \begin{bmatrix} -1/4 & -1/4 \end{bmatrix}, \ \mathbf{F}_{22} = \begin{bmatrix} 1 \end{bmatrix}, \mathbf{F}_{23} = \begin{bmatrix} -1/4 & -1/4 \end{bmatrix},$$

$$\mathbf{F}_{31} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \mathbf{F}_{32} = \begin{bmatrix} -1/2 \\ -1/2 \end{bmatrix}, \mathbf{F}_{33} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix};$$

$$\mathbf{D}_{11} = \begin{bmatrix} -1/2 \\ 0 \end{bmatrix}, \mathbf{D}_{12} = \begin{bmatrix} 0 \\ -1/2 \end{bmatrix}, \mathbf{D}_{21} = \begin{bmatrix} 0 \end{bmatrix}, \mathbf{D}_{22} = \begin{bmatrix} 0 \end{bmatrix}, \mathbf{D}_{31} = \begin{bmatrix} -1/2 \\ 0 \end{bmatrix},$$

$$\textit{and} \quad \mathbf{D}_{32} = \begin{bmatrix} 0 \\ -1/2 \end{bmatrix}.$$

Conditions (16)–(18) *give:*

$$\begin{aligned} &\operatorname{rank} \begin{bmatrix} \lambda \mathbf{I}_2 + \mathbf{F}_{11} & \mathbf{F}_{12} & \mathbf{F}_{13} & \mathbf{D}_{11} & \mathbf{D}_{12} & \mathbf{D}_{13} \end{bmatrix} \\ &\qquad = \operatorname{rank} \begin{bmatrix} \lambda+1 & 0 & -1/2 & 0 & 0 & -1/2 & 0 \\ 0 & \lambda+1 & -1/2 & 0 & 0 & 0 & -1/2 \end{bmatrix} = 2, \\ &\operatorname{rank} \begin{bmatrix} \mathbf{F}_{21} & \lambda \mathbf{I}_1 + \mathbf{F}_{22} & \mathbf{F}_{23} & \mathbf{D}_{21} & \mathbf{D}_{22} & \mathbf{D}_{23} \end{bmatrix} \\ &\qquad = \operatorname{rank} \begin{bmatrix} -1/4 & -1/4 & \lambda+1 & -1/4 & -1/4 & 0 & 0 \end{bmatrix} = 1, \\ &\operatorname{rank} \begin{bmatrix} \mathbf{F}_{31} & \mathbf{F}_{32} & \lambda \mathbf{I}_2 + \mathbf{F}_{33} & \mathbf{D}_{31} & \mathbf{D}_{32} & \mathbf{D}_{33} \end{bmatrix} \\ &\qquad = \operatorname{rank} \begin{bmatrix} 0 & 0 & -1/2 & \lambda+1 & 0 & -1/2 & 0 \\ 0 & 0 & -1/2 & 0 & \lambda+1 & 0 & -1/2 \end{bmatrix} = 2, \end{aligned}$$

for every $\lambda \in \mathbb{C}$*, showing the validity of conditions in Theorem 4.1 (and hence of Corollary 4.1). However, we find that the matrix in (14) becomes:*

$$\left[\begin{array}{cccccccc} (\mu+1) & 0 & -1/2 & 0 & 0 & -1/2 & 0 & 0 \\ 0 & (\mu+1) & -1/2 & 0 & 0 & 0 & -1/2 & 0 \\ -1/4 & -1/4 & (\mu^2+\mu+1) & -1/4(\mu+1) & -1/4(\mu+1) & 0 & 0 & 0 \\ 0 & 0 & -1/2(\mu+1) & (\mu^3+\mu^2+\mu+1) & 0 & -1/2 & 0 & 0 \\ 0 & 0 & -1/2(\mu+1) & 0 & (\mu^3+\mu^2+\mu+1) & 0 & 0 & -1/2 \end{array}\right]$$

and it is rank-deficient for certain $\mu \in \mathbb{C}$, for example when $\mu = -1$. This means that the multi-agent network is not leader–follower controllable under the protocol (2).

