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Robustness of solutions of an infinite-dimensional algebraic Sylvester equation under bounded perturbations*

by

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Abstract: The output regulation problem for distributed parameter systems, see e.g. Byrnes, Lauko, Gillian and Shubov (2000) and Paunonen (2011), and the observer design problem for such systems, see e.g. Emirsajłow (2012), are examples of important control problems, where analysis of infinite-dimensional algebraic Sylvester equations plays a crucial role. This paper studies bounded perturbations of the unbounded operators of the algebraic infinite-dimensional Sylvester equation. We derive some estimate on the perturbation operator under which the algebraic Sylvester equation preserves a unique solution or, in the control systems terminology, a solution is robust under small bounded perturbations. In our approach we employ the concept of an implemented semigroup, see e.g. Alber (2001) and Emirsajłow (2012), which is a special case of the so-called bi-continuous semigroup, e.g. Farkas (2004).

Keywords: distributed parameter systems, algebraic Sylvester equation, bounded perturbations, implemented semigroup.

1. Introduction

There are several control problems for distributed parameter systems, where analysis of infinite-dimensional algebraic Sylvester equations plays a crucial role. The robustness issues in such problems lead to the study of infinite-dimensional Sylvester equations under various perturbations. For examples see Byrnes et al. (2000), Paunonen (2011) and Emirsajłow (2012). This paper is devoted to the class of general linear bounded perturbations. For this class we derive a sufficient condition on the perturbation operator norm, which guarantees that the Sylvester equation preserves a unique solution. As the main mathematical tool we employ the implemented semigroup concept, which seems to be the

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right framework for the infinite-dimensional Sylvester equations both in the differential and the algebraic form (see, e.g. Emirsajłow and Townley, 2005, and Emirsajłow, 2012, and references cited therein).

We start with a short introduction of the concept of an implemented semigroup (see, e.g. Alber, 2001, Emirsajłow and Townley, 2005, Emirsajłow, 2012). For this purpose we need the following notation and assumptions:

- H^A and H^E are Hilbert spaces (identified with their duals) with scalar products $\langle\cdot,\cdot\rangle^A$ and $\langle\cdot,\cdot\rangle^E$.
- $\mathcal{H} := \mathscr{L}(H^E, H^A)$ is a Banach space of linear bounded operators from H^E into H^A with the norm $\|\cdot\|$. $(\mathcal{H}, \|\cdot\|)$ stands for $\mathscr{L}(H^E, H^A)$ equipped with the uniform operator topology (induced by $\|\cdot\|$) and $(\mathcal{H}, \tau)$ stands for $\mathscr{L}(H^E, H^A)$ equipped with the strong operator topology τ, i.e. topology induced by the family of seminorms $\mathcal{P} = \{p_h\}_{h\in B_1(H^E)}$, where $p_h(X) = \|Xh\|^A$ for $X \in \mathcal{H}$ and $h \in H^E$, and $B_1(H^E)$ is a unit ball in H^E.
- A is a linear, unbounded operator on H^A generating a strongly continuous semigroup of operators $(T(t))_{t\geq 0} \subset \mathscr{L}(H^A)$. $H_1^A = \mathscr{D}(A)$ is a Hilbert space with the scalar product $\langle\cdot,\cdot\rangle_1^A = \langle(\lambda I - A)(\cdot), (\lambda I - A)(\cdot)\rangle^A$ and the induced norm $\|\cdot\|_1^A$, where $\lambda \in \rho(A)$.
- E is a linear, unbounded operator on H^E generating a strongly continuous semigroup of operators $(S(t))_{t\geq 0} \subset \mathscr{L}(H^E)$. Analogously as above, we define $H_1^E = \mathscr{D}(E)$.

Using the two strongly continuous semigroups $(T(t))_{t\geq 0} \subset \mathscr{L}(H^A)$ and $(S(t))_{t\geq 0} \subset \mathscr{L}(H^E)$ generated by A and E, respectively, we can define another semigroup.

DEFINITION 1.1 *The family $(\mathcal{U}(t))_{t\geq 0} \subset \mathscr{L}(\mathcal{H})$, defined as follows*

$$\mathcal{U}(t)X = T(t)XS(t), \quad X \in \mathcal{H}, \quad t \geq 0\,, \tag{1}$$

is called the implemented semigroup.

It turns out that the family $(\mathcal{U}(t))_{t\geq 0} \subset \mathscr{L}(\mathcal{H})$ is a semigroup and for every $X \in \mathcal{H}$ satisfies the continuity condition $\mathcal{U}(\cdot)X \in C([0,\infty); (\mathcal{H}, \tau))$. Such a family is said to be strongly τ-continuous. In general this family is not a C_0-semigroup (strongly $\|\cdot\|$-continuous in our terminology) unless both operators A and E are bounded. However, in the infinite-dimensional systems and control theory the really interesting case is when both A and E are unbounded.

DEFINITION 1.2 *The* infinitesimal generator $\mathcal{A}$ *of the implemented semigroup $(\mathcal{U}(t))_{t\geq 0} \subset \mathscr{L}(\mathcal{H})$ is defined as the limit*

$$\mathcal{A}X = \tau\text{-}\lim_{t\searrow 0} \frac{\mathcal{U}(t)X - X}{t}, \quad X \in \mathscr{D}(\mathcal{A})\,, \tag{2}$$

where $\mathscr{D}(\mathcal{A}) \subset \mathcal{H}$ is the domain *of $\mathcal{A}$ defined as follows*

$$\mathscr{D}(\mathcal{A}) = \{X \in \mathcal{H} :\ \tau\text{-}\lim_{t\searrow 0} \frac{\mathcal{U}(t)X - X}{t} \ \text{ exists }\}\,. \tag{3}$$

For the domain $\mathscr{D}(\mathcal{A}) \in \mathcal{H}$ and the generator $\mathcal{A}$ we have:

(a) $X \in \mathcal{H}$ belongs to $\mathscr{D}(\mathcal{A})$ if and only if the restriction of X to H_1^E belongs to $\mathscr{L}(H_1^E, H_1^A)$ and an extension of $(AX + XE) \in \mathscr{L}(H_1^E, H^A)$ to H^E belongs to $\mathcal{H}$.

(b) $\mathcal{A}$ has the following explicit representation

$$(\mathcal{A}X)h = AXh + XEh\,,\ X \in \mathscr{D}(\mathcal{A})\,,\ h \in H_1^E\,,$$

where by (a) the right hand side of this equality is well-defined in H^A.

Basic properties of the implemented semigroup can be summarized as follows:

(c) If $X \in \mathscr{D}(\mathcal{A})$, then $(\mathcal{U}(t)X)_{t\geq 0} \subset \mathscr{D}(\mathcal{A})$ and is τ-continuously differentiable in t, i.e. $\mathcal{U}(\cdot)X \in C^1([0,\infty);(\mathcal{H},\tau))$, and

$$\frac{d}{dt}\mathcal{U}(t)X = \mathcal{A}(\mathcal{U}(t)X) = \mathcal{U}(t)(\mathcal{A}X)\,,\ t \geq 0\,. \tag{4}$$

(d) The domain $\mathscr{D}(\mathcal{A})$ is sequentially dense in $(\mathcal{H},\tau)$, which means that for every $X \in \mathcal{H}$ there exists a $\|\cdot\|$-bounded sequence $(X_n)_{n\in\mathbb{N}} \subset \mathscr{D}(\mathcal{A})$ which is convergent to X in $(\mathcal{H},\tau)$. It should be emphasized that in general $\mathscr{D}(\mathcal{A})$ is not dense in $(\mathcal{H},\|\cdot\|)$.

(e) The operator $(\mathcal{A},\mathscr{D}(\mathcal{A}))$ is sequentially closed in $(\mathcal{H},\tau)$, which means that for all sequences $(X_n)_{n\in\mathbb{N}} \subset \mathscr{D}(\mathcal{A})$ such that $(X_n)_{n\in\mathbb{N}}$ is $\|\cdot\|$-bounded and $\tau\text{-}\lim_{n\to\infty} X_n = X \in \mathcal{H}$ and $(\mathcal{A}X_n)_{n\in\mathbb{N}}$ is $\|\cdot\|$-bounded and $\tau\text{-}\lim_{n\to\infty} \mathcal{A}X_n = Y \in \mathcal{H}$, we have $X \in \mathscr{D}(\mathcal{A})$ and $Y = \mathcal{A}X$.

