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Nonlinear Surface Elastic Waves in Materials

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Abstract

This paper is devoted to analysis of the surface nonlinear elastic harmonic waves of four types (Rayleigh and Sto neley harmonic waves within the framework of plane strain state; Love and Mozhaev harmonic wave within the framework of anti-plane strain state). The nonlinear model is based on introducing the Murnaghan elastic poten tial, which includes both geometrical and physical nonlinearities. Each type of surface waves is discussed in four steps: statement of the problem, nonlinear wave equations, approximate solution (first two approximations), so-me conclusions. A nonlinear analysis of waves required many novelties: new variants of the Murnaghan poten-tial, new nonlinear wave equations and new nonlinear boundary conditions. The nonlinear wave equations were solved by the method of successive approximations. A new approach to analyze the boundary conditions is offe- red. Some new nonlinear wave effects are observed theoretically: a wave picture is reached by the 2nd harmonic and becomes changing in time of propagation, the wave numbers become depending on the initial amplitude.

Keywords: surface nonlinear elastic harmonic waves, Rayleigh wave, Love wave, Stoneley wave, Mozhaev wave

1. Introduction

The theory of elastic harmonic waves forms the big fragment both linear and nonlinear theory of elasticity. Chronologically, the free waves were first studied. Their main characteristics is that they propagate in the space without boundaries. The plane harmonic waves have to be referred just to this type of waves [7]. Further, the waves with curvilinear fronts (cylindrical, spherical and so on) were studied, where the curvilinear boundary is presented, on which the waves are generating and then passing to infinity. The surface waves present the next, third, group in complexity of theoretical analysis [8]. A necessary here allowance for the influence of interface and a condition of quick attenuation of the wave amplitudes while being gone from the boundary, form a more complicate wave picture.

An intrinsic logics of development of the theory of elastic waves was dictated, at least, three lines of the subsequent study of elastic waves. The 1st line consists in complication of the model of elastic deformation (for example, transition from the structural model of the 1st order to the models of the 2nd order - micropolar, elastic mixture, micromorphic and so forth. The 2nd line includes allowance for the initial stresses what is impossible in the framework of the linear theory and has many applications. The 3rd line is associated with the full allowance for a nonlinearity of deformation and can be divided on different sub-lines, part of each is pure theoretical, whereas other one is more applied. Among the theoretical sublines, the Moscow, Tallinn, Nizhnii Novgorod, and Kyiv ones can be outlined. The shown in this paper analysis is related to the 4th subline. It is based on introduction in-to the model a nonlinearity, described by the Murnaghan elastic potential. Here, some re-sults from analysis of the Rayleigh, Love, Stoneley, and Mozhaev waves are shown. The Rayleigh and Stoneley waves are related to the surface waves and can be analyzed in the 3D approach. The 2D analysis (a statement in the framework of the plane strain approach) seems the only more convenient for the pilot consideration. The Love and Mozhaev waves are related to the surface wave and are analyzed in the 2D approach (a statement in the fra-mework of the anti-plane strain approach).

2. Nonlinear elastic surface Rayleigh wave

2.1. Statement of the problem

The case is considered, when an interface is the plane. Then the Cartesian coordinates are chosen in the way that interface is described by equation $x_3 = 0$ and an elastic material occupies the upper half-space. Let the material is isotropic and the wave propagates along the axis Ox_1. In this case, the motion becomes not depen-ding on the coordinate x_2. The mechanical state becomes plane strain state. Consider now the problem of nonlinear Rayleigh waves within an approach based on the Murnaghan model of description of nonlinearity of elastic deformation. The starting point is then the variant of Murnaghan potential [5,7] is chosen

$$\begin{gathered} W = (1/2)\lambda\left(u_{1,1}+u_{3,3}\right)^2 + \mu\left\{\left(u_{1,1}\right)^2 + \left(u_{3,3}\right)^2 + (1/2)\left(u_{1,3}+u_{3,1}\right)^2\right\} + \cdots \\ \cdots + (1/3)A\left[\left(u_{1,1}\right)^3 + \left(u_{3,3}\right)^3 + (3/4)\left(u_{1,3}+u_{3,1}\right)\left(u_{1,1}+u_{3,3}\right)\right] + \\ + B\left(u_{1,1}+u_{3,3}\right)\left[\left(u_{1,1}\right)^2 + \left(u_{3,3}\right)^2 + \left(u_{1,3}+u_{3,1}\right)^2\right] + (1/3)C\left(u_{1,1}+u_{3,3}\right)^3 . \end{gathered} \tag{1}$$

The next basic formulas represent the components of the Kirchhoff stress tensor, that are evaluated from (1) using the rule $t_{nm} = (\partial W/\partial u_{m,n})$.