Example 5.3 *This example illustrates the fact that system (*11) *and* (12) *can be controllable without requiring the communication topology of the network to be controllable. The interconnection graph of the multi-agent network is depicted in Fig. 6.*

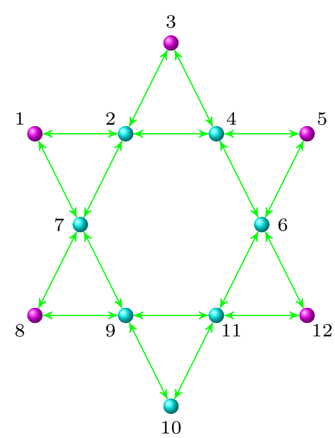


Figure 6. Interconnection graph of the multi-agent network in Example 5.3

Let $\{1, 2, 3, 4\}$ *constitute the set of first-order integrator agents,* $\{5, 6, 7, 8, 9\}$ *the set of second-order integrator agents, and* $\{10, 11, 12\}$ *the set of third-order integrator agents. Suppose* 1, 3, 5, 8, *and* 12 *are the leaders, while the rest act as followers. The random-walk normalised Laplacian matrix of the interconnection graph is given by:*

$$\mathcal{L}^{rw} = \left[\begin{array}{cccc:ccccc:ccc}
1 & -1/2 & 0 & 0 & 0 & 0 & -1/2 & 0 & 0 & 0 & 0 & 0 \\
-1/4 & 1 & -1/4 & -1/4 & 0 & 0 & -1/4 & 0 & 0 & 0 & 0 & 0 \\
0 & -1/2 & 1 & -1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1/4 & -1/4 & 1 & -1/4 & -1/4 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hdashline
0 & 0 & 0 & -1/2 & 1 & -1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1/4 & -1/4 & 1 & 0 & 0 & 0 & 0 & -1/4 & -1/4 \\
-1/4 & -1/4 & 0 & 0 & 0 & 0 & 1 & -1/4 & -1/4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1/2 & 1 & -1/2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1/4 & -1/4 & 1 & -1/4 & -1/4 & 0 \\
\hdashline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/2 & 1 & -1/2 & 0 \\
0 & 0 & 0 & 0 & 0 & -1/4 & 0 & 0 & -1/4 & -1/4 & 1 & -1/4 \\
0 & 0 & 0 & 0 & 0 & -1/2 & 0 & 0 & 0 & 0 & -1/2 & 1
\end{array}\right].$$

From this, we obtain $\mathbf{F}_{ij}$ *and* $\mathbf{D}_{ij}$ *as follows:*

$$\mathbf{F}_{11} = \begin{bmatrix} 1 & -1/4 \\ -1/4 & 1 \end{bmatrix}, \mathbf{F}_{12} = \begin{bmatrix} 0 & -1/4 & 0 \\ -1/4 & 0 & 0 \end{bmatrix}, \mathbf{F}_{13} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\mathbf{F}_{21} = \begin{bmatrix} 0 & -1/4 \\ -1/4 & 0 \\ 0 & 0 \end{bmatrix}, \mathbf{F}_{22} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1/4 \\ 0 & -1/4 & 1 \end{bmatrix}, \mathbf{F}_{23} = \begin{bmatrix} 0 & -1/4 \\ 0 & 0 \\ -1/4 & -1/4 \end{bmatrix},$$

$$\mathbf{F}_{31} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \mathbf{F}_{32} = \begin{bmatrix} 0 & 0 & -1/2 \\ -1/4 & 0 & -1/4 \end{bmatrix}, \mathbf{F}_{33} = \begin{bmatrix} 1 & -1/2 \\ -1/4 & 1 \end{bmatrix};$$

$$\mathbf{D}_{11} = \begin{bmatrix} -1/4 & -1/4 \\ 0 & -1/4 \end{bmatrix}, \mathbf{D}_{12} = \begin{bmatrix} 0 & 0 \\ -1/4 & 0 \end{bmatrix}, \mathbf{D}_{13} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mathbf{D}_{21} = \begin{bmatrix} 0 & 0 \\ -1/4 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\mathbf{D}_{22} = \begin{bmatrix} -1/4 & 0 \\ 0 & -1/4 \\ 0 & -1/4 \end{bmatrix}, \mathbf{D}_{23} = \begin{bmatrix} -1/4 \\ 0 \\ 0 \end{bmatrix}, \mathbf{D}_{31} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \mathbf{D}_{32} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

and

$$\mathbf{D}_{33} = \begin{bmatrix} 0 \\ -1/4 \end{bmatrix},$$

so that

$$\mathbf{F} = \begin{bmatrix} 1 & -1/4 & 0 & -1/4 & 0 & 0 & 0 \\ -1/4 & 1 & -1/4 & 0 & 0 & 0 & 0 \\ 0 & -1/4 & 1 & 0 & 0 & 0 & -1/4 \\ -1/4 & 0 & 0 & 1 & -1/4 & 0 & 0 \\ 0 & 0 & 0 & -1/4 & 1 & -1/4 & -1/4 \\ 0 & 0 & 0 & 0 & -1/2 & 1 & -1/2 \\ 0 & 0 & -1/4 & 0 & -1/4 & -1/4 & 1 \end{bmatrix}$$

