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EXTREMAL LENGTH AND DIRICHLET PROBLEM ON KLEIN SURFACES

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Abstract. The object of this paper is to extend the method of extremal length to Klein surfaces by solving conformally invariant extremal problems on the complex double. Within this method we define the extremal length, the extremal distance, the conjugate extremal distance, the modulus, the reduced extremal distance on a Klein surface and we study their dependences on arcs.

Keywords: Klein surface, extremal length, extremal distance.

Mathematics Subject Classification: 30F50, 35J05, 31A15.

1. INTRODUCTION

In this paper we extend the method of extremal length to Klein surfaces. The technique is to apply the classical method to the complex double of a Klein surface, which is a Riemann surface endowed with an antianalytic involution (see [4]).

Practically, using a technique in which objects on Klein surfaces are lifting to symmetric objects on the complex double, we define some numerical dianalytic invariants related to a Klein surface X and a family of rectifiable arcs on X and we study their dependence on the corresponding arcs. These invariants are global lengths and are obtained by solving conformally invariant extremal problems on the complex double. For this purpose we define the extremal length, the extremal distance, the conjugate extremal distance, the reduced extremal distance and the modulus on a Klein surface.

Their usefulness result from the fact that they satisfy a principle of majorization and certain extremal lengths or their upper and lower bounds can be computed in terms of classical conformal invariants. Ahlfors [2] introduced a formula that calculates extremal length through the Dirichlet integral of a certain harmonic measure. This formula expressed the capacity of a set in the plane in terms of extremal length. A classical result for harmonic functions shows that every closed set of zero newtonian capacity is removable for the class of bounded harmonic functions.

Serrin [20] has generalized this theorem and found the notion of capacity useful in connection with the question of removable singularities of solutions to linear second order elliptic partial differential equations and proved similar results for a class of nonlinear equations (see [21]).

The method of extremal length is based on the well known length-area principle and had a huge contribution to the development of the theory of quasiconformal mappings which is closely related to the nonlinear elasticity theory (see [6] and [7]).

The generalization of the method to Riemann surfaces, by introducing the extremal length as a conformally invariant measure of arcs families, is due to A. Beurling and was developed in collaboration with L. Ahlfors (see [2]). Teichmuller considered special cases of extremal problems on nonorientable surfaces by passing to the complex double. He introduced the notions of meromorphic functions, n-differentials and divisors on bordered nonorientable Riemann surfaces (see [22]). Spencer and Schiffer [18] extended the investigation of finite Riemann surfaces from the point of view of functional analysis. Basic function theory on Klein surfaces and the relation between compact Klein surfaces and real algebraic functions are developed in [4]. Morphisms of Klein surfaces are studied in [4] and [5]. Boundary value problems on nonorientable surfaces are studied in [9] and [16].

2. PRELIMINARIES

A connected topological Hausdorff space X is a surface with boundary if every point $\widetilde{P} \in X$ has an open neighborhood $\widetilde{U}$, which is homeomorphic to a relatively open subset of the closed upper half-plane. A homeomorphism $h : \widetilde{U} \to h(\widetilde{U})$ is called a local parameter at the point $\widetilde{P} \in \widetilde{U}$. The pair $(\widetilde{U}, h)$ is called a chart.

Let A and B be nonempty open sets in the closed upper half-plane. A continuous map of A into B is analytic on A (resp., antianalytic on A) if it extends to an analytic (resp., antianalytic) function on some neighborhood of A in $\mathbb{C}$ into $\mathbb{C}$. If f or the complex conjugate of f is analytic on each connected component of the set A, then f is called dianalytic on A (see [4]).

An atlas of the surface X is a family $A = \{(\widetilde{U}_i, h_i) \,|i \in I\}$ of charts, where $(\widetilde{U}_i)_{i\in I}$ is an open cover of X. An atlas is dianalytic if all of its transition functions are dianalytic. Two dianalytic atlases are called equivalent if their union is a dianalytic atlas as well. An equivalence class A of dianalytic atlases of X is called a dianalytic structure on X.

A Klein surface is a surface with boundary endowed with a dianalytic structure A and will be denoted with X. Observe that a classical Riemann surface is an orientable Klein surface with empty boundary.

In our study we are using the complex double of a Klein surface. A study of this concept can be found in [18] and some of its applications are developed in [10] and [19].

We present a theorem which relates a Klein surface to its complex double. We refer to [4] for the proof and more details.

Theorem 2.1. *Let X be a Klein surface and O_2 the complex double of the surface X. If $\pi : O_2 \to X$ is the double cover of the Klein surface X by O_2, then there is an antianalytic involution $k : O_2 \to O_2$, with $\pi \circ k = \pi$. More, X is dianalytically equivalent with $O_2/\langle k\rangle$, where $\langle k\rangle$ is the group generated by k. Also, given a Riemann surface O_2 and an antianalytic involution k, the orbit space $O_2/\langle k\rangle$ can be endowed with a unique structure of Klein surface, such that $f : O_2 \to O_2/\langle k\rangle$ is a morphism of Klein surfaces.*

Thus we can identify the Klein surface X with the orbit space $O_2/\langle k\rangle$. For more details see [4].

The involution k is called a symmetry of O_2.The pair (O_2, k) has been called a symmetric Riemann surface and the two corresponding points P and $k(P)$, with $P \in O_2$ are called symmetric points.

By definition, a set Δ of O_2 is called symmetric if $k(\Delta) = \Delta$.

A function f defined on a symmetric set is called a symmetric function if $f = f \circ k$.

To simplify the notation, we identify the points of O_2, respectively X, with their images on $\mathbb{C}$ from the corresponding local parameters, with respect to the relation between the dianalytic atlas on X and the analytic atlases on O_2.