2.2. Nonlinear wave equations

Substitution of these components into the motion equations $t_{11,1} + t_{31,3} = \rho\ddot{u}_1$; $t_{13,1} + t_{33,3} = \rho\ddot{u}_3$ gives two nonlinear equations of Lame type

$$\begin{gathered} \rho\ddot{u}_1 - (\lambda+2\mu)u_{1,11} - (\lambda+\mu)u_{3,13} - \mu u_{1,33} = \left[3(\lambda+2\mu) + 2(A+3B+C)\right]u_{1,1}u_{1,11} + \\ + \left[\mu + ((A/2)+B)\right]\left(u_{1,1}u_{1,33} + u_{1,3}u_{3,11} + u_{1,3}u_{3,33} + u_{3,3}u_{1,33}\right) + \\ + \left[2(\lambda+\mu) + \left(\frac{A}{2} + 3B + 2C\right)\right]\left(u_{1,1}u_{3,13} + u_{3,3}u_{3,13}\right) + \left[(\lambda+3\mu) + A + 2B\right]\left(u_{1,3}u_{1,33} + u_{3,1}u_{1,13}\right) + \\ + \left[(\lambda+2\mu) + (A/2) + B\right]\left(u_{3,1}u_{3,11} + u_{3,1}u_{3,33}\right) + \left[\lambda + 2(B+C)\right]u_{3,3}u_{1,11}. \end{gathered} \tag{2}$$

The second equation can be obtained from the first one by a change of indexes $1 \Leftrightarrow 3$.

Each equation involves 12 nonlinear summands. The total number of distinguishing sum-mands is 24. A similar increase of nonlinear summands is typical for cylindrical waves [8].

2.3. Approximate solutions (first two approximations)

The linear analysis is based on in-troduction of two new functions (potentials), which can be determined as solutions of the mutually independent linear wave equations. In the nonlinear case, the wave equations are nonlinear and coupled ones. To analyze the nonlinear case, let us introduce two potentials by the classical scheme [5,8]

$$u_1(x_1,x_3,t)=\left[\varphi(x_1,x_3,t)\right]_{,1}+\left[\psi(x_1,x_3,t)\right]_{,3};u_3(x_1,x_3,t)=\left[\varphi(x_1,x_3,t)\right]_{,3}-\left[\psi(x_1,x_3,t)\right]_{,1}. \quad (3)$$

In the first approximation, these potentials have the form corresponding to harmonic wave with frequency ω, wave number k_{Rlin} and decaying by the exponential law, when being moved away from the plane $x_1 = 0$ [4]

$$\varphi^{(1)}(x_1,x_3,t)=A_\varphi EE_L,\quad \psi^{(1)}(x_1,x_3,t)=A_\psi EE_T,$$
$$E=e^{i(k_{Rlin}x_1-\omega t)},\ E_L=e^{-k_\varphi x_3}=e^{-\sqrt{k_{Rlin}^2-k_L^2}},\ E_T=e^{-k_\psi x_3}=e^{-\sqrt{k_{Rlin}^2-k_T^2}}. \quad (4)$$

The final expressions for the second approximation solution are as follows (below only one potential is shown)