and

$$\mathbf{D} = \begin{bmatrix} -1/4 & -1/4 & 0 & 0 & 0 \\ 0 & -1/4 & -1/4 & 0 & 0 \\ 0 & 0 & -1/4 & 0 & -1/4 \\ -1/4 & 0 & 0 & -1/4 & 0 \\ 0 & 0 & 0 & -1/4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1/4 \end{bmatrix}.$$

One can easily check that the communication topology $(\mathbf{F}, \mathbf{D})$ *is uncontrollable. Furthermore, the computation of matrices in (14) and (15), respectively, gives:*

$$
\left[\begin{array}{cccccccc|cccccccc}
\mu+1 & -1/4 & 0 & -1/4 & 0 & 0 & 0 & -1/4 & -1/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1/4 & \mu+1 & -1/4 & 0 & 0 & 0 & 0 & 0 & -1/4 & -1/4 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1/4 & \mu^2+\mu+1 & 0 & 0 & 0 & (-1/4)(\mu+1) & 0 & 0 & 0 & 0 & -1/4 & 0 & 0 & -1/4 & 0 \\
-1/4 & 0 & 0 & \mu^2+\mu+1 & (-1/4)(\mu+1) & 0 & 0 & -1/4 & 0 & 0 & 0 & 0 & -1/4 & 0 & 0 & 0 \\
0 & 0 & 0 & (-1/4)(\mu+1) & \mu^2+\mu+1 & (-1/4)(\mu+1) & (-1/4)(\mu+1) & 0 & 0 & 0 & 0 & 0 & -1/4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & (-1/2)(\mu+1) & \mu^3+\mu^2+\mu+1 & (-1/2)(\mu^2+\mu+1) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & (-1/4)(\mu+1) & 0 & (-1/4)(\mu+1) & (-1/4)(\mu^2+\mu+1) & \mu^3+\mu^2+\mu+1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/4
\end{array}\right]
$$

and

$$\left[\begin{array}{cccccc|cccccc}
\mu+1 & -1/4 & 0 & -1/4 & 0 & 0 & 0 & -1/4 & -1/4 & 0 & 0 & 0\\
-1/4 & \mu+1 & -1/4 & 0 & 0 & 0 & 0 & 0 & -1/4 & -1/4 & 0 & 0\\
0 & -1/4 & \mu^2-\mu+1 & 0 & 0 & 0 & -1/4 & 0 & 0 & -1/4 & 0 & -1/4\\
-1/4 & 0 & 0 & \mu^2-\mu+1 & -1/4 & 0 & 0 & -1/4 & 0 & 0 & -1/4 & 0\\
0 & 0 & 0 & -1/4 & \mu^2-\mu+1 & -1/4 & -1/4 & 0 & 0 & 0 & -1/4 & 0\\
0 & 0 & 0 & 0 & -1/2 & \mu^3-\mu^2-\mu+1 & -1/2 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & -1/4 & 0 & -1/4 & -1/4 & \mu^3-\mu^2-\mu+1 & 0 & 0 & 0 & 0 & -1/4
\end{array}\right],$$

both of which possess the full row rank $= 7 = (12-5) = N - l$ *for any* $\mu \in \mathbb{C}$. *This means that the network is leader–follower controllable under both protocols,* (2) *and* (3).

Example 5.4 *This example illustrates that the system* (11) *and* (12) *can be controllable even when each of the pairs* $(\mathbf{F}_{ii}, \mathbf{D}_{ii})$ *for* $i = 1, 2, 3$ *is uncontrollable. The interconnection graph of the multi-agent network is plotted in Fig. 7. Let* $\{1, 2, 3\}$ *be the set of first-order integrator agents,* $\{4, 5, 6\}$ *be the set of*