(f) The following equality holds

$$\|\mathcal{U}(t)\|_{\mathscr{L}(\mathcal{H})} = \|T(t)\|^A \|S(t)\|^E\,, \tag{5}$$

where $t \geq 0$, and if $\omega_0(T)$ is the growth bound of $(T(t))_{t\geq 0} \subset \mathscr{L}(H^A)$, $\omega_0(S)$ is the growth bound of $(S(t))_{t\geq 0} \subset \mathscr{L}(H^E)$ and $\omega_0(\mathcal{U})$ is the growth bound of $(\mathcal{U}(t))_{t\geq 0} \subset \mathscr{L}(\mathcal{H})$, then

$$\omega_0(\mathcal{U}) = \omega_0(T) + \omega_0(S)\,. \tag{6}$$

(g) The following inclusion holds

$$\mathbb{C}_{\omega_0(T)+\omega_0(S)} \subset \rho(\mathcal{A})\,, \tag{7}$$

where we use the notation

$$\mathbb{C}_\omega := \{\lambda \in \mathbb{C} :\ \operatorname{Re}\lambda > \omega\}\,,$$

$\rho(\mathcal{A})$ denotes the resolvent set of $\mathcal{A}$, and for $\lambda \in \mathbb{C}_{\omega_0(T)+\omega_0(S)}$ the resolvent is explicitly given by

$$\begin{aligned} \mathcal{R}(\lambda,\mathcal{A})X &:= (\lambda I - \mathcal{A})^{-1}X \\ &= \int_0^\infty e^{-\lambda t}\mathcal{U}(t)X\,dt \\ &= \int_0^\infty e^{-\lambda t}\,T(t)XS(t)\,dt\,,\ X \in \mathcal{H}\,, \end{aligned} \tag{8}$$

where the integrals are convergent in $(\mathcal{H}, \tau)$. Moreover, for every $\lambda \in \mathbb{C}_{\omega_0(\mathcal{U})}$, $\omega \in (\omega_0(\mathcal{U}), \operatorname{Re}\lambda)$ and some $C_\omega \geq 1$ we have

$$\|\mathcal{U}(t)\|_{\mathscr{L}(\mathcal{H})} \leq C_\omega e^{\omega t} \quad \text{for all } t \geq 0 \tag{9}$$

and

$$\|\mathcal{R}(\lambda, \mathcal{A})\|_{\mathscr{L}(\mathcal{H})} \leq \frac{C_\omega}{\operatorname{Re}\lambda - \omega}\,. \tag{10}$$

- Throughout the rest of the paper we assume that $\lambda \in \mathbb{C}_{\omega_0(T)+\omega_0(S)}$. This condition is always satisfied for sufficiently large $\operatorname{Re}\lambda \in \mathbb{R}$.
- $\mathcal{H}_1 = \mathscr{D}(\mathcal{A})$ denotes the Banach space with the norm

$$\| X\|_1 = \|(\lambda\mathcal{I} - \mathcal{A})X\|\,, \quad X \in \mathscr{D}(\mathcal{A})\,. \tag{11}$$

In $\mathcal{H}_1$ we distinguish the uniform operator topology induced by the norm $\|\cdot\|_1$ and the strong operator topology τ_1 induced by a family of seminorms $\mathcal{P}_1 = \{p_{1h}\}_{h\in B_1(H^E)}$, where

$$p_{1h}(X) = p_h((\lambda\mathcal{I} - \mathcal{A})X) = \|(\lambda\mathcal{I} - \mathcal{A})Xh\|^A$$

for $X \in \mathscr{D}(\mathcal{A})$ and $h \in H^E$.

2. Algebraic Sylvester equation

It follows from (4) that if $Z_0 \in \mathcal{H}_1$, then the expression

$$Z(t) = \mathcal{U}(t)Z_0 = T(t)Z_0S(t)\,, \quad t \geq 0\,, \tag{12}$$

satisfies the following conditions

$$Z(t) \in \mathcal{H}_1\,,\ \dot{Z}(t) = \mathcal{A}(\mathcal{U}(t)Z_0) \in \mathcal{H}\,,\ Z(0) = Z_0\,,\ t \geq 0\,, \tag{13}$$

which show that $(Z(t))_{t\geq 0}$ can be viewed as a solution to the initial value problem (13). However, the differentiation in (13) is understood in $(\mathcal{H}, \tau)$ and in general does not make sense in $(\mathcal{H}, \|\cdot\|)$. Thus, the expression (12) can be regarded as a strong solution of the initial value problem

$$\dot{Z}(t) = \mathcal{A}Z(t)\,, \quad t \geq 0\,, \quad Z(0) = Z_0\,, \tag{14}$$

but we have to consider this problem on $(\mathcal{H}, \tau)$.

We refer to (14) as the *homogeneous Cauchy problem* and since we have $(Z(t))_{t\geq 0} \subset \mathcal{H}_1$, then we can rewrite the differential equation (14) in the more explicit form

$$\dot{Z}(t)h = AZ(t)h + Z(t)Eh\,, \quad t \geq 0\,,\ Z(0) = Z_0\,,\ h \in H_1^E\,, \tag{15}$$

where the equality holds in H^A. In this case we refer to the differential equation (15) as the *homogeneous differential Sylvester equation* (see Emirsajłow, 2012; Phong, 1991).

The implemented semigroup approach can be further used to study algebraic operator equations corresponding to differential operator equations (14) and (15).

PROPOSITION 2.1 *Let* $\omega > \omega_0(\mathcal{U}) = \omega_0(T) + \omega_0(S)$*, then for every* $F \in \mathcal{H}$ *the following algebraic equation*

$$(\omega I - \mathcal{A})Z = F\,, \tag{16}$$

equivalently, the algebraic Sylvester equation *(see Phong, 1991; Emirsajłow, 2012)*

$$\omega Z - AZh - ZEh = Fh\,, \quad h \in H_1^E\,, \tag{17}$$

which holds in H^A*, has a unique solution* $Z \in \mathcal{H}_1$*. Moreover, this solution is explicitly given by the expression*

$$\begin{aligned} Z = \mathcal{R}(\omega, \mathcal{A})F &= \int_0^\infty e^{-\omega t}\mathcal{U}(t)F\,dt \\ &= \int_0^\infty e^{-\omega t}T(t)FS(t)\,dt\,. \end{aligned} \tag{18}$$

PROOF 2.2 *The assumption* $\omega > \omega_0(\mathcal{U})$ *implies that* $\omega \in \varrho(\mathcal{A})$ *and hence the operator* $\omega I - \mathcal{A}$ *has a bounded inverse* $\mathcal{R}(\omega, \mathcal{A}) = (\omega I - \mathcal{A})^{-1} \in \mathscr{L}(\mathcal{H})$*. This means that for every* $F \in \mathcal{H}$ *the equation* (16) *has a unique solution* $Z \in \mathcal{H}_1$ *given by* $Z = \mathcal{R}(\omega, \mathcal{A})F$ *and since for* $\omega > \omega_0(\mathcal{U})$ *the resolvent* $\mathcal{R}(\omega, \mathcal{A})$ *admits integral representation* (8)*, we obtain* (18).

In the case when the implemented semigroup $(\mathcal{U}(t))_{t\geq 0} \subset \mathscr{L}(\mathcal{H})$ is exponentially stable, we can complement Proposition 2.1.

COROLLARY 2.3 *Let* $\omega_0(\mathcal{U}) = \omega_0(T) + \omega_0(S) < 0$*, then for every* $F \in \mathcal{H}$ *the following algebraic equation*

$$-\mathcal{A}Z = F\,, \tag{19}$$

equivalently, the algebraic Sylvester equation

$$-AZh - ZEh = Fh\,, \quad h \in H_1^E\,, \tag{20}$$

which holds in H^A*, has a unique solution* $Z \in \mathcal{H}_1$*. Moreover, this solution is explicitly given by the expression*

$$Z = (-\mathcal{A})^{-1}F = \int_0^\infty \mathcal{U}(t)F\,dt = \int_0^\infty T(t)FS(t)\,dt\,. \tag{21}$$

3. Additive bounded perturbations

In this section we develop a framework for perturbations of the implemented semigroup $(\mathcal{U}(t)_{t\geq 0} \subset \mathscr{L}(\mathcal{H})$ and for this we need to introduce the following class of functions:

- For every $t_0 > 0$ we define $\mathcal{B}_{t_0}$ to be the set of all functions $(\mathcal{V}(t))_{t\in[0,t_0]} \subset \mathscr{L}(\mathcal{H})$ which satisfy the following two conditions:

  (i) $\mathcal{V}(\cdot)$ is strongly τ-continuous, i.e., for every $Z \in \mathcal{H}$ we have $\mathcal{V}(\cdot)Z \in C([0,t_0];(\mathcal{H},\tau))$,

  (ii) $\mathcal{V}(\cdot)$ is bi-equicontinuous, i.e., for every $\|\cdot\|$-bounded sequence $(Z_n)_{n\in\mathbb{N}} \subset \mathcal{H}$, which is τ-convergent to $Z \in \mathcal{H}$, every $p_h \in \mathcal{P}$ and every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\sup_{0\leq t\leq t_0} p_h(\mathcal{V}(t)(Z_n - Z)) < \varepsilon\,, \quad n \geq n_0\,.$$

It is clear that for every $h \in H^E$ and $Z \in \mathcal{H}$ we have $(\mathcal{V}(\cdot)Z)h \in C([0,t_0];H^A)$, and it follows from the uniform boundedness principle that

$$\sup_{0\leq t\leq t_0} \|\mathcal{V}(t)\|_{\mathscr{L}(\mathcal{H})} < \infty\,. \tag{22}$$

One can prove that for every $t_0 > 0$ the space $\mathcal{B}_{t_0}$ (with properties (i) and (ii)) endowed with the norm

$$\|\mathcal{V}\|_{\mathcal{B}_{t_0}} := \sup_{0\leq t\leq t_0} \|\mathcal{V}(t)\|_{\mathscr{L}(\mathcal{H})}\,, \tag{23}$$

is a Banach space.