Suppose $\widetilde{\Delta}$ is a parametric disk of X.Then $\pi^{-1}(\widetilde{\Delta}) = \Delta \cup k(\Delta)$ is a pair of symmetric disks of O_2. If z is the local parameter on Δ, then $k(z)$ is the local parameter on $k(\Delta)$ and $\widetilde{z} = \pi(z) = \{z, k(z)\}$ is the local parameter on $\widetilde{\Delta}$.

If $F : X \to \overline{\mathbb{C}}$ is a complex function on X, that can take the value ∞ only on finite sets, then its lifting f to O_2 is given by

$$f(z) = f(k(z)) = F(\widetilde{z}),\ z \in O_2,\ \widetilde{z} = \pi(z). \tag{2.1}$$

More, to any function $g : O_2 \to \overline{\mathbb{C}}$, we can associate a function $f = g + g \circ k$ which is a symmetric function on O_2 and (2.1) defines a function F on X.

Any Riemann surface O_2 of class C^1 is endowed with a Riemannian metric determined by the line element

$$ds = \rho\,|dz + \mu d\overline{z}|\ ,$$

where $\rho \geq 0$. If μ is identically zero, then the metric

$$ds(z) = \rho\,|dz|$$

and the local parameter z are called isothermal.

We modify this metric and define a symmetric metric on O_2 by

$$d\sigma = \frac{1}{2}\,(ds + ds \circ k)\,.$$

Then

$$d\Sigma(\widetilde{z}) = d\sigma(z) = d\sigma(k(z)), \quad \widetilde{z} = \pi(z) \in X$$

is a metric on X. The metric $d\Sigma$ is invariant with respect to the group of conformal or anticonformal transition functions of X.

The area element dA on the Klein surface X is given by

$$dA(\widetilde{z}) = dA(z) = dA(k(z)),$$

where dA is the symmetric area element on O_2. For a complete treatment of the integration theory on Klein surfaces, see [8].

It is known that the isothermal metric ds defines a natural analytic structure on O_2. Similar to the orientable case, the isothermal metric $d\sigma$ defines a dianalytic structure on the Klein surface X. See [3] for details.

Let γ be a σ-rectifiable Jordan arc γ, parametrized in terms of the arc σ-length. Let $\gamma : z = z(s) = x(s)+iy(s)$, $s \in [0, l]$ parameterized by the σ- arc length l of γ. The unit inward normal vector to γ at $z(s)$ is $n_\sigma = (-\frac{dy}{d\sigma}, \frac{dx}{d\sigma})$ and we denote by $\frac{\partial}{\partial n_\sigma}$ the inward normal derivative, with respect to the symmetric metric $d\sigma$. A transformation formula for the normal derivative with respect to $d\sigma$ can be found in [9].

Let O_2 be a region in the complex plane, bounded by a finite number of analytic Jordan curves. Then $\overline{O_2} = O_2 \cup \partial O_2$ can be conceived as a bordered Riemann surface (see [18]). Since X is dianalytically equivalent with $O_2/\langle k\rangle$, there is a pair of sheets (D_1, D_2) of O_2 over X, such that $O_2 = D_1 \cup D_2$ where $D_2 = k(D_1)$. The sets $D_1$and D_2 are complete sets of representatives of the factor set $O_2/\langle k\rangle$ .We use the representation of O_2 as a symmetric region D.

3. THE EXTREMAL LENGTH

3.1. THE EXTREMAL LENGTH ON THE COMPLEX DOUBLE

Let D be a symmetric region bounded by a finite number of σ-rectifiable Jordan curves and $\Gamma \cup k(\Gamma)$ a symmetric family whose elements are σ-rectifiable arcs in D. We define a number which is invariant under conformal mappings, with respect to the lengths of the arcs and the family of Riemannian metrics which are also conformal to the euclidean metric.

Let Φ be a linear density on D. Then Φ is defined such that in terms of the local variables the metric $|\varphi(z)|\,|dz|$ is conformally invariant. Without loss the generality, we can assume that $\varphi \geq 0$ and φ is lower semicontinuous. For more details, see [3].

The Φ-lengths of two arcs γ and $k \circ \gamma$, where $\gamma \in \Gamma$, that is their lengths with respect to the Φ-metric are

$$l_\Phi(\gamma) = \int_\gamma \varphi(z)\,|dz| \quad \text{and} \quad l_\Phi(k(\gamma)) = \int_\gamma \widehat{\varphi}(k(z))\,|dk(z)|\,,$$

where $\widehat{\varphi}(k(z))\,|dk(z)|$ is the local representation of the linear density Φ, in $k(\Delta)$. For details, see [8].

Remark 3.1. We supposed that γ is contains in a parametric disk Δ. We identifed γ and $k \circ \gamma$, with their images in the complex plane, from the corresponding charts.When Δ is an open set, a σ-rectifiable curve γ which does not go through any critical point can be subdivided into intervals each one of which lies in a parametric disk. Therefore, we can consider Γ a nonempty family of curves, where each curve consists of countably many arcs. For a treatment of the notion of extremal length for a family of chains on a Riemann surface, see [15].

We can introduce a conformal parameter w, in terms of which the representation of Φ is identically equal to one. The natural parameter is given by the integral $w = \psi(z) = \int \varphi(z)dz$. The arc γ is mapped by ψ onto an arc γ'. The Φ-length of γ can be computed by means of the differential

$$|dw| = \varphi(z)\,|dz|\,,$$

which is called the length element of the Φ-metric. Then in terms of the local parameter w, we get

$$l_\Phi(\gamma) = \int\limits_{\gamma'} |dw| = \int\limits_{\gamma} \varphi(z)\,|dz|\,.$$

Thus, the Φ-length of γ is the Euclidean length of γ'.