$$\varphi^{(2)}(x_1,x_3,t)=\frac{\rho}{4(\lambda+2\mu)}x_1x_3\left(A_\varphi\right)^2e^{2i(k_{Rlin}x_1-\omega t)}\left\{-\frac{1}{4k_L^2}\frac{\sqrt{k_{Rlin}^2-k_L^2}x_1+ik_{Rlin}x_3}{\left(\sqrt{k_{Rlin}^2-k_L^2}x_1\right)^2+\left(k_{Rlin}x_3\right)^2}\times\right.$$
$$\times M_\varphi^Le^{-2\sqrt{k_{Rlin}{}^2-k_L{}^2}x_3}-\frac{1}{4k_T^2}\frac{\sqrt{k_{Rlin}^2-k_T^2}x_1+ik_{Rlin}x_3}{\left(\sqrt{k_{Rlin}^2-k_T^2}x_1\right)^2+\left(k_{Rlin}x_3\right)^2}M_\varphi^Te^{-2\sqrt{k_{Rlin}{}^2-k_T{}^2}x_3}+ \quad (5)$$
$$\left.\frac{1}{\sqrt{\left(k_{Rlin}^2-k_L^2\right)\left(k_{Rlin}^2-k_T^2\right)}-k_{Rlin}^2}\frac{2x_1\left(\sqrt{k_{Rlin}^2-k_L^2}+\sqrt{k_{Rlin}^2-k_T^2}\right)+4ik_{Rlin}x_3}{4x_1^2\left(\sqrt{k_{Rlin}^2-k_L^2}+\sqrt{k_{Rlin}^2-k_T^2}\right)^2+16\left(k_{Rlin}x_3\right)^2}M_{\varphi\psi}^{LT}e^{-\sqrt{\left(k_{Rlin}{}^2-k_L{}^2\right)\left(k_{Rlin}{}^2-k_T{}^2\right)}x_3}\right\};$$

2.4. Some conclusions

Conclusion 1. The 2nd approximation includes the 2nd harmonic, that is, it includes the 2nd harmonic relative to the harmonic wave propagating in direction of the horizontal coordina te and to the exponential decay of the wave along the vertical coordinate. New harmonics have amplitudes, which depend nonlinearly on coordinates and then increase with increasing the time of Rayleigh wave propagation. As a result, the 1st harmonic distorts.
Conclu-sion 2. The dependence of amplitude of the 2nd harmonic on the squared corresponding amplitude of the 1st harmonic is standard for the used method within an approach that the nonlinearity is weak. It has some consequence relative to the 2nd harmonic distortion.
Conclusion 3. For the pure surface wave ($x_3 = 0$) the 2nd approximation is at beginning the zeroth, but for the near-the-surface wave this approximation can introduce the essential contribution into the wave picture.

3. Nonlinear elastic surface Stoneley wave

3.1. Statement of the problem

Consider the case, when two nonlinear elastic half-spaces with different densities and mechanical properties are separated by a plane and are joined according to the condition of full mechanical contact. Choose also the Cartesian coordinates $Ox_1\ x_2\ x_3$ and assume that an interface is the coordinate plane and is described by equa- tion $x_3 = 0$ [9]. Suppose further that the mechanical state does not depend on coordinate x_2 and the transverse horizontal displacement u_2 is absent. Then the problem is reduced to analysis of two half-planes (upper and lower) with the straight interface. This exhausts the geometrical part of statement of the problem on Stoneley wave. The mechanical part consists in using the equations of motion for the present case of absence of the transverse horizontal displacements.

This approach is based on introduction of nonlinearity of deformations of both half-planes by use of the Cauchy-Green nonlinear strain tensor and the Murnaghan potential

$$W^{U(L)} = \frac{1}{2}\lambda_{U(L)}\left(u_{1,1}^{U(L)} + u_{3,3}^{U(L)}\right)^2 + \mu_{U(L)}\left\{\left(u_{1,1}^{U(L)}\right)^2 + \left(u_{3,3}^{U(L)}\right)^2 + \frac{1}{2}\left(u_{1,3}^{U(L)} + u_{3,1}^{U(L)}\right)^2\right\} + \cdots \quad (6)$$

$$\cdots + B_{U(L)}\left(u_{1,1}^{U(L)} + u_{3,3}^{U(L)}\right)\left[\left(u_{1,1}^{U(L)}\right)^2 + \left(u_{3,3}^{U(L)}\right)^2 + \left(u_{1,3}^{U(L)} + u_{3,1}^{U(L)}\right)^2\right] + \frac{1}{3}C_{U(L)}\left(u_{1,1}^{U(L)} + u_{3,3}^{U(L)}\right)^3,$$

where the superscript U (upper) is used for the upper half-plane and the superscript L (lower) is used for the lower half-plane.