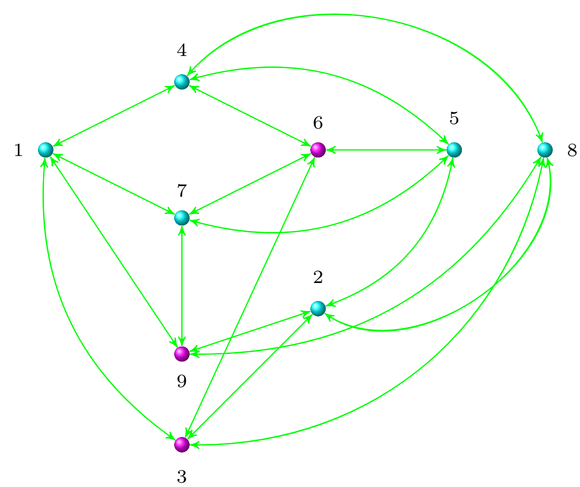


Figure 7. Interconnection graph of the multi-agent network in Example 5.4

second-order integrator agents, and $\{7, 8, 9\}$ *be the set of third-order integrator agents. Suppose* 3, 6, *and* 9 *are the leaders and the remaining are followers. The random-walk normalised Laplacian of the interconnection graph is given by:*

$$\mathcal{L}^{rw} = \left[\begin{array}{ccc:ccc:ccc} 1 & 0 & -1/4 & -1/4 & 0 & 0 & -1/4 & 0 & -1/4 \\ 0 & 1 & -1/4 & 0 & -1/4 & 0 & 0 & -1/4 & -1/4 \\ -1/4 & -1/4 & 1 & 0 & 0 & -1/4 & 0 & -1/4 & 0 \\ \hdashline -1/4 & 0 & 0 & 1 & -1/4 & -1/4 & 0 & -1/4 & 0 \\ 0 & -1/4 & 0 & -1/4 & 1 & -1/4 & -1/4 & 0 & 0 \\ 0 & 0 & -1/4 & -1/4 & -1/4 & 1 & -1/4 & 0 & 0 \\ \hdashline -1/4 & 0 & 0 & 0 & -1/4 & -1/4 & 1 & 0 & -1/4 \\ 0 & -1/4 & -1/4 & -1/4 & 0 & 0 & 0 & 1 & -1/4 \\ -1/4 & -1/4 & 0 & 0 & 0 & 0 & -1/4 & -1/4 & 1 \end{array}\right],$$

from which we obtain $\mathbf{F}_{ij}$ *and* $\mathbf{D}_{ij}$ *as follows:*

$$\mathbf{F}_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \mathbf{F}_{12} = \begin{bmatrix} -1/4 & 0 \\ 0 & -1/4 \end{bmatrix},$$
$$\mathbf{F}_{13} = \begin{bmatrix} -1/4 & 0 \\ 0 & -1/4 \end{bmatrix},$$

$$\mathbf{F}_{21} = \begin{bmatrix} -1/4 & 0 \\ 0 & -1/4 \end{bmatrix},$$
$$\mathbf{F}_{22} = \begin{bmatrix} 1 & -1/4 \\ -1/4 & 1 \end{bmatrix}, \mathbf{F}_{23} = \begin{bmatrix} 0 & -1/4 \\ -1/4 & 0 \end{bmatrix},$$

$$\mathbf{F}_{31} = \begin{bmatrix} -1/4 & 0 \\ 0 & -1/4 \end{bmatrix},$$
$$\mathbf{F}_{32} = \begin{bmatrix} 0 & -1/4 \\ -1/4 & 0 \end{bmatrix}, \mathbf{F}_{33} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix};$$

$$\mathbf{D}_{11} = \begin{bmatrix} -1/4 \\ -1/4 \end{bmatrix}, \mathbf{D}_{12} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$
$$\mathbf{D}_{13} = \begin{bmatrix} -1/4 \\ -1/4 \end{bmatrix},$$

$$\mathbf{D}_{21} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mathbf{D}_{22} = \begin{bmatrix} -1/4 \\ -1/4 \end{bmatrix},$$
$$\mathbf{D}_{23} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$\mathbf{D}_{31} = \begin{bmatrix} 0 \\ -1/4 \end{bmatrix}, \mathbf{D}_{32} = \begin{bmatrix} -1/4 \\ 0 \end{bmatrix},$$

and

$$\mathbf{D}_{33} = \begin{bmatrix} -1/4 \\ -1/4 \end{bmatrix}.$$

One can easily check that $(\mathbf{F}_{11}, \mathbf{D}_{11})$, $(\mathbf{F}_{22}, \mathbf{D}_{22})$, *and* $(\mathbf{F}_{33}, \mathbf{D}_{33})$ *are uncontrollable. However, the computation of the matrices in* (14) *and* (15) *gives*