The corresponding area element is $\varphi(z)dxdy$, which is invariant under a change of parameter on D. The Φ-area of D, namely the total area of D, is defined by

$$A_\Phi(D) = \iint\limits_D (\varphi(z))^2\,dxdy.$$

Following Ahlfors and Sario [3], the minimum Φ-length of the family Γ, is defined by

$$l_\Phi(\Gamma) = \inf_{\gamma\in\Gamma} l_\Phi(\gamma).$$

Proposition 3.2. *Under a conformal mapping symmetric regions are transformed into symmetric regions.*

Proof. Let D be a symmetric region bounded by a finite number of σ-rectifiable Jordan arcs and k a symmetry of D. Let $z' = h(z)$ be a conformal mapping which takes D into a region D'. We define $k' : D' \to D'$, such that $h(k(z)) = k'(h(z))$, namely $k' = h \circ k \circ h^{-1}$. Then k' is an anticonformal involution, without fixed points. Thus, k' is a symmetry of D'. For more details, see [9]. □

The extremal length of Γ in D is defined by

$$\lambda_D(\Gamma) = \sup_\Phi \frac{l_\Phi(\Gamma)^2}{A_\Phi(D)},$$

where the supremum is over all lower semicontinuous densities Φ which are not identically 0.

The extremal length satisfy the comparison principle (see [3]):

Proposition 3.3. *Let Γ_1, Γ_2 be families of curves on D. If every $\gamma_1 \in \Gamma_1$ contains a $\gamma_2 \in \Gamma_2$, then $\lambda_D(\Gamma_1) \geq \lambda_D(\Gamma_2)$.*

More, are valid two basic laws of composition:

Proposition 3.4. *Let D_1 and D_2 be disjoint open sets in D.Let Γ_1, Γ_2 consist of curves in D_1, D_2 respectively,and Γ is a family of curves in D.*

1. *If every $\gamma \in \Gamma$ contains a $\gamma_1 \in \Gamma_1$ and a $\gamma_2 \in \Gamma_2$, then*

$$\lambda_D(\Gamma) \geq \lambda_D(\Gamma_1) + \lambda_D(\Gamma_2).$$

2. *If every $\gamma_1 \in \Gamma_1$ and every $\gamma_2 \in \Gamma_2$ contains a $\gamma \in \Gamma$, then*

$$\frac{1}{\lambda_D(\Gamma)} \geq \frac{1}{\lambda_D(\Gamma_1)} + \frac{1}{\lambda_D(\Gamma_2)}.$$

A density Φ_s is extremal for the family Γ in D, if $\frac{l_{\Phi_s}(\Gamma)^2}{A_{\Phi_s}(D)} = \lambda_D(\Gamma)$. To compute an extremal length means to guess what the extremal metric should be and then to prove that the metric is indeed extremal.

By analogy, with the extremal length of Γ, since k is an involution, we define the extremal length of the family $k(\Gamma)$, by

$$\lambda_D(k(\Gamma)) = \sup_{\Phi} \frac{l_\Phi(k(\Gamma))^2}{A_\Phi(D)}.$$

Since the extremal length of Γ in O_2 is not symmetric, the families Γ and $k(\Gamma)$ may have different extremal lengths.

We define the length $\lambda_D^{(k)}(\Gamma)$ of Γ on D by

$$\lambda_D^{(k)}(\Gamma) = \lambda_D(\Gamma) + \lambda_D(k(\Gamma)). \tag{3.1}$$

Proposition 3.5. *The length $\lambda_D^{(k)}(\Gamma)$ is symmetric, namely is invariant with respect to k.*

Proof. Since k is an involution, $\lambda_D^{(k)}(\Gamma) = \lambda_D^{(k)}(k(\Gamma))$. □

The length $\lambda_D^{(k)}(\Gamma)$ is called the symmetric extremal length of Γ.

Remark 3.6. By Proposition 3.5, we obtain that the families Γ and $k(\Gamma)$ have the same symmetric extremal length.

Proposition 3.7. *The symmetric extremal length of Γ is a conformal invariant.*

Proof. Let D be a symmetric region bounded by a finite number of σ-rectifiable Jordan arcs and k a symmetry of D. Let $z' = h(z)$ be a conformal mapping which takes D into a symmetric region D' endowed with the symmetry k', such that $h(k(z)) = k'(h(z))$. Let $\Gamma \cup k(\Gamma)$ be a symmetric family, whose elements are σ-rectifiable arcs in D. Then

h takes Γ into Γ' and $k(\Gamma)$ into $k'(\Gamma')$.We can define Φ' such that in terms of the local variables $\varphi\,|dz| = \varphi'\,|dz'|$ .We obtain $l_\Phi(\gamma) = l_{\Phi'}(\gamma')$ and $l_\Phi(k(\gamma)) = l_{\Phi'}(k'(\gamma'))$ for any $\gamma \in \Gamma, \gamma' = h(\gamma)$. Then, $l_\Phi(\Gamma) = l_{\Phi'}(\Gamma')$ and $l_\Phi(k(\Gamma)) = l_{\Phi'}(k'(\Gamma'))$. Similarly, $A_\Phi(D) = A_{\Phi'}(D')$. Thus it results that $\lambda_D^{(k)}(\Gamma) = \lambda_{D'}^{(k')}(\Gamma')$. □

Using different normalizations, we can obtain equivalent definitions for the extremal length of Γ on D (see [3]). The usual normalization considers admissible densities. By definition, a density Φ is admissible for the family $\Gamma \cup k(\Gamma)$, if $l_\Phi(\gamma) \geq 1$ and $l_\Phi(k(\gamma)) \geq 1$, for every $\gamma \in \Gamma$.

