

On reconstructing unknown characteristics of a nonlinear system of differential equations

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Problems of dynamical reconstruction of unknown characteristics for nonlinear equations described the process of diffusion of innovations through results of observations of phase states are considered. Solving algorithms, which are stable with respect to informational noises and computational errors, are designed. The algorithms are based on the principle of auxiliary models with adaptive controls.

Key words: nonlinear differential equations, dynamical reconstruction.

1. Introduction. Statement of the problems

The problems of dynamical reconstruction on the basis of available information on an object are well known in engineering and scientific research. Our goal is to describe some algorithms for reconstructing unknown varying characteristics of dynamical systems described by second order differential equations. These algorithms are expected to be used in real time, in other words, they must be dynamical. The information on initial data is uncertain and, in general, time-varying. The algorithms should be regularizing in the sense that the final result improves if the input information becomes more accurate.

It should be noted that system (1) considered below was introduced in [1, 2] for describing the process of diffusion of innovations in a social medium.

So, we consider a system described by the equations:

$$\begin{aligned} \dot{x}_1(t) &= k(t)x_2(t) + x_1(t)(\lambda x_2(t) - \nu), \\ \dot{x}_2(t) &= -k(t)x_2(t) - (\lambda x_1(t) + \mu)x_2(t) + \gamma(t), \\ t \in T &= [t_0, \vartheta], \quad x_1(t_0) = x_{10}, \quad x_2(t_0) = x_{20}. \end{aligned} \tag{1}$$

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The work was supported by RFBR (project 13-01-12446-ofi-M2), by the Russian Fund for Humanities (project 14-02-00331a), and by the Program of UD RAS (project 15-7-1-22).

Received 10.02.2015.

Assume that positive constants λ, ν, μ , and a non-negative function $k(\cdot)$ are known whereas but the function $x_2(\cdot)$ or the function $\gamma(\cdot)$ are uncertain. We consider the situation when function $\gamma(t)$ (Lebesgue measurable and satisfying the condition $\gamma(t) \in P = [-\gamma, \gamma], t \in T$) acts upon the system. Here, $\gamma = \text{const} \in (0, +\infty)$. At discrete time moments

$$\tau_i \in \Delta = \{\tau_i\}_{i=0}^m, \quad \tau_{i+1} = \tau_i + \delta, \quad \tau_0 = t_0, \quad \tau_m = \vartheta,$$

some value $z(\tau_i) \in \mathbb{R}^n, n = 1$ or 2 , is inaccurately measured. Results of measurements (elements $\xi_i^h \in \mathbb{R}^n$) satisfy the inequalities

$$|z(\tau_i) - \xi_i^h|_n \leq h, \tag{2}$$

where $h \in (0, 1)$ is a level of informational noise, $|x|_1 = |x|$ is the modul of the number $x, |y|_2 = \max\{|y_1|, |y_2|\}$ is the norm of the vector $y = \{y_1, y_2\} \in \mathbb{R}^2$. We consider two cases. In the first case, we assume that the coordinate $x_1(\tau_i)$ is measured at the moment τ_i , i. e.,

$$z(\tau_i) = x_1(\tau_i), \quad \xi_i^h \in \mathbb{R}. \tag{3}$$

In the second one, the pair of coordinates $x_1(\tau_i)$ and $x_2(\tau_i)$ are measured. Then,

$$z(\tau_i) = \{x_1(\tau_i), x_2(\tau_i)\}, \quad \xi_i^h = \{\xi_{1i}^h, \xi_{2i}^h\} \in \mathbb{R}^2. \tag{4}$$

The problems under consideration consist in the following. It is required to design an algorithm allowing us to reconstruct the unknown coordinate $x_2(\cdot)$ (Problem 1) or (the second case) the input $\gamma(\cdot)$ (Problem 2). This is the meaningful statement of problems being investigated in the present paper.

Hereinafter, we assume that the following condition is fulfilled.

