OSCILLATORY RESULTS FOR SECOND-ORDER NONCANONICAL DELAY DIFFERENTIAL EQUATIONS

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Abstract. The main purpose of this paper is to improve recent oscillation results for the second-order half-linear delay differential equation

$$\left(r(t)\left(y'(t)\right)^{\gamma}\right)' + q(t)y^{\gamma}(\tau(t)) = 0, \quad t \geq t_0,$$

under the condition

$$\int_{t_0}^{\infty} \frac{\mathrm{d}t}{r^{1/\gamma}(t)} < \infty.$$

Our approach is essentially based on establishing sharper estimates for positive solutions of the studied equation than those used in known works. Two examples illustrating the results are given.

Keywords: linear differential equation, delay, second-order, noncanonical, oscillation.

Mathematics Subject Classification: 34C10, 34K11.

1. INTRODUCTION

Consider the second-order half-linear delay differential equation of the form

$$\left(r(t)\left(y'(t)\right)^{\gamma}\right)' + q(t)y^{\gamma}(\tau(t)) = 0, \quad t \geq t_0 > 0. \tag{1.1}$$

Throughout, we will assume that

(H_0) γ is a quotient of odd positive integers;
(H_1) $r \in \mathcal{C}([t_0, \infty), (0, \infty))$ satisfies

$$\pi(t_0) := \int_{t_0}^{\infty} \frac{\mathrm{d}t}{r^{1/\gamma}(t)} < \infty;$$

(H$_2$) the delay function $\tau \in \mathcal{C}^1([t_0, \infty), \mathbb{R})$ satisfies $\tau(t) \leq t$, $\tau'(t) \geq 0$ and $\lim_{t \to \infty} \tau(t) = \infty$;

(H$_3$) $q \in \mathcal{C}([t_0, \infty), [0, \infty))$ does not vanish identically on any half-line of the form $[t_*, \infty)$;

(H$_4$) there is a constant ℓ such that $\pi(\tau(t))/\pi(t) \geq \ell > 1$ for any $t \geq t_0$.

By a solution of Eq. (1.1) we understand a function $y \in \mathcal{C}([t_a, \infty), \mathbb{R})$ with $t_a = \tau(t_b)$, for some $t_b \geq t_0$, which has the property $r(y')^\gamma \in \mathcal{C}^1([t_a, \infty), \mathbb{R})$ and satisfies (1.1) on $[t_b, \infty)$. We consider only those solutions of (1.1) which exist on some half-line $[t_b, \infty)$ and satisfy the condition $\sup\{|x(t)| : t_c \leq t < \infty\} > 0$ for any $t_c \geq t_b$. As is customary, a solution $y(t)$ of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is said to be nonoscillatory. The equation itself is termed oscillatory if all its solutions oscillate.

The problem of establishing oscillation criteria for differential equations with deviating arguments has been a very active research area over the past decades and several references and reviews of known results can be found in the monographs by Agarwal *et al.* [1–4], Došlý and Řehák [6] and Győri and Ladas [11].

Usually, the equation (1.1) has been studied in so-called canonical form, i.e. when

$$\pi(t_0) = \infty. \tag{1.2}$$

On the other hand, much less efforts in this direction have been undertaken for non-canonical equations (i.e. when (H$_1$) holds). A common approach in the literature (see [5, 8–10, 12–16]) for investigation of such equations consists in extending known results for canonical ones. The objective of this paper is to study oscillatory and asymptotic properties of (1.1) in non-canonical form.

In 2017, Džurina and Jadlovská [7] established, contrary to most existing results, a single-condition oscillation criteria for (1.1). Among others, they showed that if, for all $t_1 \geq t_0$ large enough,

$$\limsup_{t \to \infty} \pi^\gamma(t) \int_{t_1}^{t} q(s) \mathrm{d}s > 1, \tag{1.3}$$

then (1.1) is oscillatory. The main purpose of this paper is to sequentially improve condition (1.3) by presenting new criteria for oscillation of (1.1). Our approach is essentially based on establishing sharper estimates for positive solutions of (1.1) than those used in the known works [5, 7, 9, 10, 12–16].

Remark 1.1. All functional inequalities are assumed to hold eventually, that is, they are satisfied for all t large enough.