The extremal length of Γ is defined by

$$\frac{1}{\lambda_D(\Gamma)} = \inf_\Phi A_\Phi(D),$$

where the infimum is taken over all admissible densities Φ (see [11]).

The quantity $M(\Gamma) = \frac{1}{\lambda_D(\Gamma)}$ is referred to as the modulus of Γ.It is known that the modulus is a conformal invariant.

Similarly, the modulus of $k(\Gamma)$ is

$$M(k(\Gamma)) = \frac{1}{\lambda_D(k(\Gamma))} = \inf_\Phi A_\Phi(k(D)),$$

where the infimum is taken over all admissible densities Φ.

Proposition 3.8. *The modulus $M(\Gamma)$ is symmetric, namely is invariant with respect to k.*

Proof. Since D is a symmetric region, we obtain $M(\Gamma) = M(k(\Gamma))$. □

3.2. THE EXTREMAL LENGTH ON KLEIN SURFACES

The Klein surface X is the factor manifold of the symmetric Riemann surface O_2 with respect to the group $\langle k \rangle$. Then, by identifying the corresponding symmetric points of D we obtain a region Ω bounded by a finite number of Σ-rectifiable Jordan arcs. We consider a family $\widetilde{\Gamma}$ whose elements are Σ-rectifiable arcs in Ω. A Σ-rectifiable arc $\widetilde{\gamma}$ in $\widetilde{\Gamma}$, has two lifts from D. If the initial point of $\widetilde{\gamma}$ is $\widetilde{z}_0 = \{z_0, k(z_0)\}$ and if γ is the lift of $\widetilde{\gamma}$ on D from z_0, then $k \circ \gamma$ is the lift of $\widetilde{\gamma}$ on $k(D)$ from $k(z_0)$. Then $\pi^{-1}(\widetilde{\gamma}) = \gamma \cup k(\gamma)$ and $\pi^{-1}(\widetilde{\Gamma}) = \Gamma \cup k(\Gamma)$, where Γ is the set of the σ-rectifiable arcs γ in D, for which there exists Σ-rectifiable arcs $\widetilde{\gamma}$ in $\widetilde{\Gamma}$, such that γ is the lift of $\widetilde{\gamma}$ on D.

The extremal length of $\widetilde{\Gamma}$ in X is defined by

$$\lambda_\Omega(\widetilde{\Gamma}) = \lambda_D^{(k)}(\Gamma) = \lambda_D^{(k)}(k(\Gamma)).$$

Remark 3.9. By Proposition 3.5, it follows that $\lambda_\Omega(\widetilde{\Gamma})$ is well-defined on Ω.

By Propositions 3.5 and 3.7 we get

Proposition 3.10. *The extremal length of $\widetilde{\Gamma}$ is a dianalytic invariant.*

Similar to Riemann surfaces, we state a comparison principle (see [3]).

Proposition 3.11. *Let $\widetilde{\Gamma_1}$ and $\widetilde{\Gamma_2}$ be families of curves in Ω, such that every $\widetilde{\gamma_1} \in \widetilde{\Gamma_1}$ contains a $\widetilde{\gamma_2} \in \widetilde{\Gamma_2}$. Then $\lambda_\Omega(\widetilde{\Gamma_1}) \geq \lambda_\Omega(\widetilde{\Gamma_2})$.*

Proof. We prove that $\lambda_{\Omega_1}(\widetilde{\Gamma_2}) \geq \lambda_{\Omega_2}(\widetilde{\Gamma_2})$, where Ω_1 and Ω_2 are subregion of Ω. Let $\pi^{-1}(\Omega_i) = D_i \cup k(D_i), i = 1, 2$ be the symmetric regions on the double cover. By hypothesis, any $\widetilde{\gamma_1} \in \widetilde{\Gamma_1}$ contains an arc $\widetilde{\gamma_2} \in \widetilde{\Gamma_2}$. Let γ_1 be the lift of $\widetilde{\gamma_1}$ on $D = \pi^{-1}(\Omega)$.Then, γ_1 contains the lift γ_2 of $\widetilde{\gamma_2}$ on D. Since the minimum lengths $l_\Phi(\Gamma_1)$ and $l_\Phi(\Gamma_2)$ are compared with the total area of D, using the comparison principle for Riemann surfaces (see [3]), we get $\lambda_D(\Gamma_1) \geq \lambda_D(\Gamma_2)$. Because k is an involution, we obtain the similar relation $\lambda_D(k(\Gamma_1)) \geq \lambda_D(k(\Gamma_2))$. Then, for the symmetric extremal lengths, we get $\lambda_D^{(k)}(\Gamma_1) \geq \lambda_D^{(k)}(\Gamma_2)$, which implies $\lambda_\Omega(\widetilde{\Gamma_1}) \geq \lambda_\Omega(\widetilde{\Gamma_2})$. □

The modulus of $\widetilde{\Gamma}$ is defined by $M(\widetilde{\Gamma}) = M(\Gamma) = M(k(\Gamma))$.

By Proposition 3.8, it follows that $M(\widetilde{\Gamma})$ is well defined on Ω.

4. THE EXTREMAL DISTANCE

4.1. THE EXTREMAL DISTANCE ON THE DOUBLE COVER

Let D be a symmetric region bounded by a finite number of σ-rectifiable Jordan arcs. We consider E_1 and E_2 two nonvoid disjoint sets on D. Take Γ the family of connected arcs in D, which join E_1 and E_2.