$$\begin{bmatrix} (\mu+1) & 0 & -1/4 & 0 & -1/4 & 0 & -1/4 & 0 & 0 & -1/4 & 0 & 0 \\ 0 & (\mu+1) & 0 & -1/4 & 0 & -1/4 & -1/4 & 0 & 0 & -1/4 & 0 & 0 \\ -1/4 & 0 & (\mu^2+\mu+1) & (-1/4)(\mu+1) & 0 & (-1/4)(\mu+1) & 0 & 0 & -1/4 & 0 & 0 & 0 \\ 0 & -1/4 & (-1/4)(\mu+1) & (\mu^2+\mu+1) & (-1/4)(\mu+1) & 0 & 0 & 0 & -1/4 & 0 & 0 & 0 \\ -1/4 & 0 & 0 & (-1/4)(\mu+1) & (\mu^3+\mu^2+\mu+1) & 0 & 0 & 0 & -1/4 & 0 & 0 & -1/4 \\ 0 & -1/4 & (-1/4)(\mu+1) & 0 & 0 & (\mu^3+\mu^2+\mu+1) & -1/4 & 0 & 0 & 0 & 0 & -1/4 \end{bmatrix}$$

and

$$\begin{bmatrix} (\mu+1) & 0 & -1/4 & 0 & -1/4 & 0 & -1/4 & 0 & -1/4 \\ 0 & (\mu+1) & 0 & -1/4 & 0 & -1/4 & -1/4 & 0 & -1/4 \\ -1/4 & 0 & (\mu^2-\mu+1) & -1/4 & 0 & -1/4 & 0 & -1/4 & 0 \\ 0 & -1/4 & -1/4 & (\mu^2-\mu+1) & -1/4 & 0 & 0 & -1/4 & 0 \\ -1/4 & 0 & 0 & -1/4 & (\mu^3-\mu^2-\mu+1) & 0 & 0 & -1/4 & -1/4 \\ 0 & -1/4 & -1/4 & 0 & 0 & (\mu^3-\mu^2-\mu+1) & -1/4 & 0 & -1/4 \end{bmatrix},$$

respectively, and both possesses the full row-rank $= 6 = (9-3) = N - l$ *for every* $\mu \in \mathbb{C}$. *Hence we conclude that the network is leader–follower controllable under both the protocols,* (2) *and* (3).

Example 5.5 *This example demonstrates that the systems* (11) *and* (12) *can be uncontrollable even when each of* $(\mathbf{F}_{ii}, \mathbf{D}_{ii})$ *for* $i = 1, 2, 3$ *is controllable. Figure 8 depicts the interconnection graph of the multi-agent network. Let* $\{1, 2, 3\}$

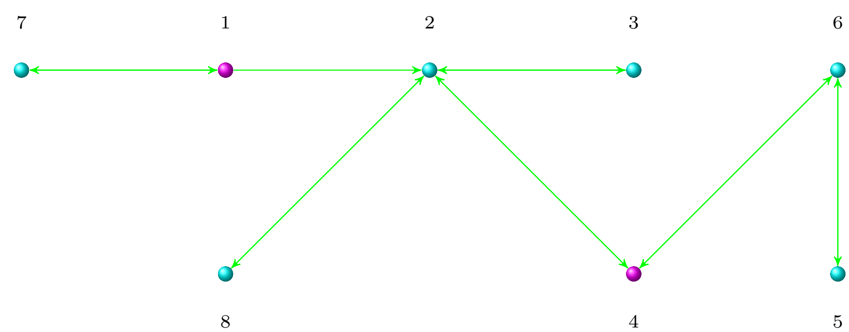


Figure 8. Interconnection graph of the multi-agent network in Example 5.5

be the set of first-order integrator agents, $\{4, 5, 6\}$ *be the set of second-order integrator agents, and* $\{7, 8\}$ *be the set of third-order integrator agents. Suppose* 1 *and* 4 *are the leaders, and the remaining agents are followers. The random-walk normalised Laplacian of the interconnection graph is given by:*

$$\mathcal{L}^{rw} = \left[\begin{array}{ccc|ccc|cc} 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ -1/4 & 1 & -1/4 & -1/4 & 0 & 0 & 0 & -1/4 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & -1/2 & 0 & 1 & 0 & -1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1/2 & -1/2 & 1 & 0 & 0 \\ \hline -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right],$$