Condition 1 a) *The real input $\gamma = \gamma(t)$ generates the solution $x(\cdot) = \{x_1(\cdot), x_2(\cdot)\} = x(\cdot; t_0, x_{10}, x_{20}, \gamma(\cdot))$ of equation (1) such that*

$$\inf_{t \in T} |k(t) + \lambda x_1(t)| \geq c > 0.$$

b) *The function $k(t)$ is Lebesgue measurable and bounded.*

Thus, Problems 1 and 2 may be formulated as follows. In the sequel, a family of partitions

$$\Delta_h = \{\tau_{i,h}\}_{h=0}^{m_h}, \quad \tau_{i+1,h} = \tau_{i,h} + \delta(h), \quad \tau_{0,h} = t_0, \quad \tau_{m_h,h} = \vartheta, \tag{5}$$

of the interval T is assumed to be fixed.

Problem 1. It is required to indicate a rule of choosing of controls u_i^h at the moments τ_i being a mapping of the form

$$U^h : \{\tau_i, \xi_{i-1}^h, \xi_i^h\} \rightarrow u_i^h \in \mathbb{R} \tag{6}$$

such that the convergence

$$\int_{t_0}^{\vartheta} |u^h(t) - x_2(t)|^2 dt \rightarrow 0 \tag{7}$$

takes place as h tends to 0. Here, $u^h(t) = u_i^h$ for $t \in \delta_{i,h} = [\tau_{i,h}, \tau_{i+1,h})$.

Problem 2. It is required to indicate a rule of choosing of controls v_i^h at the moments τ_i being a mapping of the form

$$V^h : \{\tau_i, \xi_{i-1}^h, \xi_i^h\} \rightarrow v_i^h \in \mathbb{R} \tag{8}$$

such that the convergence

$$\int_{t_0}^{\vartheta} |v^h(t) - \gamma(t)|^2 dt \rightarrow 0 \tag{9}$$

takes place as h tends to 0. Here, $v^h(t) = v_i^h$ for $t \in \delta_{h,i}$.

The analogous problem is considered in the paper [3, 4], where solving algorithms based on the method of controlled models [5–12] are presented. In this paper, we design solving algorithms without the usage of such models.

2. Solving algorithms

Further, we assume that the constants d_1, d_2, k , and γ are such that

$$|x_1(t)| \leq d_1, \quad |x_2(t)| \leq d_2, \tag{10}$$

$$|\dot{k}(t)| \leq k, \quad |\dot{\gamma}(t)| \leq \gamma \quad \text{for a.a. } t \in T. \tag{11}$$

In virtue of (10) and (11), the following inequalities

$$|k(t)| \leq k_1 = |k(t_0)| + k(\vartheta - t_0), \tag{12}$$

$$|\dot{x}_1(t)| \leq d_3 = k_1 d_2 + \lambda d_1 d_2 + \nu d_1, \tag{13}$$

$$|\dot{x}_2(t)| \leq d_4 = k_1 d_2 + \lambda d_1 d_2 + \mu d_2 + \gamma \tag{14}$$

are fulfilled.

In turn, for $t \in [\tau_{i-1}, \tau_i]$, the inequality

$$|x_1(t) - \xi_{i-1}^h| \leq h + \int_{\tau_{i-1}}^t |\dot{x}_1(\tau)| d\tau \leq h + d_3 \delta \tag{15}$$

holds. Let

$$P_T(\cdot) = \{u(\cdot) \in L_2(T; \mathbb{R}) : |u(t)| \leq d_2 \text{ for a.a. } t \in T\},$$

$$U(x_1(\cdot)) = \{u(\cdot) \in P_T(\cdot) : \dot{x}_1(t) = (k(t) + \lambda x_1(t))u(t) - \nu x_1(t) \text{ for a.a. } t \in T\}.$$