2. MAIN RESULTS

We recall that by the result [7, Theorem 2], Eq. (1.1) is oscillatory if

$$\int\limits_{t_0}^{\infty} \pi^{\gamma+1}(s)q(s)\mathrm{d}s = \infty. \tag{2.1}$$

Thus, without further mentioning, we will assume that the integral in (2.1) is convergent.

We begin with the preliminary result on the structure of nonoscillatory, let us say positive solutions of Eq. (1.1) and their asymptotic properties, which plays an essential role in the proofs of the main results.

Lemma 2.1. *Let* $(H_0)-(H_3)$ *hold. Assume that*

$$\int\limits_{t_0}^{\infty} \frac{1}{r^{1/\gamma}(t)} \left(\int\limits_{t_0}^{t} q(s)\mathrm{d}s \right)^{1/\gamma} \mathrm{d}t = \infty. \tag{2.2}$$

Furthermore, suppose that (1.1) *has a positive solution* y *on* $[t_1, \infty)$. *Then*

$$y > 0, \quad y' < 0, \quad \left(r\left(y'\right)^{\gamma}\right)' \leq 0, \quad on \ [t_1, \infty) \tag{2.3}$$

and

$$\lim_{t\to\infty} y(t) = 0. \tag{2.4}$$

Proof. The proof is similar to that of [7, Theorem 1] and hence we omit it. □

The following criterion is in fact condition (1.3), improved in the sense that the criterion does not depend on the choice of the initial constant. For the reader's convenience and further purposes, we state its complete proof here.

Theorem 2.2. *Let* $(H_0)-(H_3)$ *and* (2.2) *hold. If*

$$K := \limsup_{t\to\infty} \pi(t) \left(\int\limits_{t_0}^{t} q(s)\mathrm{d}s \right)^{1/\gamma} > 1, \tag{2.5}$$

then (1.1) *is oscillatory.*

Proof. Suppose the contrary and assume that y is a nonoscillatory solution of (1.1) on $[t_0, \infty)$. Without loss of generality, we may assume that $y(t) > 0$, $y(\tau(t)) > 0$ for

$t \in [t_1, \infty) \subseteq [t_0, \infty)$. By Lemma 2.1, y satisfies (2.3) and (2.4). Integrating (1.1) from t_1 to t, we have

$$-r(t) \left(y'(t) \right)^\gamma = -r(t_1) \left(y'(t_1) \right)^\gamma + \int_{t_1}^t q(s) y^\gamma(\tau(s)) \mathrm{d}s$$

$$\geq -r(t_1) \left(y'(t_1) \right)^\gamma + y^\gamma(\tau(t)) \int_{t_1}^t q(s) \mathrm{d}s$$

$$\geq -r(t_1) \left(y'(t_1) \right)^\gamma + y^\gamma(\tau(t)) \int_{t_0}^t q(s) \mathrm{d}s - y^\gamma(\tau(t)) \int_{t_0}^{t_1} q(s) \mathrm{d}s.$$

In view of (2.4), there is a $t_2 > t_1$ such that

$$-r(t_1) \left(y'(t_1) \right)^\gamma - y^\gamma(\tau(t)) \int_{t_0}^{t_1} q(s) \mathrm{d}s > 0$$

for $t \geq t_2$. Thus,

$$-r(t) \left(y'(t) \right)^\gamma \geq y^\gamma(\tau(t)) \int_{t_0}^t q(s) \mathrm{d}s \geq y^\gamma(t) \int_{t_0}^t q(s) \mathrm{d}s. \tag{2.6}$$

On the other hand, using the monotonicity of $r^{1/\gamma} y'$, we have

$$y(t) \geq - \int_t^\infty r^{-1/\gamma}(s) r^{1/\gamma}(s) y'(s) \mathrm{d}s \geq -r^{1/\gamma}(t) y'(t) \pi(t), \tag{2.7}$$

which gives

$$-r(t) \left(y'(t) \right)^\gamma \geq -r(t) \left(y'(t) \right)^\gamma \pi^\gamma(t) \int_{t_0}^t q(s) \mathrm{d}s.$$

Taking the limsup on both sides of the above inequality, we arrive at contradiction with (2.5). The proof is complete. □