The extremal length of the family Γ is called the extremal distance of E_1 and E_2 in D and is denoted by $\lambda_D(E_1, E_2)$. Then, by definition

$$\lambda_D(E_1, E_2) = \lambda_D(\Gamma).$$

An extensive study of the extremal distance ona Riemann surface is developed in [3].

It is easy to see that $k(\Gamma)$ represents the family of connected arcs in D, which join $k(E_1)$ and $k(E_2)$. Therefore,

$$\lambda_D(k(E_1), k(E_2)) = \lambda_D(k(\Gamma)).$$

Next, we define the symmetric extremal distance of E_1 and E_2 on D, by

$$\lambda_D^{(k)}(E_1, E_2) = \lambda_D^{(k)}(\Gamma). \tag{4.1}$$

Proposition 4.1. $\lambda_D^{(k)}(E_1, E_2) = \lambda_D^{(k)}(k(E_1), k(E_2))$.

Proof. Since k is an involution, using (3.5) we obtain

$$\lambda_D^{(k)}(E_1, E_2) = \lambda_D^{(k)}(\Gamma) = \lambda_D^{(k)}(k(\Gamma)) = \lambda_D^{(k)}(k(E_1), k(E_2)).$$ □

Remark 4.2. By Proposition 3.7 and Proposition 4.1, we obtain that the symmetric extremal distance is a dianalytical invariant.

The extremal distance is associated with another conformal invariant, the Dirichlet integral.

We consider the surface O_2 given as the interior of the compact bordered Riemann surface $\overline{O_2}$. We choose a boundary neighborhood $O_2' \subset O_2$, with compact complement.

Following Ahlfors and Sario ([3]) for a continuous function f on the boundary of O_2', there exists a unique harmonic function in O_2', which has the boundary values f on the boundary of O_2' and whose normal derivative vanishes on the border of O_2.

Without loss of generality, we can assume that the sets E_1 and E_2 consist of a finite number of arcs which are contained in the boundary of D (see [3]).

Let u be the unique bounded harmonic function in $D - (E_1 \cup E_2)$, which is 0 on E_1, 1 on E_2 and whose normal derivative vanishes on $\partial D - (E_1 \cup E_2)$.

The existence and uniqueness of u results from the construction of the normal operator L_0 (see [3]).

The Dirichlet integral of u over the region D is defined by

$$D(u) = \iint\limits_D |\nabla u|^2 \, dxdy.$$

According to a result due to Ahlfors and Sario [3], we find:

Proposition 4.3. *The extremal distance, $\lambda_D(E_1, E_2)$, between E_1 and E_2 is equal to the reciprocal of the Dirichlet integral $D(u)$.*

Thus the "extremal metric", namely a Φ for which the supremum is attained in the definition of the corresponding extremal length is given by $\varphi(z)\,|dz| = |\nabla u|\,|dz|$.

Let $\widehat{u}(z) = u(k(z))$ be the unique bounded harmonic function in $D - (k(E_1) \cup k(E_2))$, which is 0 on $k(E_1)$, 1 on $k(E_2)$ and whose normal derivative vanishes on $\partial D - (k(E_1) \cup k(E_2))$. Similar with Proposition 4.3, we get the following result.

Proposition 4.4. *The extremal distance, $\lambda_D(k(E_1), k(E_2))$, between $k(E_1)$ and $k(E_2)$ is*

$$\lambda_D(k(E_1), k(E_2) = \frac{1}{D(\widehat{u})}.$$

Theorem 4.5. *The symmetric extremal distance of E_1 and E_2 on D is*

$$\lambda_D^{(k)}(E_1, E_2) = \frac{1}{D(u)} + \frac{1}{D(\widehat{u})}.$$

Proof. By (3.1), (4.1) and Propositions 4.3 and 4.4 we get the above result. □

Another conformal invariant can be defined if we consider the family $\overline{\Gamma}$ of arcs that separate E_1 and E_2. Then, an element $\overline{\gamma} \in \overline{\Gamma}$ is a countable union of arcs in O_2 and the sets E_1 and E_2 are contains in different components of $\overline{O_2} - \overline{\gamma}$.

The conjugate extremal distance of E_1 and E_2 on D is the extremal length of the family $\overline{\Gamma}$ (see [3]). We denote it by $\overline{\lambda}_D(E_1, E_2)$.By definition

$$\overline{\lambda}_D(E_1, E_2) = \lambda_D(\overline{\Gamma}).$$

It is known to hold the following result (see [3]).

Proposition 4.6. $\overline{\lambda}_D(E_1, E_2) = \frac{1}{\lambda_D(\Gamma)}$.

It is easy to see that $k(\overline{\Gamma})$ represents the family of arcs in D, which separate $k(E_1)$ and $k(E_2)$. Therefore,

$$\overline{\lambda}_D(k(E_1), k(E_2)) = \frac{1}{\lambda_D(k(\Gamma))}.$$

Next, we define the symmetric conjugate extremal distance of E_1 and E_2 on D, by

$$\overline{\lambda}_D^{(k)}(E_1, E_2) = \frac{1}{\lambda_D(\Gamma)} + \frac{1}{\lambda_D(k(\Gamma))}.$$

Proposition 4.7. $\overline{\lambda}_D^{(k)}(E_1, E_2) = \overline{\lambda}_D^{(k)}(k(E_1), k(E_2))$.

Proof. Since k is an involution, using Proposition 3.5, we obtain

$$\lambda_D^{(k)}(E_1, E_2) = \frac{\lambda_D^{(k)}(\Gamma)}{\lambda_D(\Gamma)\lambda_D(k(\Gamma))} = \frac{\lambda_D^{(k)}(k(\Gamma))}{\lambda_D(\Gamma)\lambda_D(k(\Gamma))} = \lambda_D^{(k)}(k(E_1), k(E_2)). \qquad \square$$

Remark 4.8. By Propositions 3.7 and 4.7, we deduce that the symmetric conjugate extremal distance is a dianalytic invariant.