3.2 Nonlinear equations of motion

These equations are written through the nonsymmet-ric Kirchhoff stress tensor $t_{nm}^{U(L)}$

$$t_{11,1}^{U(L)} + t_{31,3}^{U(L)} = \rho_{U(L)}\ddot{u}_1^{U(L)};\ t_{13,1}^{U(L)} + t_{33,3}^{U(L)} = \rho_{U(L)}\ddot{u}_3^{U(L)}, \quad (7)$$

The Kirchhoff tensors are determined by the formula $t_{nm}^{U(L)} = \left(\partial W^{U(L)} / \partial u_{m,n}^{U(L)}\right)$.

Let us introduce the potentials like the case (3)

$$\begin{aligned} u_1^{U(L)}(x_1,x_3,t) &= \left[\varphi^{U(L)}(x_1,x_3,t)\right]_{,1} + \left[\psi^{U(L)}(x_1,x_3,t)\right]_{,3}; \\ u_3^{U(L)}(x_1,x_3,t) &= \left[\varphi^{U(L)}(x_1,x_3,t)\right]_{,3} - \left[\psi^{U(L)}(x_1,x_3,t)\right]_{,1}. \end{aligned} \quad (8)$$

Substitute representations (8) into (7) and obtain a system of two geometrically nonlinear equations relative to potentials

$$\begin{aligned} &\left[\rho_{B(H)}\ddot{\varphi}^{B(H)} - \left(\lambda_{B(H)} + 2\mu_{B(H)}\right)\Delta\varphi^{B(H)}\right]_{,1} + \left[\rho_{B(H)}\ddot{\psi}^{B(H)} - \mu_{B(H)}\Delta\psi^{B(H)}\right]_{,3} = \\ &= (\lambda_{B(H)} + 2\mu_{B(H)})\left(3\varphi_{,11}^{B(H)}\varphi_{,111}^{B(H)} - \psi_{,11}^{B(H)}\psi_{,133}^{B(H)} - \psi_{,33}^{B(H)}\psi_{,111}^{B(H)}\right) + \cdots \\ &\cdots + \lambda_{B(H)}\varphi_{,33}^{B(H)}\varphi_{,111}^{B(H)} + \mu_{B(H)}\left(\psi_{,11}\psi_{,111} + \psi_{,33}\psi_{,133}\right); \end{aligned}$$

$$\begin{aligned} &\left[\rho_{B(H)}\ddot{\varphi}^{B(H)} - \left(\lambda_{B(H)} + 2\mu_{B(H)}\right)\Delta\varphi^{B(H)}\right]_{,3} - \left[\rho_{B(H)}\ddot{\psi}^{B(H)} - \mu_{B(H)}\Delta\psi^{B(H)}\right]_{,1} = \\ &= \cdots + (\lambda_{B(H)} + 3\mu_{B(H)})\left(3\varphi_{,13}\varphi_{,133} + 2\psi_{,13}\psi_{,133} + \varphi_{,13}\varphi_{,111}\right) + \cdots \\ &\cdots + \lambda_{B(H)}\varphi_{,11}\varphi_{,333} + \mu_{B(H)}\left(\psi_{,33}\psi_{,333} + \psi_{,11}\psi_{,113}\right). \end{aligned} \quad (9)$$

3.3. Approximate solutions (first two approximations)

Apply now the method of success-ive approximations and choose the 1st approximation solution in the form of the classical linear representation of the Stoneley wave. Thus, the four potentials have the form of har-monic wave with frequency ω and wave number

k. These waves attenuate by the expo-nential law, when they move away from the plane $x_1 = 0$ (different for the upper and lo-wer half-planes)

$$\varphi^{(1)B}(x_1,x_3,t) = \tilde{A}_{\varphi}^{B} e^{-\sqrt{(k_S)^2-(k_L^B)^2}\,x_3} e^{i(k_S x_1-\omega t)} \equiv \tilde{A}_{\varphi}^{B} e^{-\beta_L^B x_3} e^{i(k_S x_1-\omega t)},\ldots$$

(10)

$$\ldots\psi^{(1)H}(x_1,x_3,t) = \tilde{A}_{\psi}^{H} e^{+\sqrt{(k_S)^2-(k_T^H)^2}\,x_3} e^{i(k_S x_1-\omega t)} \equiv \tilde{A}_{\psi}^{H} e^{+\beta_T^H x_3} e^{i(k_S x_1-\omega t)}.$$

The amplitudes $A_{\varphi}^{B(H)}(x_3), A_{\psi}^{B(H)}(x_3)$ have to fulfill the condition of attenuation with increasing the distance x_3 and the wave number $k_s = (\omega/v_s)$ has to be determined from the additional considerations.