Theorem 2.3. *Let* $(H_0) - (H_3)$ *and* (2.2) *hold. If*

$$k := \liminf_{t \to \infty} \frac{1}{\pi(t)} \int_t^\infty \pi^{\gamma+1}(s) q(s) \mathrm{d}s > \gamma \tag{2.8}$$

or

$$k \leq \gamma \quad and \quad K > 1 - \frac{k}{\gamma}, \tag{2.9}$$

then (1.1) *is oscillatory.*

Proof. Suppose the contrary and assume that y is a nonoscillatory solution of (1.1) on $[t_0, \infty)$. Without loss of generality, we may assume that $y(t) > 0$, $y(\tau(t)) > 0$ for $t \in [t_1, \infty) \subseteq [t_0, \infty)$. By Lemma 2.1, y satisfies (2.3) and (2.4). Employing the identity

$$\left(y(t) + r^{1/\gamma}(t)y'(t)\pi(t) \right)' = \pi(t) \left(r^{1/\gamma}(t)y'(t) \right)'$$

and the chain rule

$$\left(r(t) \left(y'(t) \right)^{1/\gamma} \right)' = \gamma \left(r^{1/\gamma}(t)y'(t) \right)^{\gamma-1} \left(r^{1/\gamma}(t)y'(t) \right)'$$

in (1.1), we get

$$\begin{aligned}
x'(t) &= \frac{\pi(t)}{\gamma} \left(r^{1/\gamma}(t)y'(t) \right)^{1-\gamma} \left(r(t) \left(y'(t) \right)^{\gamma} \right)' \\
&= -\frac{\pi(t)}{\gamma} \left(r^{1/\gamma}(t)y'(t) \right)^{1-\gamma} q(t)y^{\gamma}(\tau(t)),
\end{aligned} \tag{2.10}$$

where we set $x(t) = y(t) + r^{1/\gamma}(t)y'(t)\pi(t)$. From (2.7), it is easy to see that x is positive. Integrating (2.10) from t to ∞, we arrive at

$$\begin{aligned}
x(t) &\geq \int_t^{\infty} \frac{\pi(s)}{\gamma} \left(r^{1/\gamma}(s)y'(s) \right)^{1-\gamma} q(s)y^{\gamma}(\tau(s)) \mathrm{d}s \\
&\geq \int_t^{\infty} \frac{\pi(s)}{\gamma} \left(r^{1/\gamma}(s)y'(s) \right)^{1-\gamma} q(s)y^{\gamma}(s) \mathrm{d}s \\
&\geq \int_t^{\infty} \frac{\pi(s)}{\gamma} \left(r^{1/\gamma}(s)y'(s) \right)^{1-\gamma} q(s) \left(-\pi(s)r^{1/\gamma}(s)y'(s) \right)^{\gamma-1} y(s) \mathrm{d}s \\
&\geq \frac{1}{\gamma} \int_t^{\infty} \pi^{\gamma}(s)q(s)y(s) \mathrm{d}s \geq \frac{y(t)}{\gamma\pi(t)} \int_t^{\infty} \pi^{\gamma+1}(s)q(s) \mathrm{d}s,
\end{aligned} \tag{2.11}$$

that is,

$$y(t) + r^{1/\gamma}(t)y'(t)\pi(t) \geq \frac{y(t)}{\gamma\pi(t)} \int_t^{\infty} \pi^{\gamma+1}(s)q(s) \mathrm{d}s,$$

or

$$y(t) \left(1 - \frac{1}{\gamma\pi(t)} \int_t^{\infty} \pi^{\alpha+1}(s)q(s) \mathrm{d}s \right) \geq -\pi(t)r^{1/\gamma}(t)y'(t) > 0. \tag{2.12}$$

By virtue of (2.8), there exists $\varepsilon > 0$ such that

$$k - \varepsilon > \gamma.$$

From the definition of k, we have

$$1 - \frac{1}{\gamma\pi(t)} \int\limits_t^\infty \pi^{\alpha+1}(s)q(s)\mathrm{d}s \leq 1 - \frac{k-\varepsilon}{\gamma} < 0,$$

which in view of (2.12) contradicts the positivity of y.