from which we obtain $\mathbf{F}_{ij}$ *and* $\mathbf{D}_{ij}$ *as follows:*

$$\mathbf{F}_{11} = \begin{bmatrix} 1 & -1/4 \\ -1 & 1 \end{bmatrix}, \mathbf{F}_{12} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\mathbf{F}_{13} = \begin{bmatrix} 0 & -1/4 \\ 0 & 0 \end{bmatrix}, \mathbf{F}_{21} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\mathbf{F}_{22} = \begin{bmatrix} 1 & -1 \\ -1/2 & 1 \end{bmatrix}, \mathbf{F}_{23} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\mathbf{F}_{31} = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}, \mathbf{F}_{32} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\mathbf{F}_{33} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix};$$

$$\mathbf{D}_{11} = \begin{bmatrix} -1/4 \\ 0 \end{bmatrix}, \mathbf{D}_{12} = \begin{bmatrix} -1/4 \\ 0 \end{bmatrix},$$

$$\mathbf{D}_{21} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mathbf{D}_{22} = \begin{bmatrix} 0 \\ -1/2 \end{bmatrix},$$

$$\mathbf{D}_{31} = \begin{bmatrix} -1 \\ 0 \end{bmatrix},$$

and

$$\mathbf{D}_{32} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

It can be easily verified that both $(\mathbf{F}_{11}, \mathbf{D}_{11})$ *and* $(\mathbf{F}_{22}, \mathbf{D}_{22})$ *are controllable. (Since* $\mathbf{D}_{33}$ *does not exist, there is no need to check the controllability of* $(\mathbf{F}_{33}, \mathbf{D}_{33})$*).) However, the computation of the matrices in* (14) *and* (15) *yields the following matrices:*

$$\begin{bmatrix}(\mu+1) & -1/4 & 0 & 0 & 0 & -1/4 & -1/4 & -1/4 & 0\\ -1 & (\mu+1) & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & (\mu^2+\mu+1) & -(\mu+1) & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & (-1/2)(\mu+1) & (\mu^2+\mu+1) & 0 & 0 & 0 & 0 & -1/2\\ 0 & 0 & 0 & 0 & (\mu^3+\mu^2+\mu+1) & 0 & -1 & 0 & 0\\ -1 & 0 & 0 & 0 & 0 & (\mu^3+\mu^2+\mu+1) & 0 & 0 & 0\end{bmatrix}$$

and

$$\begin{bmatrix}(\mu+1) & -1/4 & 0 & 0 & 0 & -1/4 & -1/4 & -1/4\\ -1 & (\mu+1) & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & (\mu^2-\mu+1) & -1 & 0 & 0 & 0 & 0\\ 0 & 0 & -1/2 & (\mu^2-\mu+1) & 0 & 0 & 0 & -1/2\\ 0 & 0 & 0 & 0 & (\mu^3-\mu^2-\mu+1) & 0 & -1 & 0\\ -1 & 0 & 0 & 0 & 0 & (\mu^3-\mu^2-\mu+1) & 0 & 0\end{bmatrix},$$

respectively, and both are rank-deficient for certain $\mu \in \mathbb{C}$ (for instance when $\mu = -1$). Hence, this network is not leader–follower controllable under both of the protocols, (2) *and* (3)*.*

6. Concluding remarks

In this study, we have examined a multi-agent network with a directed, unweighted, cooperative, and time-invariant communication topology. The agents in this network follow a diverse set of linear dynamics, including first-, second-, and third-order ordinary differential equations on a continuous time scale. Two different kinds of neighbour-based linear distributed control protocols are introduced: one utilising average feedback from relative velocities/relative accelerations, while the other utilising feedback from absolute velocities/absolute accelerations. The dynamical rule exploits the random-walk normalised Laplacian matrix of the network's graph, resulting in the agents achieving asymptotic consensus in their states (positions, velocities, and accelerations).

Subsequently, we have conducted an analysis of leader–follower controllability within the network by categorising the agents as leaders and followers. Leaders are defined as agents whose states are independently regulated within an admissible space $\mathfrak{L}^2(\cdot)$, while followers are subject to the direct influence of leaders. Utilising the PBH rank criterion, we have derived easily verifiable necessary and sufficient algebraic tests to assess the controllability of followers' states through the leaders. Furthermore, we have established necessary controllability conditions in terms of eigenvalues and eigenvectors of system matrices. This is illustrated with an example showing that network's leader - follower controllability remains achievable even when the communication topology of the network itself is not controllable. We have presented several numerical examples to support our theoretical results. The inference diagrams offer deeper insights into how leader-follower interactions impact network controllability within the leader - follower framework.

It is worth noting that our results have broad applicability, as the controllability conditions we have developed can be extended to scenarios involving not only the case of heterogeneous networks comprising first- and second-order, or first- and third-order, or second- and third-order integrator agents, but also the case of homogeneous variants of $\mathfrak{N}$, consisting only of first-order or only of second-order or only of third-order integrator agents.