Next, we define a symmetric reduced extremal distance for the complex double. This extremal distance is computed in terms of harmonic functions with prescribed singularities and boundary behaviour, whose existence follow from the main existence theorem of the theory of normal operators (see [3, Theorem 3A], and [15]).

Let D be a symmetric region bounded by a finite number of σ-rectifiable Jordan curves. Let $E \cup k(E)$ be the union of a finite number of symmetric closed arcs on the boundary C. We consider z_0 and $k(z_0)$ two symmetric points inside D. Let U_1 and U_2 be two symmetric, disjoint parametric disks corresponding to z_0 and $k(z_0)$, respectively. Suppose that z is a local parameter defined in a neighborhood of z_0.Then $k(z)$ is a local parameter defined in a neighborhood of $k(z_0)$. For all sufficiently small positive values of r we consider $C_1(r) = \{z \in D : |z - z_0| = r\}$ and $C_2(r) = \{w \in D : |w - k(z_0)| = r\}$ the boundaries of the parametric disks U_1 and U_2, respectively.

Let $\Gamma(r)$ be the family of connected arcs on $D \setminus \overline{U_1}$ which join $C_1(r)$ and E.The extremal length of the arcs family $\Gamma(r)$ is the extremal distance between $C_1(r)$ and E. Using the first composition law (Proposition 3.4) we get the following result:

Proposition 4.9. $\lambda_D(\Gamma(r)) + \frac{1}{2\pi}\log r$ *is a decreasing function of* r.

Proof. Suppose $0 < r < r'$ and let $\Gamma(r, r')$ be the family of all connected arcs in the annular region $U_1(r') \setminus \overline{U_1(r)}$ which join $C_1(r)$ and $C_1(r')$. By Proposition 3.4,

$$\lambda_D(\Gamma(r)) \geq \lambda_D(\Gamma(r')) + \lambda_D(\Gamma(r, r')) \geq \lambda_D(\Gamma(r')) + \frac{1}{2\pi}\log\frac{r'}{r},$$

where $\frac{1}{2\pi}\log\frac{r'}{r}$ is the extremal distance of $C_1(r)$ and $C_1(r')$ (see [17]). $\square$

The preceding result guarantees that $\lambda_D(\Gamma(r)) + \frac{1}{2\pi}\log r$ approaches a finite limit as $r \to 0$, namely the limit is the infimum taken over r positive and sufficiently small. However, the number

$$d(z_0, E) = \lim_{r\to 0}\left(\lambda_D(\Gamma(r)) + \frac{1}{2\pi}\log r\right)$$

it is neither positive nor a conformal invariant.

To eliminate these shortcomings, we define the number

$$\lambda_D(z_0, E) = d(z_0, E) - d(z_0, C)$$

as the reduced extremal distance between z_0 and E.

Similarly, the reduced extremal distance between $k(z_0)$ and $k(E)$, is

$$\lambda_D(k(z_0), k(E)) = d(k(z_0), k(E)) - d(k(z_0), C).$$

We define the symmetric reduced extremal distance between z_0 and E by

$$\lambda_D^{(k)}(z_0, E) = \lambda(z_0, E) + \lambda(k(z_0), k(E)).$$

Since k is an involution, we get that $\lambda^{(k)}(z_0, E)$ is symmetric, namely is invariant with respect to k.

Proposition 4.10. $\lambda_D^{(k)}(z_0, E) = \lambda_D^{(k)}(k(z_0), k(E))$.

We can relate the symmetric reduced extremal distance with certain invariants numbers.

Let $g = g(z, z_0)$ be the L_0-principal function having a negative logarithmic singularity in z_0. Thus g is harmonic on $D \setminus \{z_0\}$, g is 0 on E and $g = L_0 g$ on the rest of the boundary, namely g has a vanishing normal derivative on the rest of the boundary. Then

$$g(z, z_0) = -\log|z - z_0| + \gamma(E) + \varepsilon_1(z),$$

where $\gamma(E)$ is a constant and $\varepsilon_1(z) \to 0$, as $z \to z_0$. Following Ahlfors [1] and using Proposition 4.3, we obtain that

$$d(z_0, E) = \frac{1}{2\pi}\left[-\log r + \gamma(E) + \varepsilon_1(z)\right]$$

and

$$\lambda_D(z_0, E) = \frac{1}{2\pi}\left[\gamma(E) - \gamma(C)\right],$$

where $\gamma(C)$ is the Robin constant relative to z_0. Similarly, $g_k = g(z, k(z_0))$ is the L_0-principal function having a negative logarithmic singularity in $k(z_0)$. We obtain

$$\lambda_D(k(z_0), E) = \frac{1}{2\pi}\left[\gamma(k(E)) - \gamma(C_k)\right],$$

where $\gamma(C_k)$ is the Robin constant relative to $k(z_0)$.

Consequently,

$$\lambda_D^{(k)}(z_0, E) = \frac{1}{2\pi}\left[\gamma(E) + \gamma(k(E))\right] - \frac{1}{2\pi}\left[\gamma(C) + \gamma(C_k)\right].$$

and the right side is a dianalytic invariant.

4.2. THE EXTREMAL DISTANCE ON THE KLEIN SURFACE

The Klein surface X is the factor manifold of the symmetric Riemann surface O_2 with respect to the group $\langle k\rangle$. Then, Ω is obtained from the symmetric region D by identifying the corresponding symmetric points.