Now assume that $k \leq \gamma$. Proceeding as in the proof of Theorem 2.2, we obtain (2.6), that is,

$$-r^{1/\gamma}(t)y'(t) \geq y(\tau(t)) \left(\int\limits_{t_0}^t q(s)\mathrm{d}s \right)^{1/\gamma} \geq y(t) \left(\int\limits_{t_0}^t q(s)\mathrm{d}s \right)^{1/\gamma}. \qquad (2.13)$$

Then, in view of (2.12), we have

$$-r^{1/\gamma}(t)y'(t) \left(1 - \frac{1}{\gamma\pi(t)} \int\limits_t^\infty \pi^{\alpha+1}(s)q(s)\mathrm{d}s \right)$$

$$\geq y(t) \left(1 - \frac{1}{\gamma\pi(t)} \int\limits_t^\infty \pi^{\alpha+1}(s)q(s)\mathrm{d}s \right) \left(\int\limits_{t_0}^t q(s)\mathrm{d}s \right)^{1/\gamma}$$

$$\geq -r^{1/\gamma}(t)y'(t)\pi(t) \left(\int\limits_{t_0}^t q(s)\mathrm{d}s \right)^{1/\gamma}$$

and therefore

$$\pi(t) \left(\int\limits_{t_0}^t q(s)\mathrm{d}s \right)^{1/\gamma} \leq 1 - \frac{1}{\gamma\pi(t)} \int\limits_t^\infty \pi^{\alpha+1}(s)q(s)\mathrm{d}s.$$

Taking limit superior on both sides of the last inequality, we get

$$\limsup_{t\to\infty} \pi(t) \left(\int\limits_{t_1}^t q(s)\mathrm{d}s \right)^{1/\gamma} \leq 1 - \liminf_{t\to\infty} \frac{1}{\gamma\pi(t)} \int\limits_t^\infty \pi^{\alpha+1}(s)q(s)\mathrm{d}s,$$

that is,

$$K \leq 1 - \frac{k}{\gamma},$$

which contradicts (2.9). The proof is complete. $\qquad\qquad\square$

The next considerations are intended to improve criteria given in Theorems 2.2 and 2.3 by incorporating the value of delay argument.

Lemma 2.4. *Let* $(H_0)-(H_3)$ *and* (2.2) *hold. Assume that* (1.1) *has an eventually positive solution* y *on* $[t_1, \infty)$. *Then there exists* $T \geq t_1$ *such that for any* $\varepsilon > 0$

$$\frac{y}{\pi^{K-\varepsilon}} \downarrow \quad on \ [T, \infty). \tag{2.14}$$

Proof. Pick $t_1 \in [t_0, \infty)$ such that $y(\tau(t)) > 0$. By Lemma 2.1, y satisfies (2.3) and (2.4). Proceeding as in the proof of Theorem 2.2, we arrive at (2.6), that is,

$$-r^{1/\gamma}(t)y'(t) \geq y(t) \left(\int_{t_0}^{t} q(s)\mathrm{d}s \right)^{1/\gamma},$$

which holds for any $t \geq t_2$, where $t_2 \geq t_1$ is large enough. Let $\tilde{K} = K - \varepsilon$, where $\varepsilon > 0$ is arbitrary. In view of the definition of K, we have

$$\pi(t) \left(\int_{t_0}^{t} q(s)\mathrm{d}s \right) > \tilde{K}$$

eventually, say for $t \geq t_3 \geq t_2$. Consequently,

$$\begin{aligned}
\left(\frac{y(t)}{\pi^{\tilde{K}}(t)} \right)' &= \frac{r^{1/\gamma}(t)y'(t)\pi^{\tilde{K}}(t) + \tilde{K}y(t)\pi^{\tilde{K}-1}(t)}{r^{1/\gamma}(t)\pi^{2\tilde{K}}(t)} \\
&\leq \frac{y(t)\pi^{\tilde{K}-1}(t) \left(\tilde{K} - \pi(t) \left(\int_{t_0}^{t} q(s)\mathrm{d}s \right)^{1/\gamma} \right)}{r^{1/\gamma}(t)\pi^{2\tilde{K}}(t)} < 0
\end{aligned} \tag{2.15}$$

on $t \in [t_3, \infty)$. The proof is complete. $\qquad\square$

The following two results serve as an improvement of Theorems 2.2 and 2.3, respectively, when $K \leq 1$.