Our proposed model has certain limitations. For instance, in this study, we assumed that all communication weights are set to unity. In many real-life scenarios, network models incorporate arbitrary weights, including negative ones, reflecting both cooperative and competitive communications. In such cases, our proposed model may exhibit instability. Furthermore, our research exclusively considers a time-invariant communication topology, which is one of its limitations. Future investigations may encompass scenarios involving time-varying communication topologies. Generalising our research to high-order dynamical systems holds theoretical significance. Additionally, exploring the discrete-time variant of the proposed model for consensus analysis and leader–follower controllability is a promising avenue. Considering these aspects, there is ample room for further research building upon our work.

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References

ALDOUS, D. (1991) Applications of random walks on finite graphs. *IMS Lecture Notes-Monograph Series* 18, 12–26. DOI: https://www.jstor.org/stable/4355644

BONDY, J. A. and MURTY, U. S. R. (2008) *Graph Theory.* Springer-Verlag, London.

CHEN, F. and REN, W. (2019) On the control of multi-agent systems: A survey. *Foundations and Trends in Systems and Control* **6** (4), 339–499. DOI: https://doi.org/10.1561/2600000019

COPPERSMITH, D., DOYLE, P., RAGHAVAN, P. and SNIR, M. (1993) Random walks on weighted graphs and applications to on-line algorithms. *Journal of the ACM* **40** (3), 421–453. DOI: https://doi.org/10.1145/174130.17413110.1145/174130.174131

DEFOORT, M., POLYAKOV A., DEMESURE, G., DJEMAI, M. and VELUVOLU, K. (2015) Leader–follower fixed-time consensus for multi-agent systems with unknown non-linear inherent dynamics. *IET Control Theory & Applications* **9** (14), 2165–2170. DOI: https://doi.org/10.1049/iet-cta.2014.1301

FRIEDBERG, S. H., INSEL, A. J. and SPENCE, L. W. (1989) *Linear Algebra.* Prentice Hall, New Jersey.

GENG, H., WU, H., MIAO, J., HOU, S. and CHEN, Z. (2022) Consensus of heterogeneous multi-agent systems under directed topology. *IEEE Access* 10, 5936–5943. DOI: https://doi.org/10.1109/ACCESS.2022.3142539

GREGUŠ, M. (1987) *Third Order Linear Differential Equations*. D. Reidel Publishing Company, Dordrecht.

GUAN, Y., JI, Z., ZHANG, L. and WANG, L. (2016) Controllability of heterogeneous multi-agent systems under directed and weighted topology. *International Journal of Control* **89** (5), 1009–1024. DOI: https://doi.org/10.1080/00207179.2015.1110756

JAYARAMAN, G., PADMANABHAN, N. and MEHROTRA, R. (1986) Entry flow into a circular tube of slowly varying cross section. *Fluid Dynamics Research* **1** (2), 131–144. DOI: https://doi.org/10.1016/0169-5983(86)90013-4

KURRAS, S. (2016) *Variants of the Graph Laplacian with Applications in Machine Learning. Ph.D. dissertation*. Universität Hamburg, Hamburg.

LELEUX, P., COURTAIN, S., FRANCOISSE, K. and SAERENS, M. (2022) Design of biased random walks on a graph with application to collaborative recommendation. *Physica A: Statistical Mechanics and its Applications* 590, 126752. DOI: https://doi.org/10.1016/j.physa.2021.126752

LIU, J., AN, B. and WU, H. (2018) Consensus of third-order multi-agent systems with communication delay. *2018 Chinese Control And Decision Conference*, 1428–1432. DOI: https://doi.org/10.1109/CCDC.2018.8407351

LIU, Y.-Y. and BARABÁSI, A.-L. (2016) Control principles of complex systems. *Reviews of Modern Physics* **88** (3), 035006. DOI: https://doi.org/10.1103/RevModPhys.88.035006

LIU, B., CHU, T., WANG, L. and XIE, G. (2008) Controllability of a leader–follower dynamic network with switching topology. *IEEE Transactions on Automatic Control* **53** (4), 1009–1013. DOI: https://doi.org/10.1109/TAC.2008.919548

LIU, K., XIE, G. and WANG, L. (2012) Consensus for multi-agent systems under double integrator dynamics with time-varying communication delays. *International Journal of Robust and Nonlinear Control* **22** (17), 1881–1898. DOI: https://doi.org/10.1002/rnc.1792

LOZANO, R., SPONG, M. W., GUERRERO, J. A. and CHOPRA, N. (2008) Controllability and observability of leader-based multi-agent systems. *47th IEEE Conference on Decision and Control*, 3713–3718.
DOI: https://doi. org/10.1109/CDC.2008.4739071

MESBAHI, M. and EGERSTEDT, M. (2010) *Graph Theoretic Methods in Multiagent Networks*. Princeton University Press, Princeton.