The second approximation is as follows (only one potential is shown)

$$\begin{aligned}
\varphi^{B(H)(2)}(x_1,x_3,t) &= \frac{1}{4} x_1 x_3 \left(A_{\varphi}^{B(H)(1)}\right)^2 E^2 \frac{\rho_{B(H)}}{\lambda_{B(H)}+2\mu_{B(H)}} \times \\
&\times \Bigg\{ -\frac{1}{4\left(k_L^{B(H)}\right)^2} \frac{k_{\varphi}^{B(H)} x_1 + i k_{lin}^{B(H)} x_3}{\left(\left(k_{\varphi}^{B(H)} x_1\right)^2 + \left(k_{lin}^{B(H)} x_3\right)^2\right)^2} M_{\varphi}^{B(H)L} \left(E_L^{B(H)}\right)^2 - \\
&- \frac{1}{4\left(k_T^{B(H)}\right)^2} \frac{k_{\psi} x_1 + i k_{lin} x_3}{\left(\left(k_{\psi}^{B(H)} x_1\right)^2 + \left(k_{lin}^{B(H)} x_3\right)^2\right)^2} M_{\varphi}^{B(H)T} \left(E_T^{B(H)}\right)^2 + \\
&+ \frac{1}{k_{\varphi}^{B(H)} k_{\psi}^{B(H)} - \left(k_{lin}^{B(H)}\right)^2} \frac{2x_1\left(k_{\varphi}+k_{\psi}\right) + 4 i k_{lin} x_3}{\left(\left[2x_1\left(k_{\varphi}^{B(H)}+k_{\psi}^{B(H)}\right)\right]^2 + 16\left(k_{lin}^{B(H)} x_3\right)^2\right)^2} M_{\varphi\psi}^{B(H)LT} E_L^{B(H)} E_T^{B(H)} \Bigg\}.
\end{aligned} \tag{11}$$

3.4. Some conclusions

Conclusion 1. The 2nd approximation solutions include the 2nd harmonic relative the 1st (linear) approximation, that is, it includes the 2nd harmonic relative to harmonic waves propagating in direction of the horizontal coordinate and to the exponential decay of the wave along the vertical coordinate. New harmonics have amplitudes, which depend non-linearly on coordinates and then increase with increasing the Stoneley wave propagation time. As a result, the 1st harmonic distorts.

Conclusion 2. The characteristic feature of non- linearity is dependence of the 2nd approximation on squared amplitudes and coordinates. This means that the 2nd harmonic can dominate with time.

4. Nonlinear elastic surface Love wave

4.1. Statement of the problem

Consider the problem on the Love elastic wave in the clas- sical statement under additional assumption on nonlinearity of deformation process. From the geometrical point of view, the nonlinear problem statement coincides in many parts with the linear one and consists in that the system is considered: the layer of constant thickness defined by condition $-h \le x_1 \le 0$ and the upper half-space $x_1 \ge 0$ are described by Cartesian coordinates $Ox_1x_2x_3$ (the abscissa axis is directed deep into the half-space, the ordinate axis is directed along the interface) [1].

From point of view of mechanics, the problem includes some initial assumptions: (1) It is supposed that the half-space and the layer are filled by nonlinearly elastic materials with distinguishing properties (further, the quantities describing the layer and half-space are assigned the indexes L and H, respectively). (2) Materials are deformed by the Murnaghan model and, therefore, the properties include density $\rho_{L(H)}$ and five elastic constants $\lambda_{L(H)}$, $\mu_{L(H)}$, $\mathrm{A}_{L(H)}$, $\mathrm{B}_{L(H)}$, $\mathrm{C}_{L(H)}$. (3) It is supposed also that the half-space and the layer are in con ditions of full mechanical contact (equality of displacements and stresses at the interface) and the layer lower plane $x_1 = -h$ is free of stresses.