Theorem 2.5. *Let* $(H_0)-(H_4)$ *and* (2.2) *hold. If*

$$\ell^K K > 1 \tag{2.16}$$

then (1.1) *is oscillatory.*

Proof. Proceeding as in the proof of Theorem 2.2, we arrive at (2.6), which holds for any $t \geq t_2$, where $t_2 \geq t_1$ is large enough. By (2.14), we obtain

$$y(\tau(t)) \geq y(t) \left(\frac{\pi(\tau(t))}{\pi(t)} \right)^{K-\varepsilon} \geq y(t)\ell^{K-\varepsilon}. \tag{2.17}$$

Using the above estimate in (2.6), we have

$$-r(t)(y'(t))^{\gamma} \geq y^{\gamma}(t)\ell^{\gamma(K-\varepsilon)} \int_{t_0}^{t} q(s)\mathrm{d}s, \tag{2.18}$$

which implies

$$-r(t)\left(y'(t)\right)^{\gamma} \geq -r(t)\left(y'(t)\right)^{\gamma}\ell^{\gamma(K-\varepsilon)}\pi^{\gamma}(t)\int\limits_{t_0}^{t}q(s)\mathrm{d}s.$$

Taking the lim sup on both sides of the latter inequality, we obtain

$$1 \geq \ell^{K-\varepsilon}K.$$

Since ε is arbitrary, the above condition contradicts (2.16). The proof is complete. $\quad\square$

Theorem 2.6. *Let* $(H_0)-(H_4)$ *and* (2.2) *hold. If* (2.8) *or*

$$k \leq \gamma \quad and \quad \ell^K K > 1 - \frac{k}{\gamma}, \tag{2.19}$$

then (1.1) *is oscillatory.*

Proof. We proceed as in the proof of Theorem 2.3 with (2.13) replaced by (2.18) to obtain

$$-r^{1/\gamma}(t)y'(t)\left(1 - \frac{1}{\gamma\pi(t)}\int\limits_{t}^{\infty}\pi^{\alpha+1}(s)q(s)\mathrm{d}s\right)$$

$$\geq \ell^{K-\varepsilon}y(t)\left(1 - \frac{1}{\gamma\pi(t)}\int\limits_{t}^{\infty}\pi^{\alpha+1}(s)q(s)\mathrm{d}s\right)\left(\int\limits_{t_0}^{t}q(s)\mathrm{d}s\right)^{1/\gamma}$$

$$\geq -\ell^{K-\varepsilon}r^{1/\gamma}(t)y'(t)\pi(t)\left(\int\limits_{t_0}^{t}q(s)\mathrm{d}s\right)^{1/\gamma}.$$

Obviously,

$$\limsup_{t\to\infty}\ell^{K-\varepsilon}\pi(t)\left(\int\limits_{t_0}^{t}q(s)\mathrm{d}s\right)^{1/\gamma} \leq 1 - \liminf_{t\to\infty}\frac{1}{\gamma\pi(t)}\int\limits_{t}^{\infty}\pi^{\alpha+1}(s)q(s)\mathrm{d}s,$$

that is,

$$\ell^{K-\varepsilon}K \leq 1 - \frac{k}{\gamma}.$$

Since ε is arbitrary, the above inequality is in contradiction with (2.19). The proof is complete. $\quad\square$

Theorem 2.7. *Assume* $(H_0)-(H_3)$. *If*

$$k > \left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1}, \tag{2.20}$$

then (1.1) *is oscillatory.*

Proof. Suppose the contrary and assume that y is a nonoscillatory solution of (1.1) on $[t_0, \infty)$. Without loss of generality, we may assume that $y(t) > 0$, $y(\tau(t)) > 0$ for $t \in [t_1, \infty) \subseteq [t_0, \infty)$. By Lemma 2.1, y satisfies (2.3) and (2.4). First, we show that