MUNI, V. S., RAFEEK, K. V. M., ATHIRA, V. S. and REDDY, G. J. (2023) Controllability of consensus of multi-agent networks over heterogeneous dynamics. *Results in Control and Optimization* 12, 100272. DOI: https://doi.org/10.1016/j.rico.2023.100272

MUNI, V. S., RAFEEK, K. V. M., REDDY, G. J. and GEORGE, R. K. (2022) On the selection of leaders for the controllability of multi-agent networks.

Bulletin of the Iranian Mathematical Society **48** (6), 3141–3183. DOI: https://doi.org/10.1007/s41980-022-00683-2

PADHI, S. and PATI, S. (2014) *Theory of Third-Order Differential Equations*. Springer, New Delhi.

REN, W. (2007) Consensus strategies for cooperative control of vehicle formations. *IET Control Theory & Applications* **1** (2), 505–512. DOI: https://doi.org/10.1049/iet-cta:20050401

REN, W. and BEARD, R. W. (2005) Consensus seeking in multiagent systems under dynamically changing interaction topologies. *IEEE Transactions on Automatic Control* **50** (5), 655–661.
DOI: https://doi.org/10.1109/TAC. 2005.846556

REYNOLDS, D. W. (1989) Bifurcation of harmonic solutions of an integro-differential equation modelling resonant sloshing. *SIAM Journal of Applied Mathematics* **49** (2), 362–372.
DOI: https://doi.org/10.1137/01490 22

SABER, R. O. and MURRAY, R. M. (2004) Consensus problems in networks of agents with switching topology and time-delays. *IEEE Transactions on Automatic Control* **49** (9), 1520–1533.
DOI: https://doi.org/10.1109/TAC. 2004.834113

TANNER, H. G. (2004) On the controllability of nearest neighbour interconnections. *43rd IEEE Conference on Decision and Control*, 2467–2472.
DOI: https://doi.org/10.1109/CDC.2004.1428782

TERRELL, W. J. (2009) *Stability and Stabilization: An Introduction*. Princeton University Press, Princeton.

XIE, D., YUAN, D., LU, J. and ZHANG, Y. (2013) Consensus control of second-order leader–follower multi-agent systems with event-triggered strategy. *Transactions of the Institute of Measurement and Control* **35** (4), 426–436.
DOI: https://doi.org/10.1177/0142331212454046

YAZICIOĞLU, A. Y. and EGERSTEDT, M. (2013) Leader selection and network assembly for controllability of leader–follower networks. *American Control Conference*, 3802–3807.
DOI: https://doi.org/10.1109/ACC.2013. 6580419

YU, W., REN, W., ZHENG, W. X., Chen, G. and LÜ, J. (2013) Distributed control gains design for consensus in multi-agent systems with second-order nonlinear dynamics. *Automatica* **49** (7), 2107–2115.
DOI: https:// doi.org/10.1016/j.automatica.2013.03.005

ZHANG, L., SUN, J. and YANG, Q. (2021) Distributed model-based event-triggered leader–follower consensus control for linear continuous-time multiagent systems. *IEEE Transactions on Systems, Man, and Cybernetics: Systems* **51** (10), 6457–6465.
DOI: https://doi.org/10.1109/TSMC.2019. 2962735

ZHENG, Y. and WANG, L. (2012) Finite-time consensus of heterogeneous multi-agent systems with and without velocity measurements. *Systems & Control Letters* **61** (8), 871–878.
DOI: https://doi.org/10.1016/j.sysconle. 2012.05.009

ZHENG, Y. and WANG, L. (2012) Distributed consensus of heterogeneous multi-agent systems with fixed and switching topologies. *International Journal of Control* **85** (12), 1967–1976.
DOI: https://doi.org/10.1080/0020 7179.2012.713986

ZHENG, Y., ZHU, Y. and WANG, L. (2011) Consensus of heterogeneous multi-agent systems. *IET Control Theory & Applications* 5 (16), 1881–1888.
DOI: https://doi.org/10.1049/iet-cta.2011.0033