The possibility of propagation of the harmonic plane vertically polarized transverse wave is studied under the condition of absence of displacements u_1, u_2 in longitudinal and horizontal directions, respectively.

The form of Murnaghan potential corresponding to the stated problem is as follows

$$W = \frac{1}{4}\lambda\left[\left(u_{3,1}\right)^2 + \left(u_{3,2}\right)^2\right]^2 + \mu\left[\frac{1}{2}\left(u_{3,1}\right)^2 + \frac{1}{2}\left(u_{3,2}\right)^2 + \frac{1}{4}\left(u_{3,1}\right)^4 + \frac{1}{4}\left(u_{3,2}\right)^4 + \frac{1}{4}\left(u_{3,1}u_{3,2}\right)^2\right] + \cdots$$

$$+\frac{1}{8}B\left[2\left(u_{3,1}\right)^2 + 2\left(u_{3,2}\right)^2 + \left(u_{3,1}\right)^4 + \left(u_{3,2}\right)^4 + \left(u_{3,1}u_{3,2}\right)^2\right]\left[\left(u_{3,1}\right)^2 + \left(u_{3,2}\right)^2\right] + \cdots \qquad (12)$$

The main feature of representation (12) is only occurrence even degrees of nonzeroth components $u_{3,1}$, $u_{3,2}$: the 2[nd] degrees (corresponding to the linear approach), the 4[th] degrees (corresponding to the cubically nonlinear approach), and the 6[th] degrees (corresponding to nonlinearity of the 5[th] order) are presented in (12).

4.2. Nonlinear wave equation

The stress tensor is determined by the classical formula $t_{ik} = (\partial W/\partial u_{k,i})$. Only two t_{13}, t_{23} of nine components of the stress tensor are nonzeroth.

Note the goal is stated to analyze the possibility of propagation in direction Ox_1 (at the neighborhood of interface) of the wave with unknown amplitude $\hat{u}_3^{L(H)}\left(x_1\right)$ and wave number k. Then the wave can be represented in the form $u_3^{L(H)} = \hat{u}_3^{L(H)}\left(x_1\right)e^{i\left(kx_2 - \omega t\right)}$.

If the requirement is formulated that the wave is localized near the interface, that is, it has the maximal amplitude at the interface and the amplitude decays essentially with increase of the absolute values of x_1, then the statement in the framework of linear theory of elasticity corresponds to the nonlinear statement of the problem on Love wave. Two of three equations of motion are degenerated into identities in this problem, whereas the third one has a form $t_{13,1} + t_{23,2} = \rho\ddot{u}_3$, which can be transformed into the next nonlinear wave equation

$$\rho\ddot{u}_3 - \mu\left(u_{3,11} + u_{3,22}\right) = T_1\left(u_{3,1}\right)^2 u_{3,11} + T_2\left(u_{3,2}\right)^2 u_{3,11} + T_1\left(u_{3,2}\right)^2 u_{3,22} +$$

$$+T_2\left(u_{3,1}\right)^2 u_{3,22}+4T_2 u_{3,1}u_{3,2}u_{3,12}+F_1\left(u_{3,1}\right)^4 u_{3,11}+F_1\left(u_{3,2}\right)^4 u_{3,22}+$$
$$+F_2\left(u_{3,2}\right)^4 u_{3,11}+F_2\left(u_{3,1}\right)^4 u_{3,22}+F_3\left(u_{3,1}\right)^3 u_{3,2}u_{3,12}+F_3 u_{3,1}\left(u_{3,2}\right)^3 u_{3,12}+ \qquad (13)$$
$$+F_4\left(u_{3,1}\right)^2\left(u_{3,2}\right)^2 u_{3,11}+F_4\left(u_{3,2}\right)^2\left(u_{3,1}\right)^2 u_{3,22},$$

$$T_1=3\left[(\lambda+\mu)+(1/4)A+(1/2)B\right],\ T_2=(1/2)\left[(\lambda+\mu)+A+B\right],\ F_1=(5/4)(A+B+C),$$
$$F_2=A+(1/4)B+(1/4)C,\ F_3=2A+(3/2)B+2C,\ F_4=(3/4)(2A+B+2C).$$

The equation (13) contains the nonlinear summands of the 3rd (five summands) and the 5th (eight summands) orders. This feature of absence of even order summands is the consequence of the problem statement. A similar situation was arisen in the study of pla-ne transverse wave in the 3rd approximation [7].