In this section we examine the perturbed operator $\mathcal{A}^{\mathcal{P}} = \mathcal{A} + \mathcal{P}$, where $\mathcal{A}$ is the generator of the implemented semigroup $(\mathcal{U}(t))_{t\geq 0} \subset \mathscr{L}(\mathcal{H})$ and the unknown perturbation operator $\mathcal{P}$ satisfies the following condition:

- $\mathcal{P} \in \mathscr{L}(\mathcal{H})$ and is τ-continuous on $\|\cdot\|$-bounded sets.

Our goal is to find conditions under which the operator $\mathcal{A}^{\mathcal{P}}$ generates a strongly τ-continuous semigroup on $\mathscr{L}(\mathcal{H})$ and we approach this problem by looking for a strongly τ-continuous solution $(\mathcal{V}(t))_{t\geq 0} \subset \mathscr{L}(\mathcal{H})$ of the following integral equation

$$\mathcal{V}(t)Z = \mathcal{U}(t)Z + \int_0^t \mathcal{U}(t-r)(\mathcal{P}(\mathcal{V}(r)Z))\,dr\,,\ t \geq 0\,, \tag{24}$$

where $Z \in \mathcal{H}$ and the integral is understood in $(\mathcal{H},\tau)$.

Let us notice that the above defined perturbation operator $\mathcal{P}$ really covers the case we are interested in, i.e.,

$$\mathcal{H} \ni Z \mapsto \mathcal{P}Z = P^A Z + Z P^E \in \mathcal{H}\,, \tag{25}$$

where $P^A \in \mathscr{L}(H^A)$ and $P^E \in \mathscr{L}(H^E)$. In this case

$$\begin{aligned}\mathcal{A}^{\mathcal{P}} Z &= (\mathcal{A}+\mathcal{P})Z \\ &= (A+P^A)Z + Z(E+P^E)\,, \quad Z \in \mathscr{D}(\mathcal{A})\,,\end{aligned} \tag{26}$$

and τ-continuity of $\mathcal{P}$ follows from the relation

$$p_h(\mathcal{P}Z) \le \|P^A\|_{\mathscr{L}(H^A)} p_h(Z) + p_g(Z)\,, \quad Z \in \mathcal{H}\,,$$

where $h \in H^E$ and $g = P^E h$.

In order to deal with the equation (24) it is convenient to define, for every $t_0 > 0$, an operator $\mathcal{M}^{\mathcal{P}}$ as follows

$$(\mathcal{M}^{\mathcal{P}}\mathcal{V})(t)Z := \int_0^t \mathcal{U}(t-r)(\mathcal{P}\mathcal{V}(r)Z)\,dr\,, \quad t \in [0,t_0]\,, \tag{27}$$

where $Z \in \mathcal{H}$, $\mathcal{V}(\cdot) \in \mathcal{B}_{t_0}$ and the integral is in $(\mathcal{H},\tau)$.

Lemma 3.1 *For every $t_0 > 0$, the operator $\mathcal{M}^{\mathcal{P}}$ satisfies*

$$\mathcal{M}^{\mathcal{P}} \in \mathscr{L}(\mathcal{B}_{t_0}) \tag{28}$$

and we have the following estimate

$$\|(\mathcal{M}^{\mathcal{P}})^n\|_{\mathscr{L}(\mathcal{B}_{t_0})} \le C^n \|\mathcal{P}\|^n_{\mathscr{L}(\mathcal{H})} \frac{t_0^n}{n!}\,, \quad n \in \mathbb{N}\,, \tag{29}$$

where $C := \sup_{0\le t\le t_0} \|\mathcal{U}(t)\|_{\mathscr{L}(\mathcal{H})}$, and

$$r(\mathcal{M}^{\mathcal{P}}) := \limsup_{n\to\infty} \sqrt[n]{\|(\mathcal{M}^{\mathcal{P}})^n\|_{\mathscr{L}(\mathcal{B}_{t_0})}} = 0\,, \tag{30}$$

where $r(\mathcal{M}^{\mathcal{P}})$ is the spectral radius of $\mathcal{M}^{\mathcal{P}} \in \mathscr{L}(\mathcal{B}_{t_0})$. We also have

$$\|((\mathcal{M}^{\mathcal{P}})^n\mathcal{U})(t)\|_{\mathscr{L}(\mathcal{H})} \le C_\omega e^{\omega t} C_\omega^n \|\mathcal{P}\|^n_{\mathscr{L}(\mathcal{H})} \frac{t^n}{n!}\,,\ t \ge 0\,, \tag{31}$$

for $n \in \mathbb{N}$, where ω and C_ω are the constants from (9).

The proof of this lemma is given in the Appendix.

The main result on bounded perturbations of the implemented semigroup reads as follows.

Theorem 3.2 *Let $(\mathcal{U}(t))_{t\ge0} \subset \mathscr{L}(\mathcal{H})$ be an implemented semigroup, $\mathcal{A}$ denote its generator and $\mathcal{P} \in \mathscr{L}(\mathcal{H})$ be τ-continuous on $\|\cdot\|$-bounded sets. There exists a unique strongly τ-continuous family $(\mathcal{V}(t))_{t\ge0} \subset \mathscr{L}(\mathcal{H})$ satisfying the integral equation*

$$\mathcal{V}(t)Z = \mathcal{U}(t)Z + \int_0^t \mathcal{U}(t-r)(\mathcal{P}(\mathcal{V}(r)Z))\,dr\,, \quad t \ge 0\,,\ Z \in \mathcal{H}\,, \tag{32}$$

such that:

(i) The family $(\mathcal{V}(t))_{t\geq 0} \subset \mathscr{L}(\mathcal{H})$ *admits the following representation*

$$\mathcal{V}(t) = \sum_{n=0}^{\infty}((\mathcal{M}^{\mathcal{P}})^n\mathcal{U})(t) = \sum_{n=0}^{\infty}\mathcal{W}_n(t)\,, \tag{33}$$

where $t \geq 0$ *and the series is convergent in the norm of* $\mathscr{L}(\mathcal{H})$*, uniformly in* t *on every finite interval, and*

$$\begin{aligned} \mathcal{W}_0(t) &:= \mathcal{U}(t)\,, \\ \mathcal{W}_{n+1}(t) &:= (\mathcal{M}^{\mathcal{P}}\mathcal{W}_n)(t) \\ &= \int_0^t \mathcal{U}(t-r)\mathcal{P}\mathcal{W}_n(r)\,dr\,, \quad t\geq 0\,, \quad n\in\mathbb{N}\,, \end{aligned}$$

where the integral is understood in $(\mathcal{H},\tau)$*.*

(ii) $(\mathcal{V}(t))_{t\geq 0} \subset \mathscr{L}(\mathcal{H})$ *is a strongly* τ*-continuous semigroup and its generator* $(\mathcal{A}^{\mathcal{P}}, \mathscr{D}(\mathcal{A}^{\mathcal{P}}))$ *(defined as in Definition 1.2) is given by*

$$\mathcal{A}^{\mathcal{P}} = \mathcal{A} + \mathcal{P}\,, \quad \mathscr{D}(\mathcal{A}^{\mathcal{P}}) = \mathscr{D}(\mathcal{A})\,. \tag{34}$$