Introduce a family of sets

$$U^h(\cdot) = \left\{ u(\cdot) \in P_T(\cdot) : \right.$$

$$\left. u(t) = u_i \text{ for a.a. } t \in [\tau_{i-1}, \tau_i), u_i \in \tilde{U}^h(\tau_i, \xi_{i-1}^h, \xi_i^h), i \in [1 : m] \right\},$$

where $\tau_i = \tau_{i,h}$, $m = m_h$,

$$\tilde{U}^h(\tau_i, \xi_{i-1}^h, \xi_i^h) = \left\{ u \in \mathbb{R} : \right.$$

$$\left. |u| \leq d_2, |(k(\tau_{i-1}) + \lambda \xi_{i-1}^h)u - \nu \xi_{i-1}^h - (\xi_i^h - \xi_{i-1}^h)\delta^{-1}| \leq \sigma_h^{(1)} \right\},$$

$$\sigma_h^{(1)} = 2h\delta^{-1} + K_1\delta + K_2h, \quad K_1 = \nu d_3 + (k + \lambda d_3)d_2, \quad K_2 = \nu + \lambda d_2.$$

Lemma 1 *Let $u(\cdot) \in U(x_1(\cdot))$. Then, the inclusion*

$$\delta^{-1} \int_{\tau_{i-1}}^{\tau_i} u(t) dt \in \tilde{U}^h(\tau_i, \xi_{i-1}^h, \xi_i^h), \quad i \in [1 : m]$$

takes place.

Proof. Let $f_1(t, x_1, x_2) = (k(t) + \lambda x_1)x_2 - \nu x_1$. Then, for all $q \in \mathbb{R}$, $|q| \leq d_2$, and $t \in [\tau_{i-1}, \tau_i]$, by (10), (11), and (15), we obtain

$$\begin{aligned} & |f_1(t, x_1(t), q) - f_1(\tau_{i-1}, \xi_{i-1}^h, q)| \leq |k(t) - k(\tau_{i-1})||q| \\ & + \lambda|x_1(t) - \xi_{i-1}^h||q| + \nu|x_1(t) - \xi_{i-1}^h| \leq k\delta|q| + \lambda(h + d_3\delta)|q| + \nu(h + d_3\delta) \\ & \leq \nu(h + d_3\delta) + \{\lambda h + (k + \lambda d_3)\delta\}d_2 = K_1\delta + K_2h. \end{aligned} \tag{16}$$

In this case, for any $u(\cdot) \in U(x_1(\cdot))$, the inequality

$$\begin{aligned} & \left| \delta^{-1} \int_{\tau_{i-1}}^{\tau_i} f_1(t, x_1(t), u(t)) dt - \delta^{-1} \int_{\tau_{i-1}}^{\tau_i} f_1(\tau_{i-1}, \xi_{i-1}^h, u(t)) dt \right| \\ & \leq K_1\delta + K_2h, \quad i \in [1 : m], \end{aligned} \tag{17}$$

is fulfilled. In turn, from

$$\delta^{-1} \int_{\tau_{i-1}}^{\tau_i} f_1(t, x_1(t), u(t)) dt = \delta^{-1}(x_1(\tau_i) - x_1(\tau_{i-1})),$$

we derive

$$\left| \frac{\xi_i^h - \xi_{i-1}^h}{\delta} - \delta^{-1} \int_{\tau_{i-1}}^{\tau_i} f_1(t, x_1(t), u(t)) dt \right| \leq 2\frac{h}{\delta}. \tag{18}$$

The statement of the lemma follows from (17) and (18). The lemma is proved. □

Lemma 2 For any $t_1 \in T$ and $u(\cdot) \in U^h(\cdot)$, the inequality

$$\left| \int_{t_0}^{t_1} \{k(t) + \lambda x_1(t)\} (x_2(t) - u(t)) dt \right| \leq c_1 h + c_2 \delta + c_3 h \delta^{-1}$$

is fulfilled. Here,

$$c_1 = 2 + (\vartheta - t_0)(v + \lambda d_2 + K_2),$$

$$c_2 = k_1 + d_1 + d_3 + \lambda(1 + d_1)d_2 + (\vartheta - t_0)(v d_3 + K_1 + k + \lambda d_2 d_3), \quad c_3 = 2(\vartheta - t_0).$$