By Theorem 2.1, if E is a set in the symmetric region D, then $k(E)$ is a set in $k(D) = D$ and $E \cup k(E)$ projects by π into a set $\widetilde{E}$ in Ω. Conversely, for every set $\widetilde{E}$ in Ω, the set $\pi^{-1}(\widetilde{E})$ is a symmetric set in D.

We consider $\widetilde{E_1}$ and $\widetilde{E_2}$ two nonvoid disjoint sets on Ω. Let $\widetilde{\Gamma}$ be the family of connected arcs in Ω, which join $\widetilde{E_1}$ and $\widetilde{E_2}$ and $\widetilde{\overline{\Gamma}}$ the family of arcs that separate $\widetilde{E_1}$ and $\widetilde{E_2}$. Then $\pi^{-1}(\widetilde{E_i}) = E_i \cup k(E_i), i = 1, 2$, are symmetric sets in D. Also, $\pi^{-1}(\widetilde{\Gamma}) = \Gamma \cup k(\Gamma)$ and $\pi^{-1}(\widetilde{\overline{\Gamma}}) = \overline{\Gamma} \cup k(\overline{\Gamma})$ are the corresponding symmetric sets of arcs in the symmetric region D.

We denote by $\lambda_\Omega(\widetilde{E_1}, \widetilde{E_2})$ the extremal distance of $\widetilde{E_1}$ and $\widetilde{E_2}$ in Ω.

We define the extremal distance of $\widetilde{E_1}$ and $\widetilde{E_2}$ in Ω by

$$\lambda_\Omega(\widetilde{E_1}, \widetilde{E_2}) = \lambda_D^{(k)}(E_1, E_2). \tag{4.2}$$

Remark 4.11. By Remark 4.2, the extremal distance in Ω is well defined.

We obtain the following explicit formula for the extremal distance in Ω.

Theorem 4.12. *The extremal distance of $\widetilde{E_1}$ and $\widetilde{E_2}$ in Ω is*

$$\lambda_\Omega(\widetilde{E_1}, \widetilde{E_2}) = \frac{1}{D(u)} + \frac{1}{D(\widehat{u})}.$$

Proof. We are using definition (4.2) and Theorem 4.5. □

We denote by $\overline{\lambda}_\Omega(\widetilde{E_1}, \widetilde{E_2})$ the conjugate extremal distance of $\widetilde{E_1}$ and $\widetilde{E_2}$ in Ω.

We define the conjugate extremal distance of $\widetilde{E_1}$ and $\widetilde{E_2}$ in Ω by

$$\overline{\lambda}_\Omega(\widetilde{E_1}, \widetilde{E_2}) = \overline{\lambda}_D^{(k)}(E_1, E_2). \tag{4.3}$$

Remark 4.13. By Remark 4.8, the conjugate extremal distance in Ω is well defined.

Let $\widetilde{z_0}$ be a point in Ω. The reduced extremal distance between $\widetilde{z_0}$ and $\widetilde{E}$ is denoted by $\lambda_\Omega(\widetilde{z_0}, \widetilde{E})$ and by definition,

$$\lambda_\Omega(\widetilde{z_0}, \widetilde{E}) = \lambda_D^{(k)}(z_0, E).$$

Because the Klein surfaces X and $O_2/\langle k\rangle$ are dianalytically equivalent, a boundary value problem on Ω is equivalent with a similar problem on the symmetric region D (see [16]). Klein surfaces are the most general two-manifolds that support harmonic functions.

Assume that the sets $\widetilde{E_1}$ and $\widetilde{E_2}$ consist of a finite number of arcs which are contained in the boundary of Ω.

Consider the mixed Dirichlet–Neumann problem on X for harmonic functions

$$\begin{cases} \Delta U = 0 & \text{on } \Omega \setminus (\widetilde{E_1} \cup \widetilde{E_2}), \\ U = 0 & \text{on } \widetilde{E_1}, \\ U = 1 & \text{on } \widetilde{E_2}, \\ \frac{\partial U}{\partial n_\Sigma} = 0 & \text{on } \partial\Omega \setminus (\widetilde{E_1} \cup \widetilde{E_2}). \end{cases} \tag{4.4}$$

We define $D = \pi^{-1}(\Omega)$. Then D is a symmetric region bounded by a finite number of σ-rectifiable Jordan curves.

The mixed Dirichlet–Neumann problem (4.4) on X is equivalent with the following mixed Dirichlet–Neumann problem for harmonic functions on O_2

$$\begin{cases} \Delta u = 0 & \text{on } D \setminus \{E_i \cup k(E_i) \,|\, i \in \{1,2\}\}, \\ u = 0 & \text{on } E_1 \cup k(E_1), \\ u = 1 & \text{on } E_2 \cup k(E_2), \\ \frac{\partial u}{\partial n_\sigma} = 0 & \text{on } \partial D \setminus \{E_i \cup k(E_i) \,|\, i \in \{1,2\}\}. \end{cases} \tag{4.5}$$

The Dirichlet problem on Klein surfaces is studied in [16].

Using the maximum principle for harmonic functions, it follows that the mixed Dirichlet–Neumann problem has a unique solution for the region D.

Various topics at the interplay between complex analysis and partial differential equations are developed in [12] and [14].

Since k is an antianalytic involution, the symmetry of D implies that the prescribed values of the normal derivative satisfy the compatibility condition

$$\int_{\partial D} \frac{\partial u}{\partial n_\sigma} d\sigma = 0.$$

The symmetric conditions on the boundary imply symmetric solutions for Problem 4.5.