(iii) The following estimate holds

$$\|\mathcal{V}(t)\|_{\mathscr{L}(\mathcal{H})} \leq C_\omega\, e^{(\omega + C_\omega\|\mathcal{P}\|_{\mathscr{L}(\mathcal{H})})t}\,, \quad t\geq 0\,, \tag{35}$$

where C_ω *and* ω *are constants from* (9)*.*

(iv) The following holds

$$\mathbb{C}_{(\omega + C_\omega\|\mathcal{P}\|_{\mathscr{L}(\mathcal{H})})} \subset \varrho(\mathcal{A}^{\mathcal{P}})\,, \tag{36}$$

where

$$\mathbb{C}_{(\omega + C_\omega\|\mathcal{P}\|_{\mathscr{L}(\mathcal{H})})} = \{\lambda\in\mathbb{C}:\ \operatorname{Re}\lambda > \omega + C_\omega\|\mathcal{P}\|_{\mathscr{L}(\mathcal{H})}\}\,,$$

and for $\lambda \in \mathbb{C}_{(\omega + C_\omega\|\mathcal{P}\|_{\mathscr{L}(\mathcal{H})})}$ *we have*

$$\begin{aligned} \mathcal{R}(\lambda,\mathcal{A}^{\mathcal{P}}) &= \mathcal{R}(\lambda,\mathcal{A})(I - \mathcal{P}\mathcal{R}(\lambda,\mathcal{A}))^{-1} \\ &= (I - \mathcal{R}(\lambda,\mathcal{A})\mathcal{P})^{-1}\mathcal{R}(\lambda,\mathcal{A})\,, \end{aligned}$$

and

$$\|\mathcal{R}(\lambda,\mathcal{A}^{\mathcal{P}})\|_{\mathscr{L}(\mathcal{H})} \leq \frac{C_\omega}{\operatorname{Re}\lambda - \omega - C_\omega\|\mathcal{P}\|_{\mathscr{L}(\mathcal{H})}}\,, \tag{37}$$

where ω *i* C_ω *are constants from* (9)*.*

Proof 3.3 *Let us rewrite the equation* (32) *in the simplified form as an equation in* $\mathscr{L}(\mathcal{H})$

$$\mathcal{V}(t) = \mathcal{U}(t) + (\mathcal{M}^{\mathcal{P}}\mathcal{V})(t)\,, \quad t\geq 0\,, \tag{38}$$

and apply the method of successive approximation to show that it has a unique solution. For this purpose we define a sequence of functions $(\mathcal{V}_k(t))_{t\geq 0} \subset \mathscr{L}(\mathcal{H})$*,* *where*

$$\begin{aligned}\mathcal{V}_0(t) &:= \mathcal{U}(t)\\ \mathcal{V}_1(t) &:= \mathcal{U}(t) + (\mathcal{M}^{\mathcal{P}}\mathcal{V}_0)(t) = \mathcal{U}(t) + (\mathcal{M}^{\mathcal{P}}\mathcal{U})(t)\\ &\vdots\\ \mathcal{V}_k(t) &:= \mathcal{U}(t) + (\mathcal{M}^{\mathcal{P}}\mathcal{V}_{k-1})(t) = \sum_{n=0}^{n=k}((\mathcal{M}^{\mathcal{P}})^n\mathcal{U})(t)\end{aligned}$$

and show that there exists the limit

$$\begin{aligned}\|\cdot\|_{\mathscr{L}(\mathcal{H})}\text{-}\lim_{k\to\infty}\mathcal{V}_k(t) &= \sum_{n=0}^{\infty}((\mathcal{M}^{\mathcal{P}})^n\mathcal{U})(t)\\ &= \mathcal{V}(t)\in\mathscr{L}(\mathcal{H})\,,\quad t\geq 0\,.\end{aligned}\tag{39}$$

Using the estimate (31) *we can majorize the sequence* $(\mathcal{V}_k(t))_{k\geq 0}$ *as follows*

$$\begin{aligned}\|\mathcal{V}_k(t)\|_{\mathscr{L}(\mathcal{H})} &= \|\sum_{n=0}^{n=k}((\mathcal{M}^{\mathcal{P}})^n\mathcal{U})(t)\|_{\mathscr{L}(\mathcal{H})}\\ &\leq C_\omega e^{\omega t}\sum_{n=0}^{n=k}C_\omega^n\|\mathcal{P}\|^n_{\mathscr{L}(\mathcal{H})}\frac{t^n}{n!}\,,\quad t\geq 0\,,\end{aligned}$$

which implies the required (pointwise in t) convergence of (39) *with*

$$\begin{aligned}\|\mathcal{V}(t)\|_{\mathscr{L}(\mathcal{H})} &= \|\sum_{n=0}^{\infty}((\mathcal{M}^{\mathcal{P}})^n\mathcal{U})(t)\|_{\mathscr{L}(\mathcal{H})}\\ &\leq C_\omega e^{\omega t}\sum_{n=0}^{\infty}C_\omega^n\|\mathcal{P}\|^n_{\mathscr{L}(\mathcal{H})}\frac{t^n}{n!}\\ &= C_\omega e^{\omega t}e^{C_\omega\|\mathcal{P}\|_{\mathscr{L}(\mathcal{H})}t}\,,\quad t\geq 0\,.\end{aligned}\tag{40}$$

It also follows from the properties of the operator $\mathcal{M}^{\mathcal{P}}$ *that every function* $(\mathcal{V}_k(t))_{t\geq 0}\subset\mathscr{L}(\mathcal{H})$ *is strongly* τ*-continuous and bi-equicontinuous. Moreover, it follows from the two above estimates that in fact the convergence* (39) *is uniform in* t *on every finite interval* $[0,t_0]$*, i.e., for every* $t_0>0$ *we have convergence in the norm of* $\mathcal{B}_{t_0}$*. Consequently, for every* $t_0>0$ *the limit function* $(\mathcal{V}(t))_{t\geq 0}\subset\mathscr{L}(\mathcal{H})$ *restricted to the interval* $[0,t_0]$ *satisfies* $\mathcal{V}(\cdot)\in\mathcal{B}_{t_0}$*, which means that* $(\mathcal{V}(t))_{t\geq 0}\subset\mathscr{L}(\mathcal{H})$ *is also strongly* τ*-continuous and bi-equicontinuous.*

In order to show that the function

$$\mathcal{V}(t) = \sum_{n=0}^{\infty}((\mathcal{M}^{\mathcal{P}})^n\mathcal{U})(t)\,,\quad t\geq 0\tag{41}$$

(defined by (39)*) satisfies the integral equation* (38) *let us apply* $\mathcal{M}^{\mathcal{P}}$ *to both sides of* (41) *and then we get*

$$\begin{aligned}(\mathcal{M}^{\mathcal{P}}\mathcal{V})(t) &= (\mathcal{M}^{\mathcal{P}}\mathcal{U})(t) + ((\mathcal{M}^{\mathcal{P}})^2\mathcal{U})(t) + ((\mathcal{M}^{\mathcal{P}})^3\mathcal{U})(t) + \cdots \\ &= -\mathcal{U}(t) + \mathcal{U}(t) + (\mathcal{M}^{\mathcal{P}}\mathcal{U})(t) + ((\mathcal{M}^{\mathcal{P}})^2\mathcal{U})(t) + \cdots \\ (\mathcal{M}^{\mathcal{P}}\mathcal{V})(t) &= -\mathcal{U}(t) + \mathcal{V}(t)\,, \quad t \geq 0\,,\end{aligned}$$

as required.

Finally, to prove the uniqueness of $(\mathcal{V}(t))_{t\geq 0}$ *let us assume that* $(\mathcal{W}(t))_{t\geq 0}$ *is another solution of* (38) *and hence the difference* $\mathcal{D}(t) := \mathcal{V}(t) - \mathcal{W}(t)$ *satisfies the equation* $\mathcal{D}(t) = (\mathcal{M}^{\mathcal{P}}\mathcal{D})(t)$ *for* $t \geq 0$*, which in fact implies that*

$$\mathcal{D}(t) = ((\mathcal{M}^{\mathcal{P}})^n\mathcal{D})(t)\,, \quad t \geq 0\,, \quad n \in \mathbb{N}\,.$$

Using now the estimate (29) *we see that for every* $t_0 > 0$ *we have*

$$\begin{aligned}\|\mathcal{D}\|_{\mathcal{B}_{t_0}} &\leq \limsup_{n\to\infty} \|(\mathcal{M}^{\mathcal{P}})^n\|_{\mathscr{L}(\mathcal{B}_{t_0})}\|\mathcal{D}\|_{\mathcal{B}_{t_0}} \\ &\leq \limsup_{n\to\infty} C^n\|\mathcal{P}\|^n_{\mathscr{L}(\mathcal{H})}\frac{t_0^n}{n!}\|\mathcal{D}\|_{\mathcal{B}_{t_0}} = 0\,,\end{aligned}$$

i.e., $\mathcal{D}(t) = 0$ *for every* $t \geq 0$*. This completes the proof of the first part of the lemma.*

(i) Since $\mathcal{W}_n(t) = (\mathcal{M}^{\mathcal{P}})^n\mathcal{U})(t)$ *for* $t \geq 0$*, then this part follows immediately from the above considerations.*