Let us save in (13) only the cubic nonlinearity and search the solution by the method of successive approximations.

4.3. Approximate solutions (first two approximations)

The solution in the framework of first two approximations is as follows

$$\text{for}\ \ x_2\in(-\infty,\infty),\ x_1\in[0;\infty):\qquad u_3^H\left(x_1,x_2,t\right)=u_3^{H(1)}+u_3^{H(2)}= \qquad (14)$$

$$=L_H e^{-\sqrt{\left[1-\left(v/v_T^H\right)^2\right]}kx_1}e^{i(kx_2-\omega t)}+\frac{x_1x_2\left[\sqrt{1-\left(v/v_T^H\right)^2}\,x_2+ix_1\right]}{\left[1-\left(v/v_T^H\right)^2\right]\left(x_2\right)^2+\left(x_1\right)^2}K_H^{(2)}e^{-3\beta_H kx_1}e^{i3(kx_2-\omega t)};$$

$$\text{for}\ \ x_2\in(-\infty,\infty),\ x_1\in[-h;0]:\quad u_3^L\left(x_1,x_2,t\right)=u_3^{L(1)}+u_3^{L(2)}= \qquad (15)$$

$$=\left\{\left[-L_H\frac{\mu_\Pi}{\mu_C}\frac{\sqrt{1-\left(v/v_T^H\right)^2}}{\sqrt{\left(v/v_T^L\right)^2-1}}+x_1x_2\frac{\left(L_H\right)^3k^3}{24}K_{1s}\right]\sin\sqrt{\left[\left(v/v_T^L\right)^2-1\right]}kx_1+\right.$$
$$\left.+\left[L_\Pi+x_1x_2\frac{\left(L_H\right)^3k^3}{24}K_{1c}\right]\cos\sqrt{\left[\left(v/v_T^L\right)^2-1\right]}kx_1\right\}e^{i(kx_2-\omega t)}+$$
$$\left.+K_{3s}\sin 3\sqrt{\left[\left(v/v_T^L\right)^2-1\right]}kx_1+K_{3c}\cos 3\sqrt{\left[\left(v/v_T^L\right)^2-1\right]}kx_1\right\}e^{3i(kx_2-\omega t)}.$$

The solutions (14) and (15) contain the unknown parameters: amplitude L_H and wa-ve number k. If the amplitude can be assumed to be arbitrary according to the fact that the Love wave is the running surface wave, then the wave number should be determined from the boundary conditions. But for the nonlinear statement these conditions are alrea-dy nonlinear what enables allowance for effect of nonlinearity on the wave number.

4.4. Some conclusions

Conclusion 1. The wave is dispersive one, because analysis of boundary conditions testifies the nonlinear dependence of phase velocity v on wave number k: (1) For zero value of wave number (for infinite wave length), the velocity is equal to the phase velocity of plane transverse waves in the half-space v_T^H . (2) With increasing the wave number, the velocity decreases.

Conclusion 2. The 2nd approximation includes the 3rd harmonic relati-ve the 1st (linear) approximation, that is, it includes the 3rd harmonic relative to the harmo-nic wave propagating along the horizontal coordinate and to the exponential decay of the wave along the ver tical coordinate. These new harmonics have amplitudes, which depend nonlinearly on coordinates and then increase with increasing the time of Love wave propa-gation. As a result, the 1st harmonic distorts.

Conclusion 3. The dependence of amplitu- des of the 2nd harmonic on the cubed corresponding amplitudes of the 1st harmonic is standard for the used method within an approach that the nonlinearity is weak [7].

5. Nonlinear elastic surface Mozhaev wave

Analysis of the nonlinear elastic surface wave propagating within the condition of antiplane strain state in the half-plane (in contrast to the case of Love wave, where presence of layer is predicted) is proposed in [3]. In [2], such a virtual wave was called the Mozhaev wave. Unfortunately, the presented in this lecture four cases of harmonic and solitary waves do not certificate existence of such a wave.

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