Proposition 4.14. *A solution u of the problem 4.5 is a symmetric function on D.*

Proof. Let u be a solution of Problem 4.5. We define $u_k : \overline{D} \to \mathbb{R}$ by $u_k = \frac{1}{2}(u + u \circ k)$. Then $\Delta u_k = 0$ on $D \setminus \{E_i \cup k(E_i) \,|\, i \in \{1,2\}\}$, $u_k = 0$ on $E_1 \cup k(E_1)$, $u_k = 1$ on $E_2 \cup k(E_2)$ and $\frac{\partial u_k}{\partial n_\sigma} = 0$ on $\partial D \setminus \{E_i \cup k(E_i) \,|\, i \in \{1,2\}\}$. Therefore, u_k is also a solution of the problem 4.5. The uniqueness of the solution yields $u_k = u$ on D, therefore $u = u \circ k$ on D. □

Consequently, we obtain the solution of the mixed Dirichlet–Neumann problem on the region Ω.

Theorem 4.15. *The solution of the mixed Dirichlet–Neumann problem 4.4 is the function U defined on $\overline{\Omega}$, by the relation $u = U \circ \pi$, where π is the canonical projection of O_2 on X and u is the solution of the mixed Dirichlet–Neumann problem 4.5 on the symmetric region D.*

Proof. By definition, $\Delta U(\widetilde{\zeta}) = \Delta u(\zeta) = 0$, for all $\widetilde{\zeta} \in \Omega \setminus (\widetilde{E_1} \cup \widetilde{E_2})$, where $\widetilde{\zeta} = \pi(\zeta)$. Thus, U is a harmonic function on $\Omega \setminus (\widetilde{E_1} \cup \widetilde{E_2})$. Also,

$$U(\widetilde{\zeta}) = u(\zeta) = u(k(\zeta)) = 0 \text{ on } \widetilde{E_1},$$

$$U(\widetilde{\zeta}) = u(\zeta) = u(k(\zeta)) = 1 \text{ on } \widetilde{E_2}$$

and

$$\frac{\partial U}{\partial n_\Sigma}(\widetilde{\zeta}) = \frac{\partial U}{\partial n_\sigma}(\zeta) = \frac{\partial U}{\partial n_\sigma}(\zeta) = 0$$

on $\partial\Omega \setminus (\widetilde{E_1} \cup \widetilde{E_2})$. Due to the uniqueness of the solution, the function U defined on $\overline{\Omega}$ by

$$U(\widetilde{\zeta}) = u(\zeta) = u(k(\zeta)),$$

for all $\widetilde{\zeta}$ in $\overline{\Omega}$, where $\widetilde{\zeta} = \pi(\zeta)$, is the solution of the mixed Dirichlet–Neumann problem 4.4 on Ω. □

We conclude the following result:

Proposition 4.16. *The extremal distance $\lambda_D(E_1 \cup k(E_1), E_2 \cup k(E_2))$ of $E_1 \cup k(E_1)$ and $E_2 \cup k(E_2)$ is $1/(D(u))$.*

Since k is an involution, we get

Proposition 4.17. *The extremal distance $\lambda_D(E_1 \cup k(E_1), E_2 \cup k(E_2))$ is a dianalytical invariant.*

4.3. THE EXTREMAL DISTANCE ON THE MÖBIUS STRIP

Let $\overline{A_r}$ be the annulus represented as

$$\overline{A_r} = \left\{ z \in \mathbb{C} : \frac{1}{r} \leq |z| \leq r \right\}, \quad r > 1.$$

The Möbius strip, denoted by $\overline{M_r}$, is obtained from $\overline{A_r}$ by identifying the points z and $-1/\overline{z}$. Let $k : \overline{A_r} \to \overline{A_r}$ defined by $k(z) = -1/\overline{z}$ an antianalytic involution without fixed points of $\overline{A_r}$. Then $(\overline{A_r}, k)$ is a symmetric Riemann surface and the orbit space $\overline{A_r}/\langle k \rangle$ is the Möbius strip

$$\overline{M_r} = \left\{ \widetilde{z} : \widetilde{z} = \{z, k(z)\}, z \in \overline{A_r} \right\}.$$

The Möbius strip is obtained by cutting the ring along the real axis in the z-plane and joining the two halves together along corresponding boundaries. Thus, the annulus $\overline{A_R}$ with points z and $-1/\overline{z}$ identified is a canonical form for the Möbius strip.

If $E_1 = \{z \in \mathbb{C} : |z| = r\}$ and $E_2 = \left\{z \in \mathbb{C} : |z| = \frac{1}{r}\right\}$ are the two contours of $\overline{A_r}$, then the extremal distance between E_1 and E_2 (see [3]) is

$$\lambda_D(E_1, E_2) = \frac{1}{\pi} \log r.$$

Since $E_2 = k(E_1)$ and $E_1 = k(E_2)$, we obtain $\lambda_D(k(E_1), k(E_2)) = \lambda_D(E_1, E_2)$.

Therefore the extremal distance of $\widetilde{E_1}$ and $\widetilde{E_2}$ in Ω is $\frac{2}{\pi} \log r$.

The conjugate extremal distance $\overline{\lambda}_D(E_1, E_2)$ is the extremal length of the family of closed curves that separate the contours E_1 and E_2. We get

$$\overline{\lambda}_D(E_1, E_2) = \frac{\pi}{\log r}.$$

Again, the conditions $E_2 = k(E_1)$ and $E_1 = k(E_2)$ imply

$$\overline{\lambda}_D(k(E_1), k(E_2)) = \overline{\lambda}_D(E_1, E_2).$$

Therefore the conjugate extremal distance of $\widetilde{E_1}$ and $\widetilde{E_2}$ in Ω is $\frac{2\pi}{\log r}$.

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