(ii)+(iii)+(iv) It has been shown above that the family $(\mathcal{V}(t))_{t\geq 0} \subset \mathscr{L}(\mathcal{H})$ *is strongly* τ*-continuous. In order to prove that it is a semigroup we use the equation* (38)*. The condition* (30) *implies that for every* $t_0 > 0$ *there exists an inverse* $(I - \mathcal{M}^{\mathcal{P}})^{-1} \in \mathscr{L}(\mathcal{B}_{t_0})$ *and hence function* $(\mathcal{V}(t))_{t\geq 0} \subset \mathscr{L}(\mathcal{H})$ *can be expressed in the form*

$$\mathcal{V}(t) = ((I - \mathcal{M}^{\mathcal{P}})^{-1}\mathcal{U})(t)\,, \quad t \geq 0\,. \tag{42}$$

Let us now consider the equation

$$\mathcal{V}(t+s) = \mathcal{U}(t+s) + \int_0^{t+s} \mathcal{U}(t+s-r)\mathcal{P}\mathcal{V}(r)\,dr\,, \quad t,s \geq 0\,,$$

which, after manipulations involving semigroup properties of $(\mathcal{U}(t))_{t\geq 0} \subset \mathscr{L}(\mathcal{H})$ *and change of the integration variable, can be transformed to the form*

$$\mathcal{V}(t+s) = \mathcal{U}(t)\mathcal{V}(s) + \int_0^{t} \mathcal{U}(t-r)\mathcal{P}\mathcal{V}(r+s)\,dr\,, \quad t,s \geq 0\,. \tag{43}$$

Fixing $s \geq 0$ *and introducing* $\mathcal{V}_s(t) := \mathcal{V}(t+s)$*, where* $t \geq 0$*, we rewrite* (43) *as follows*

$$\mathcal{V}_s(t) = \mathcal{U}(t)\mathcal{V}(s) + (\mathcal{M}^{\mathcal{P}}\mathcal{V}_s)(t)\,, \quad t \geq 0\,.$$

Hence, making use of (38) *and* (42)*, we obtain*

$$\mathcal{V}_s(t) = ((I - \mathcal{M}^{\mathcal{P}})^{-1}\mathcal{U})(t)\mathcal{V}(s) = \mathcal{V}(t)\mathcal{V}(s)\,, \quad t \geq 0\,,$$

which simply means that $\mathcal{V}(t+s) = \mathcal{V}(t)\mathcal{V}(s)$ *for* $t, s \geq 0$*. By changing the role of* t *and* s *in the equation* (43) *we get* $\mathcal{V}(t+s) = \mathcal{V}(s)\mathcal{V}(t)$ *for* $t, s \geq 0$*. Since the equality* $\mathcal{V}(0) = I$ *is obvious, the semigroup properties of* $(\mathcal{V}(t))_{t\geq 0} \subset \mathscr{L}(\mathcal{H})$ *are proven.*

Let $(\mathcal{A}^{\mathcal{P}}, \mathscr{D}(\mathcal{A}^{\mathcal{P}}))$ *be the generator of the semigroup* $(\mathcal{V}(t))_{t\geq 0} \subset \mathscr{L}(\mathcal{H})$ *(defined according to Definition 1.2). If we assume* $Z \in \mathscr{D}(\mathcal{A}^{\mathcal{P}})$*, then we can differentiate both sides of* (32) *as follows*

$$\dot{\mathcal{V}}(t)Z = \mathcal{A}_{-1}(\mathcal{U}_{-1}(t)Z) + \mathcal{A}\int_0^t \mathcal{U}(t-r)(\mathcal{P}(\mathcal{V}(r)Z))\,dr + \mathcal{P}(\mathcal{V}(t)Z)\,, \quad t \geq 0\,,$$

where the equality is understood in an extrapolated space $\mathcal{H}_{-1}$ *(*$\mathcal{H}_1 \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{H}_{-1}$*). This space, together with the extrapolated (extended) implemented semigroup* $(\mathcal{U}_{-1}(t))_{t\geq 0} \subset \mathscr{L}(\mathcal{H}_{-1})$ *and its generator* $\mathcal{A}_{-1}$ *are defined, e.g., in Emirsajłow (2012). Hence, for* $t = 0$*, we get*

$$\mathcal{A}^{\mathcal{P}}Z = \mathcal{A}_{-1}Z + \mathcal{P}Z\,, \quad Z \in \mathscr{D}(\mathcal{A}^{\mathcal{P}})\,,$$

which means that $\mathcal{A}_{-1}Z \in \mathcal{H}$*, i.e.,* $Z \in \mathscr{D}(\mathcal{A})$ *and*

$$\mathcal{A}^{\mathcal{P}}Z = \mathcal{A}Z + \mathcal{P}Z\,, \quad Z \in \mathscr{D}(\mathcal{A}^{\mathcal{P}}) \subset \mathscr{D}(\mathcal{A})\,.$$

In order to show that, in fact, $\mathscr{D}(\mathcal{A}^{\mathcal{P}}) = \mathscr{D}(\mathcal{A})$*, let us assume* $\lambda \in \varrho(\mathcal{A})$ *and consider the relation*

$$\begin{aligned} \lambda I - \mathcal{A}^{\mathcal{P}} = \lambda I - \mathcal{A} - \mathcal{P} &= (I - \mathcal{P}\mathcal{R}(\lambda, \mathcal{A}))(\lambda I - \mathcal{A}) \\ &= (\lambda I - \mathcal{A})(I - \mathcal{R}(\lambda, \mathcal{A})\mathcal{P}). \end{aligned}$$

Using (10) *we see that for* $\lambda \in \mathbb{C}$ *such that* $\operatorname{Re}\lambda > \omega + C_\omega\|\mathcal{P}\|_{\mathscr{L}(\mathcal{H})}$*, we have*

$$\|\mathcal{P}\mathcal{R}(\lambda, \mathcal{A})\|_{\mathscr{L}(\mathcal{H})} = \|\mathcal{R}(\lambda, \mathcal{A})\mathcal{P}\|_{\mathscr{L}(\mathcal{H})} \leq \|\mathcal{P}\|_{\mathscr{L}(\mathcal{H})}\frac{C_\omega}{\operatorname{Re}\lambda - \omega} < 1\,, \tag{44}$$

which implies that there exist inverses $(I - \mathcal{P}\mathcal{R}(\lambda, \mathcal{A}))^{-1}$*,* $(I - \mathcal{R}(\lambda, \mathcal{A})\mathcal{P})^{-1} \in \mathscr{L}(\mathcal{H})$ *and*

$$\mathcal{R}(\lambda, \mathcal{A}^{\mathcal{P}}) = \mathcal{R}(\lambda, \mathcal{A})(I - \mathcal{P}\mathcal{R}(\lambda, \mathcal{A}))^{-1} \tag{45}$$

$$= (I - \mathcal{R}(\lambda, \mathcal{A})\mathcal{P})^{-1}\mathcal{R}(\lambda, \mathcal{A})\,, \tag{46}$$

i.e., $\mathbb{C}_{(\omega + C_\omega\|\mathcal{P}\|_{\mathscr{L}(\mathcal{H})})} \subset \varrho(\mathcal{A}^{\mathcal{P}})$*. In particular, since* $(I - \mathcal{P}\mathcal{R}(\lambda, \mathcal{A}))^{-1} \in \mathscr{L}(\mathcal{H})$ *is a bijection, then* (45) *implies that* $\mathscr{R}(\mathcal{R}(\lambda, \mathcal{A}^{\mathcal{P}})) = \mathscr{R}(\mathcal{R}(\lambda, \mathcal{A}))$*, i.e.,* $\mathscr{D}(\mathcal{A}^{\mathcal{P}}) = \mathscr{D}(\mathcal{A})$*.*

Moreover, combining (45) *(or* (46)*) with* (44) *we also obtain*

$$\|\mathcal{R}(\lambda, \mathcal{A}^{\mathcal{P}})\|_{\mathscr{L}(\mathcal{H})} \leq \frac{C_\omega}{\operatorname{Re}\lambda - \omega - C_\omega\|\mathcal{P}\|_{\mathscr{L}(\mathcal{H})}}\,. \tag{47}$$

The estimate (35) *has been already derived as relation* (40)*.*

In the important case when the implemented semigroup $(\mathcal{U}(t))_{t\geq 0} \subset \mathscr{L}(\mathcal{H})$ is exponentially stable, we immediately obtain the following result:

Corollary 3.4 *If the implemented semigroup $(\mathcal{U}(t))_{t\geq 0} \subset \mathscr{L}(\mathcal{H})$ is exponentially stable, i.e. $\omega_0(\mathcal{U}) < 0$, and the constant ω satisfies $0 > \omega > \omega_0(\mathcal{U})$, then for arbitrary perturbation $\mathcal{P} \in \mathscr{L}(\mathcal{H})$ such that*

$$\|\mathcal{P}\|_{\mathscr{L}(\mathcal{H})} < \frac{|\omega|}{C_\omega}, \tag{48}$$

the perturbed semigroup $(\mathcal{V}(t))_{t\geq 0} \subset \mathscr{L}(\mathcal{H})$ is also exponentially stable (and satisfies (35)*).*