$$\limsup_{t \to \infty} \int_{t_1}^{t} \left(\pi^\gamma(s)q(s) - \left(\frac{\gamma}{\gamma+1} \right)^{\gamma+1} \frac{1}{r^{1/\gamma}(s)\pi(s)} \right) \mathrm{d}s = \infty \qquad (2.21)$$

implies that (1.1) is oscillatory. Define the function

$$v(t) = \frac{r(t)\,(y'(t))^\gamma}{y^\gamma(t)}.$$

Using (2.7), one can easily see that

$$-1 \leq v(t)\pi^\gamma(t) < 0.$$

Differentiating v and using (1.1) with the fact that y is decreasing, we have

$$v'(t) = \frac{\left(r(t)\,(y'(t))^\gamma\right)'}{y^\gamma(t)} - \gamma\frac{r(t)\,(y'(t))^\gamma\,y'(t)}{y^{\gamma+1}(t)} \leq -q(t) - \frac{\gamma}{r^{1/\gamma}(t)}v^{1+1/\gamma}(t). \qquad (2.22)$$

Multiplying (2.22) by π^γ and integrating the resulting inequality from t_1 to t, we obtain

$$v(t)\pi^\gamma(t) - v(t_1)\pi^\gamma(t_1) + \gamma\int_{t_1}^{t} \frac{\pi^{\gamma-1}(s)}{r^{1/\gamma}(s)}v(s)\mathrm{d}s$$

$$+ \int_{t_1}^{t} \pi^\gamma(s)q(s)\mathrm{d}s + \gamma\int_{t_1}^{t} \frac{\pi^\gamma(s)}{r^{1/\gamma}(s)}v^{1+1/\gamma}(s)\mathrm{d}s \leq 0$$

Using Young's inequality

$$|ab| \leq \frac{1}{p}|a|^p + \frac{1}{q}|b|^q, \quad a, b \in \mathbb{R}, \quad p > 1, \quad q > 1, \quad \frac{1}{p} + \frac{1}{q} = 1$$

with $p = 1 + 1/\gamma$, $q = \gamma + 1$ and

$$a = (\gamma+1)^{1+1/\gamma}\pi^{\gamma^2/(\gamma+1)}v(t), \quad b = \frac{\gamma}{(\gamma+1)^{1-\gamma/\gamma+1}}\pi^{-1/(\gamma+1)}(t)$$

yields

$$-\gamma\pi^{\gamma-1}v(t) \leq \gamma\pi^\gamma(t)v^{1+1/\gamma}(t) + \left(\frac{\gamma}{\gamma+1} \right)^{\gamma+1} \frac{1}{\pi(t)}.$$

Therefore,

$$\int_{t_1}^{t} \left(\pi^\gamma(s)q(s) - \left(\frac{\gamma}{\gamma+1} \right)^{\gamma+1} \frac{1}{r^{1/\gamma}(s)\pi(s)} \right) \mathrm{d}s \leq v(t_1)\pi^\gamma(t_1) - v(t)\pi^\gamma(t)$$

$$\leq v(t_1)\pi^\gamma(t_1) + 1.$$

Taking the limsup on both sides of the above inequality, we obtain a contradiction with (2.21). Now, it is enough to show that (2.21) implies (2.20). To do this, assume that (2.21) is not satisfied. Then, there is a $t_2 \geq t_1$ such that for any $\tilde{\varepsilon} > 0$,

$$\int\limits_t^\infty \left(\pi^\gamma(s)q(s) - \left(\frac{\gamma}{\gamma+1} \right)^{\gamma+1} \frac{1}{r^{1/\gamma}(s)\pi(s)} \right) \mathrm{d}s < \tilde{\varepsilon}.$$

Since π is decreasing, we have

$$\frac{1}{\pi(t)} \int\limits_t^\infty \left(\pi^{\gamma+1}(s)q(s) - \left(\frac{\gamma}{\gamma+1} \right)^{\gamma+1} \frac{1}{r^{1/\gamma}(s)} \right) \mathrm{d}s < \tilde{\varepsilon}$$

that is,

$$\frac{1}{\pi(t)} \int\limits_t^\infty \left(\pi^{\gamma+1}(s)q(s) + \left(\frac{\gamma}{\gamma+1} \right)^{\gamma+1} \pi'(s) \right) \mathrm{d}s < \tilde{\varepsilon}.$$