4. Perturbed algebraic Sylvester equation

In this section we will specify the obtained results to the perturbed algebraic Sylvester equation. Using the notation $\mathcal{H}^E := \mathscr{L}(H^E)$ and $\mathcal{H}^A := \mathscr{L}(H^A)$, we assume throughout the section that the strongly continuous semigroups $(T(t))_{t\geq 0} \subset \mathcal{H}^A$ and $(S(t))_{t\geq 0} \subset \mathcal{H}^E$ satisfy

$$\omega_0(T) + \omega_0(S) < 0. \tag{49}$$

It follows that for all ω_1, ω_2 and C_1, C_2 such that $0 > \omega_1 + \omega_2 > \omega_0(T) + \omega_0(S)$ and

$$\|T(t)\|_{\mathcal{H}^A} \leq C_1 e^{\omega_1 t}, \quad \|S(t)\|_{\mathcal{H}^E} \leq C_2 e^{\omega_2 t}, \quad t \geq 0,$$

we have

$$\|\mathcal{U}(t)\|_{\mathscr{L}(\mathcal{H})} \leq C_1 C_2 e^{(\omega_1+\omega_2)t}, \quad t \geq 0.$$

Since the implemented semigroup is exponentially stable, then by Corollary 2.3 the algebraic Sylvester equation

$$AZh + ZEh = Fh, \quad h \in H_1^E, \tag{50}$$

where the equality is in H^A, has for every operator $F \in \mathcal{H}$ a unique solution $Z \in \mathcal{H}_1 \subset \mathcal{H}$.

Let us now consider the *perturbed algebraic Sylvester equation*

$$A^P Z^{\mathcal{P}} h + Z^{\mathcal{P}} E^P h = Fh, \quad h \in H_1^E, \tag{51}$$

where the equality is in H^A, and the perturbed operators A^P and E^P are given by

$$A^P = A + P^A, \quad E^P = E + P^E, \tag{52}$$

where $P^A \in \mathcal{H}^A$ and $P^E \in \mathcal{H}^E$ are unknown bounded perturbations. For this equation we have the following result:

Lemma 4.1 *If the perturbations $P^A \in \mathcal{H}^A$ and $P^E \in \mathcal{H}^E$ satisfy the bound*

$$\|P^A\|_{\mathcal{H}^A} + \|P^E\|_{\mathcal{H}^E} < \frac{|\,\omega_1 + \omega_2|}{C_1 C_2}\,, \tag{53}$$

then for every $F \in \mathcal{H}$ the perturbed algebraic Sylvester equation (51) *admits a unique solution $Z^{\mathcal{P}} \in \mathcal{H}_1 \subset \mathcal{H}$ such that*

$$\|Z - Z^{\mathcal{P}}\|_1 \le \frac{C_1C_2(\|P^A\|_{\mathcal{H}^A} + \|P^E\|_{\mathcal{H}^E})\|F\|}{|\,\omega_1 + \omega_2| - C_1C_2(\|P^A\|_{\mathcal{H}^A} + \|P^E\|_{\mathcal{H}^E})}\,, \tag{54}$$

where $\|\cdot\|_1 = \|-\mathcal{A}(\,\cdot\,)\|$, and Z is a solution of the unperturbed Sylvester equation (50).

Proof 4.2 *Using the notation*

$$\mathcal{A}^{\mathcal{P}} Z = \mathcal{A}Z + \mathcal{P}Z = AZ + ZE + P^A Z + ZP^E\,, \quad Z \in \mathcal{H}_1\,,$$

the equations (50) *and* (51) *can be rewritten as*

$$-\mathcal{A}Z = -F \quad \text{and} \quad -\mathcal{A}^{\mathcal{P}} Z^{\mathcal{P}} = -F\,.$$

If we transform the second equation to the form $-\mathcal{A}Z^{\mathcal{P}} = \mathcal{P}Z^{\mathcal{P}} - F$ and subtract from the first one, we get

$$-\mathcal{A}(Z - Z^{\mathcal{P}}) = -\mathcal{P}Z^{\mathcal{P}}\,. \tag{55}$$

For the equation (55) *to make sense we have to show that the equation $-\mathcal{A}^{\mathcal{P}} Z^{\mathcal{P}} = -F$ has a unique solution. For the start let us notice that we have*

$$\omega = \omega_1 + \omega_2\,, \quad C_\omega = C_1 C_2 \tag{56}$$

and

$$\|\mathcal{P}\|_{\mathscr{L}(\mathcal{H})} \le \|P^A\|_{\mathcal{H}^A} + \|P^E\|_{\mathcal{H}^E}\,. \tag{57}$$

Corollary 3.4 implies that for perturbations such that (see (53)*)*

$$0 > \omega_1 + \omega_2 + C_1C_2(\|P^A\|_{\mathcal{H}^A} + \|P^E\|_{\mathcal{H}^E})\,, \tag{58}$$

the perturbed semigroup $(\mathcal{V}(t))_{t\ge 0}) \subset \mathscr{L}(\mathcal{H})$, with generator $(\mathcal{A}^{\mathcal{P}}, \mathscr{D}(\mathcal{A}^{\mathcal{P}}) = \mathscr{D}(\mathcal{A}))$, is exponentially stable and by Corollary 2.3, the equation $-\mathcal{A}^{\mathcal{P}} Z^{\mathcal{P}} = -F$ has a unique solution. Moreover, $0 \subset \varrho(\mathcal{A}^{\mathcal{P}})$ and this solution can be expressed in the form

$$Z^{\mathcal{P}} = -\mathcal{R}(0, \mathcal{A}^{\mathcal{P}})F\,.$$

Substituting this expression into (55) *we get*

$$-\mathcal{A}(Z - Z^{\mathcal{P}}) = \mathcal{P}(\mathcal{R}(0, \mathcal{A}^{\mathcal{P}})F)\,, \tag{59}$$

and hence

$$\| -\mathcal{A}(Z - Z^{\mathcal{P}})\| = \|\mathcal{P}(\mathcal{R}(0, \mathcal{A}^{\mathcal{P}})F)\| \leq \|\mathcal{P}\|_{\mathscr{L}(\mathcal{H})}\|\mathcal{R}(0, \mathcal{A}^{\mathcal{P}})\|_{\mathscr{L}(\mathcal{H})}\|F\|. \quad (60)$$

Taking into account relations (56)-(58) *the condition* (47) *implies the inequality*

$$\|\mathcal{R}(0, \mathcal{A}^{\mathcal{P}})\|_{\mathscr{L}(\mathcal{H})} \leq \frac{C_1C_2}{-\omega_1 - \omega_2 - C_1C_2(\|P^A\|_{\mathcal{H}^A} + \|P^E\|_{\mathcal{H}^E})},$$

which, applied to (60)*, leads to the estimation* (54).

In some applications the following corollary to Lemma 4.1 may be useful.

Corollary 4.3 *Let the solution $Z \in \mathcal{H} = \mathscr{L}(H^E, H^A)$ of the Sylvester equation* (50) *have an inverse $Z^{-1} \in \mathscr{L}(H^A, H^E)$ and $C > 0$ be a constant from the inequality*

$$\|X\| \leq C\|X\|_1, \quad X \in \mathcal{H}_1 .$$

If the perturbations $P^A \in \mathcal{H}^A$ and $P^E \in \mathcal{H}^E$ satisfy the bound

$$\|P^A\|_{\mathcal{H}^A} + \|P^E\|_{\mathcal{H}^E} < \frac{|\,\omega_1 + \omega_2|}{C_1C_2}, \quad (61)$$

and, additionally, the following bound holds

$$\frac{CC_1C_2(\|P^A\|_{\mathcal{H}^A} + \|P^E\|_{\mathcal{H}^E})\|F\|}{|\,\omega_1 + \omega_2| - C_1C_2(\|P^A\|_{\mathcal{H}^A} + \|P^E\|_{\mathcal{H}^E})} < \frac{1}{\|Z^{-1}\|_{\mathscr{L}(H^A,H^E)}}, \quad (62)$$

then the solution $Z^{\mathcal{P}} \in \mathcal{H}$ of the perturbed Sylvester equation (51) *has an inverse $(Z^{\mathcal{P}})^{-1} \in \mathscr{L}(H^A, H^E)$.*

Proof 4.4 *It follows from Lemma 4.1, if we use the well-known fact from functional analysis, that if $Z \in \mathscr{L}(H^E, H^A)$ has a bounded inverse $Z^{-1} \in \mathscr{L}(H^A, H^E)$ and*

$$\|Z - Z^{\mathcal{P}}\|_{\mathscr{L}(H^E,H^A)} < \frac{1}{\|Z^{-1}\|_{\mathscr{L}(H^A,H^E)}}, \quad (63)$$

then there also exists an inverse $(Z^{\mathcal{P}})^{-1} \in \mathscr{L}(H^A, H^E)$.

We illustrate applicability of the general results with the following example.

Example 4.5 *Let us assume that H^A, A and $(T(t))_{t\geq 0}$ are the same as in Section 1. In the obvious way we also define the unbounded adjoint operator A^* on H^A and the adjoint semigroup $(T^*(t))_{t\geq 0} \subset \mathscr{L}(H^A)$. Moreover, let U be another Hilbert space - the* control space *and $B \in \mathscr{L}(U, H^A)$ - the* control operator.

Under these assumptions we consider the following control system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \in H^A, \quad (64)$$

where $(x(t))_{t\geq 0} \subset H^A$ is the state *and $u \in L^2_{loc}(0, \infty; U)$ is the* control. *For the system* (64) *we assume that it enjoys the following two properties:*

(i) the semigroup $(T(t))_{t\geq 0} \subset \mathscr{L}(H^A)$ *is exponentially stable, i.e.* $\omega_0(T) < 0$,
(ii) the system (64) *is exactly infinite-time controllable, i.e. there exists* $\gamma > 0$ *such that*

$$\langle Mh, h\rangle^A := \langle \int_0^\infty T(t)BB^*T^*(t)dt\, h, h\rangle^A \geq \gamma(\|h\|^A)^2\,, \quad h \in H^A\,, \tag{65}$$

where the operator $M \in \mathscr{L}(H^A)$ *is the* controllability gramian *of the system.*

It is rather clear that condition (ii) is equivalent to bounded invertibility of the selfadjoint and non-negative operator $M \in \mathscr{L}(H^A)$*, i.e.,*

$$M^{-1} \in \mathscr{L}(H^A)\,. \tag{66}$$

It follows from Corollary 2.3, see also Emirsajłow and Townley (2005), that if we define an implemented semigroup $(\mathcal{U})(t))_{t\geq 0} \subset \mathscr{L}(\mathscr{L}(H^A))$ *and its generator* $\mathcal{A}$ *as follows*

$$\mathcal{U}(t)Z \;=\; T(t)ZT^*(t)\,, \quad Z \in \mathscr{L}(H^A)\,, \quad t \geq 0\,, \tag{67}$$

$$(\mathcal{A}Z)h \;=\; AZh + ZA^*h\,,\; Z \in \mathscr{L}(H^A) \cap \mathcal{H}_1\,,\; h \in H_1^{A^*}\,, \tag{68}$$

then under the assumption (i) the controllability gramian $M \in \mathscr{L}(H^A)\cap\mathcal{H}_1$ *can be regarded as a unique solution of the following algebraic Sylvester equation*

$$-AMh - MA^*h = BB^*h\,, \quad h \in H_1^{A^*} = \mathscr{D}(A^*)\,, \tag{69}$$

where the equality holds in H^A, $BB^* \in \mathscr{L}(H^A)$ *and an explicit description of* $\mathcal{H}_1$ *is given as property (b) in Section 1. The above special case of the Sylvester equation, where* $E = A^*$ *and* $F = BB^*$ *is a selfadjoint and non-negative operator in* $\mathscr{L}(H^A)$*, is called the algebraic Lyapunov equation.*

Let us now assume that the system operator A *is additively perturbed by an unknown bounded perturbation* $P^A \in \mathscr{L}(H^A)$*, so that the perturbed system becomes*

$$\dot{x}(t) = A^P x(t) + Bu(t)\,, \quad x(0) = x_0 \in H^A\,, \tag{70}$$

where

$$A^P := A + P^A\,. \tag{71}$$

For the perturbed system we would like to answer the following robustness question: what perturbations $P^A \in \mathscr{L}(H^A)$ are allowed so that the perturbed system (70) preserves properties (i) and (ii), i.e., exponential stability and exact infinite-time controllability?

In order to answer this question we use the perturbed implemented semigroup $(\mathcal{U}^{\mathcal{P}})(t))_{t\geq 0} \subset \mathscr{L}(\mathscr{L}(H^A))$ *and its generator* $\mathcal{A}^{\mathcal{P}}$*, i.e.,*

$$\mathcal{U}^{\mathcal{P}}(t)Z \;=\; T^{P^A}(t)Z(T^{P^A})^*(t)\,, \quad Z \in \mathscr{L}(H^A)\,, \quad t \geq 0\,, \tag{72}$$

$$(\mathcal{A}^{\mathcal{P}}Z)h \;=\; (A + P^A)Zh + Z(A + P^A)^*h\,,\; Z \in \mathscr{L}(H^A) \cap \mathcal{H}_1\,,\; h \in H_1^{A^*} \tag{73}$$

Since we have

$$\omega_0(\mathcal{U}^{\mathcal{P}}) = \omega_0(T^{P^A}) + \omega_0((T^{P^A})^*) = 2\omega_0(T^{P^A}), \tag{74}$$

then the exponential stability of the perturbed semigroup $(T^{P^A}(t))_{t\geq 0} \in \mathscr{L}(H^A)$ *is equivalent to the exponential stability of the perturbed implemented semigroup* $(\mathcal{U}^{\mathcal{P}})(t))_{t\geq 0} \subset \mathscr{L}(\mathscr{L}(H^A))$. *Using now Corollary 3.4 we obtain the following sufficient condition for exponential stability*

$$\|P^A\|_{\mathscr{L}(H^A)} \leq \frac{|\omega|}{C_\omega^2}, \tag{75}$$

where ω *satisfies* $\omega_0(T) < \omega < 0$ *and* C_ω *is such that*

$$\|T(t)\|_{\mathscr{L}(H^A)} \leq C_\omega e^{\omega t}, \quad t \geq 0. \tag{76}$$

Under the exponential stability condition the controllability gramian $M^{\mathcal{P}}$ *of the perturbed system* (70) *is well defined by the relation*

$$M^{\mathcal{P}} := \int_0^\infty T^{P^A}(t)BB^*(T^{P^A})^*(t)dt \in \mathscr{L}(H^A), \tag{77}$$

and by Corollary 2.3 we know that $M^{\mathcal{P}} \in \mathscr{L}(H^A) \cap \mathcal{H}_1$ *and uniquely satisfies the perturbed algebraic Sylvester equation*

$$-(A + P^A)M^{\mathcal{P}}h - M^{\mathcal{P}}(A + P^A)^*h = BB^*h, \quad h \in H_1^{A^*}. \tag{78}$$

It now follows from Corollary 4.3 that the additional estimate

$$\frac{CC_\omega^2\|P^A\|_{\mathscr{L}(H^A)}\|BB^*\|_{\mathscr{L}(H^A)}}{|\omega| - C_\omega^2\|P^A\|_{\mathscr{L}(H^A)}} < \frac{1}{\|M^{-1}\|_{\mathscr{L}(H^A)}}, \tag{79}$$

provides a sufficient condition for the existence of the inverse operator $(M^{\mathcal{P}})^{-1} \in \mathscr{L}(H^A)$, *which is equivalent to the exact infinite-time controllability of the perturbed system.*

5. Final remarks

In this paper we have proved that if the infinite-dimensional algebraic Sylvester equation (50) has a unique solution $Z \in \mathcal{H}_1$, then this property is preserved under arbitrary bounded perturbations (52) as long as they remain small and the inequality (53) provides the upper bound for their norms. In this case the difference of these two solutions satisfies the estimate (54). A simple example of application of the results to the robustness of the exponential stability and the exact infinite-time controllability of an infinite-dimensional control system under bounded perturbations is given.

It seems that similar approach can be developed also for unbounded perturbations which are of great interest in output regulation as well as observer design problem for distributed parameter systems.