Hence,

$$\frac{1}{\pi(t)} \int\limits_t^\infty \pi^{\gamma+1}(s)q(s)\mathrm{d}s < \tilde{\varepsilon} + \left(\frac{\gamma}{\gamma+1} \right)^{\gamma+1}$$

for all $\tilde{\varepsilon} > 0$, which contradicts to (2.20). The proof is complete. $\qquad \square$

Theorem 2.8. *Let $(H_0)-(H_4)$ and (2.2) hold. If*

$$\ell^K k > \left(\frac{\gamma}{\gamma+1} \right)^{\gamma+1}, \tag{2.23}$$

then (1.1) is oscillatory.

Proof. We proceed as in the proof of Theorem 2.7 with (2.22) replaced by

$$v'(t) \leq -\ell^{K-\varepsilon}q(t) - \frac{\gamma}{r^{1/\gamma}(t)}v^{1+1/\gamma}(t),$$

where we used (2.17) with ε arbitrary. The rest of the proof is similar and so we omit it. $\qquad \square$

3. EXAMPLES

We illustrate the applicability of the main results by means of a couple of examples.

Example 3.1. Consider the second-order delay differential equation

$$\left(t^{\gamma+1} \left(y'(t) \right)^\gamma \right)' + q_0 y^\gamma(\lambda t) = 0, \quad t \geq 1, \quad 0 < \lambda \leq 1, \quad q_0 > 0. \tag{3.1}$$

Here,

$$r(t) = t^{\gamma+1}, \quad q(t) = q_0, \quad \tau(t) = \lambda t,$$

$$\pi(t) = \int_t^\infty r^{-1/\gamma}(s)\mathrm{d}s = \int_t^\infty s^{-(\gamma+1)/\gamma}\mathrm{d}s = \gamma t^{-1/\gamma}, \quad \text{and} \quad \ell = \lambda^{-1/\gamma}.$$

Also, it is easy to verify that

$$K = \gamma q_0^{1/\gamma} \quad \text{and} \quad k = \gamma^{\gamma+1} q_0.$$

Theorem 2.2 requires

$$q_0 > \frac{1}{\gamma^\gamma} \tag{3.2}$$

for (3.1) to be oscillatory.

Theorem 2.3 improves Theorem 2.2 in the sense that if condition (3.2) is not satisfied, then (3.1) is oscillatory if

$$q_0 > \frac{(1 - \gamma^\gamma q_0)^\gamma}{\gamma^\gamma}. \tag{3.3}$$

Theorem 2.5 improves condition (3.2) by taking the value λ into account:

$$q_0 > \frac{\lambda^{q_0}}{\gamma^\gamma}. \tag{3.4}$$

Finally, Theorem 2.6 provides a similar improvement of condition (3.3), namely,

$$q_0 > \lambda^{q_0} \frac{(1 - \gamma^\gamma q_0)^\gamma}{\gamma^\gamma}. \tag{3.5}$$

Let $\gamma = 1/3$ and $\lambda = 0.5$. Then (3.2)−(3.5) reduce to $q_0 > \sqrt[3]{3} \approx 1.44225$, $q_0 > 0.9840$, $q_0 > 0.8181$ and $q_0 > 0.70633$, respectively.

Example 3.2. We consider again Eq. (3.1). By Theorem 2.7, equation (3.1) is oscillatory if

$$q_0 > \frac{1}{(\gamma+1)^{\gamma+1}}. \tag{3.6}$$

Note that for $\gamma = 1$ and $\lambda = 1$, condition $q_0 > 1/4$ is sharp for oscillation of the Euler differential equation

$$\left(t^2 y'(t)\right)' + q_0 y(t) = 0.$$

On the other hand, Theorem 2.8 improves Theorem 2.7 in the sense that condition (3.6) is replaced by

$$q_0 > \frac{\lambda^{q_0^{1/\gamma}}}{(\gamma+1)^{\gamma+1}}. \tag{3.7}$$

Let $\gamma = 1/3$ and $\lambda = 0.5$. Then condition (3.6) reduces to $q_0 > 0.6814$. Obviously, criterion (3.7) provides a sharper result, since it requires that $q_0 > 0.4075$.

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