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A. Appendix: Proof of Lemma 3.1

Proof A.1 *It is clear that for every* $t_0 > 0$ *and* $Z \in \mathcal{H}$ *we have* $(\mathcal{M}^{\mathcal{P}}\mathcal{V})(\cdot)Z \in C([0,t_0];(\mathcal{H},\tau))$. *Moreover,*

$$\begin{aligned}\|(\mathcal{M}^{\mathcal{P}}\mathcal{V})(t)Z\| &\leq \int_0^t \|\mathcal{U}(t-r)\|_{\mathscr{L}(\mathcal{H})}\|\mathcal{P}\|_{\mathscr{L}(\mathcal{H})}\|\|\mathcal{V}(r)Z\|\,dr\\ &\leq t\,C\,\|\mathcal{P}\|_{\mathscr{L}(\mathcal{H})}\sup_{0\leq r\leq t}\|\mathcal{V}(r)\|_{\mathscr{L}(\mathcal{H})}\|Z\|\,,\end{aligned}$$

where $t \in [0,t_0]$, *and hence*

$$\|(\mathcal{M}^{\mathcal{P}}\mathcal{V})(t)\|_{\mathscr{L}(\mathcal{H})} \leq t\,C\,\|\mathcal{P}\|_{\mathscr{L}(\mathcal{H})}\|\mathcal{V}\|_{\mathcal{B}_{t_0}}\,,\ t\in[0,t_0], \tag{80}$$

which implies $\|\mathcal{M}^{\mathcal{P}}\|_{\mathcal{B}_{t_0}} < \infty$. *In order to complete the proof of* (28) *we have to show that for every* $\mathcal{V} \in \mathcal{B}_{t_0}$ *the family* $(\mathcal{M}^{\mathcal{P}}\mathcal{V})(t))_{t\in[0,t_0]} \subset \mathscr{L}(\mathcal{H})$ *is bi-equicontinuous, i.e., for every* $\|\cdot\|$*-bounded sequence* $(Z_n)_{n\in\mathbb{N}} \subset \mathcal{H}$ *which is* τ*-convergent to* $Z\in\mathcal{H}$, *every* $p_h \in \mathcal{P}$, *every and* $\varepsilon > 0$, *there exists* $n_0 \in \mathbb{N}$ *such that*

$$\sup_{0\leq t\leq t_0} p_h((\mathcal{M}^{\mathcal{P}}\mathcal{V})(t)(Z_n - Z)) < \varepsilon\,,\quad n\geq n_0\,. \tag{81}$$

Since we have

$$\begin{aligned}\sup_{0\leq t\leq t_0} p_h((\mathcal{M}^{\mathcal{P}}\mathcal{V})(t)(Z_n - Z)) &\leq \sup_{0\leq t\leq t_0}\int_0^t p_h(\mathcal{U}(t-r)(\mathcal{P}\mathcal{V}(r)(Z_n-Z)))\,dr\\ &\leq t_0\sup_{0\leq t\leq t_0} p_h(\mathcal{U}(t)(\mathcal{P}\mathcal{V}(t)(Z_n-Z)))\,,\end{aligned}$$

then (81) *will follow from sequential* τ*-continuity of the family* $(\mathcal{U}(t)\mathcal{P}\mathcal{V}(t))_{t\in[0,t_0]} \subset \mathscr{L}(\mathcal{H})$. *We prove this continuity by contradiction. For this purpose we assume that there exists a* $\|\cdot\|$*-bounded sequence* $(Z_n)_{n\in\mathbb{N}} \subset \mathcal{H}$, *which is* τ*-convergent to some* $Z \in \mathcal{H}$, *a seminorm* $p_h \in \mathcal{P}$ *and* $\varepsilon > 0$ *such that for every* $n \in \mathbb{N}$ *we can find* $t_n \in [0, t_0]$ *such that*

$$p_h(\mathcal{U}(t_n)(\mathcal{P}\mathcal{V}(t_n)(Z_n - Z))) > \varepsilon\,. \tag{82}$$

However, by (22) *and assumptions on* $\mathcal{P}$, *the sequence* $(\mathcal{P}\mathcal{V}(t_n)(Z_n - Z))_{n\in\mathbb{N}} \subset \mathcal{H}$ *is* $\|\cdot\|$*-bounded and by sequential* τ*-continuity of* $(\mathcal{V}(t))_{t\in[0,t_0]}$ *we have*

$$\lim_{n\to\infty} p_h(\mathcal{P}\mathcal{V}(t_n)(Z_n - Z)) = 0\,.$$

Thus, the bi-equicontinuity of the family $(\mathcal{U}(t))_{t\geq 0} \subset \mathscr{L}(\mathcal{H})$ *implies*

$$\lim_{n\to\infty} p_h(\mathcal{U}(t_n)(\mathcal{P}\mathcal{V}(t_n)(Z_n - Z))) = 0\,,$$

which contradicts (82) *and completes the proof of* (28).

Next we prove the relation (29) *and for this purpose we use the following estimate*

$$\|((\mathcal{M}^{\mathcal{P}})^n\mathcal{V})(t)\|_{\mathscr{L}(\mathcal{H})} \leq C^n\|\mathcal{P}\|^n_{\mathscr{L}(\mathcal{H})}\frac{t^n}{n!}\|\mathcal{V}\|_{\mathcal{B}_{t_0}}\,,\ t \in [0, t_0], \tag{83}$$

which holds for every $\mathcal{V} \in \mathcal{B}_{t_0}$ *and* $n \in \mathbb{N}$. *In order to show* (83) *we use mathematical induction. For* $n = 1$ *this estimate is just* (80). *If we now assume that it holds for some* $n \in \mathbb{N}$, *then we obtain*

$$\begin{aligned}
\|((\mathcal{M}^{\mathcal{P}})^{n+1}\mathcal{V})(t)\|_{\mathscr{L}(\mathcal{H})} &= \|\int_0^t \mathcal{U}(t-r)\mathcal{P}((\mathcal{M}^{\mathcal{P}})^n\mathcal{V})(r)\,dr\|_{\mathscr{L}(\mathcal{H})} \\
&\leq \int_0^t \|\mathcal{U}(t-r)\|_{\mathscr{L}(\mathcal{H})}\|\mathcal{P}\|_{\mathscr{L}(\mathcal{H})}\|((\mathcal{M}^{\mathcal{P}})^n\mathcal{V})(r)\|_{\mathscr{L}(\mathcal{H})}\,dr \\
&\leq \int_0^t C\|\mathcal{P}\|_{\mathscr{L}(\mathcal{H})}C^n\|\mathcal{P}\|^n_{\mathscr{L}(\mathcal{H})}\frac{r^n}{n!}\|\mathcal{V}\|_{\mathcal{B}_{t_0}}dr \\
&= C^{n+1}\|\mathcal{P}\|^{n+1}_{\mathscr{L}(\mathcal{H})}\frac{t^{n+1}}{(n+1)!}\|\mathcal{V}\|_{\mathcal{B}_{t_0}}\,,
\end{aligned}$$

i.e., it also holds for $n+1$ *and hence for every* $n \in \mathbb{N}$. *It is obvious that* (83) *implies* (29). *In turn, relation* (30) *follows easily from the estimate* (29).

It now remains to show (31). *For* $n = 1$ *we have*

$$\begin{aligned}
\|(\mathcal{M}^{\mathcal{P}}\mathcal{U})(t)\|_{\mathscr{L}(\mathcal{H})} &= \|\int_0^t \mathcal{U}(t-r)\mathcal{P}\mathcal{U}(r)\,dr\|_{\mathscr{L}(\mathcal{H})} \\
&\leq \int_0^t \|\mathcal{U}(t-r)\|_{\mathscr{L}(\mathcal{H})}\|\mathcal{P}\|_{\mathscr{L}(\mathcal{H})}\|\mathcal{U}(r)\|_{\mathscr{L}(\mathcal{H})}\,dr \\
&\leq \int_0^t C_\omega e^{\omega(t-r)}\|\mathcal{P}\|_{\mathscr{L}(\mathcal{H})}C_\omega e^{\omega r}\,dr \\
&= C_\omega e^{\omega t}C_\omega\|\mathcal{P}\|_{\mathscr{L}(\mathcal{H})}\,t\,, \quad t \geq 0\,.
\end{aligned}$$

If we now assume that (31) *holds for some* $n \in \mathbb{N}$*, then*

$$\begin{aligned}
\|((\mathcal{M}^{\mathcal{P}})^{n+1}\mathcal{U})(t)\|_{\mathscr{L}(\mathcal{H})} &= \|\int_0^t \mathcal{U}(t-r)\mathcal{P}((\mathcal{M}^{\mathcal{P}})^n\mathcal{U})(r)\,dr\|_{\mathscr{L}(\mathcal{H})} \\
&\leq \int_0^t \|\mathcal{U}(t-r)\|_{\mathscr{L}(\mathcal{H})}\|\mathcal{P}\|_{\mathscr{L}(\mathcal{H})}\|((\mathcal{M}^{\mathcal{P}})^n\mathcal{U})(r)\|_{\mathscr{L}(\mathcal{H})}\,dr \\
&\leq \int_0^t C_\omega e^{\omega(t-r)}\|\mathcal{P}\|_{\mathscr{L}(\mathcal{H})}C_\omega e^{\omega r}C_\omega^n\|\mathcal{P}\|^n_{\mathscr{L}(\mathcal{H})}\frac{r^n}{n!}\,dr \\
&= C_\omega e^{\omega t}C_\omega^{n+1}\|\mathcal{P}\|^{n+1}_{\mathscr{L}(\mathcal{H})}\frac{t^{n+1}}{(n+1)!}\,,
\end{aligned}$$

i.e., it also holds for $n+1$ *and hence for every* $n \in \mathbb{N}$.