Proof. Note that the following equality

$$\int_{t_0}^{t_1} \{k(t) + \lambda x_1(t)\} x_2(t) dt = x_1(t_1) - x_1(t_0) + v \int_{t_0}^{t_1} x_1(t) dt \quad (19)$$

is valid. Let $q = \max\{\tau_i \in \Delta : \tau_i \leq t_1\}$. Then, in virtue of (10) and (13), the inequalities

$$|x_1(t_1) - x_1(q)| \leq d_3 \delta, \quad (20)$$

$$\left| \int_{t_0}^{t_1} x_1(t) dt - \int_{t_0}^q x_1(t) dt \right| \leq d_1 \delta \quad (21)$$

hold. In this case, from (19), (20), and (21) we derive

$$\left| \int_{t_0}^{t_1} \{k(t) + \lambda x_1(t)\} x_2(t) dt - \left\{ x_1(q) - x_1(t_0) + v \int_{t_0}^q x_1(t) dt \right\} \right| \leq (d_1 + d_3) \delta. \quad (22)$$

Using the inclusion $u(\cdot) \in U^h(\cdot)$, we have

$$\begin{aligned} \left| \sum_{i=1}^q \int_{\tau_{i-1}}^{\tau_i} (k(\tau_{i-1}) + \lambda \xi_{i-1}^h) u_i dt - v \sum_{i=1}^q \delta \xi_{i-1}^h - \xi_q^h + \xi_0^h \right| &\leq \delta \sum_{i=1}^q \sigma_h^{(1)} \\ &\leq (\vartheta - t_0)(2h \delta^{-1} + K_1 \delta + K_2 h). \end{aligned} \quad (23)$$

By taking into account (15), we obtain the inequality

$$\begin{aligned} &\left| \left\{ x_1(q) - x_1(t_0) + v \int_{t_0}^q x_1(t) dt \right\} - \left\{ v \sum_{i=1}^q \delta \xi_{i-1}^h - \xi_q^h + \xi_0^h \right\} \right| \\ &\leq 2h + v \left| \sum_{i=1}^q \int_{\tau_{i-1}}^{\tau_i} \{x_1(t) - \xi_{i-1}^h\} dt \right| \leq 2h + v(\vartheta - t_0)(h + d_3 \delta). \end{aligned} \quad (24)$$

Combining (22)–(24), we deduce that

$$\left| \int_{t_0}^{t_1} (k(t) + \lambda x_1(t)) x_2(t) dt - \sum_{i=1}^q \int_{\tau_{i-1}}^{\tau_i} (k(\tau_{i-1}) + \lambda \xi_{i-1}^h) u_i dt \right| \tag{25}$$

$$\leq \{2 + (\vartheta - t_0)(K_2 + \nu)\} h + 2(\vartheta - t_0) h \delta^{-1} + \{d_1 + d_3 + (\vartheta - t_0)(\nu d_3 + K_1)\} \delta.$$

Then, we have (see (10) and (12))

$$\left| \int_{\tau_q}^{t_1} (k(\tau_q) + \lambda \xi_q^h) u_q dt \right| \leq \{k_1 + \lambda(d_1 + 1)d_2\} \delta. \tag{26}$$

Using (11), (25), and (15), we derive

$$\begin{aligned} & \left| \int_{t_0}^{t_1} (k(t) + \lambda x_1(t)) u(t) dt - \sum_{i=1}^q \int_{\tau_{i-1}}^{\tau_i} (k(\tau_{i-1}) + \lambda \xi_{i-1}^h) u_i dt \right| \\ & \leq (k_1 + \lambda(d_1 + 1)) d_2 \delta + \left| \sum_{i=1}^q \int_{\tau_{i-1}}^{\tau_i} \{k(t) - k(\tau_{i-1}) + \lambda(x_1(t) - \xi_{i-1}^h)\} u_i dt \right| \\ & \leq (k_1 + \lambda(d_1 + 1)) d_2 \delta + \delta \sum_{i=1}^q \{\delta k + \lambda(h + \delta d_3) d_2\} \\ & \leq (k_1 + \lambda(d_1 + 1)) d_2 \delta + (\vartheta - t_0) \{ (k + \lambda d_2 d_3) \delta + \lambda d_2 h \} \\ & = \{k_1 + \lambda(d_1 + 1) d_2 + (\vartheta - t_0)(k + \lambda d_2 d_3)\} \delta + (\vartheta - t_0) \lambda d_2 h. \end{aligned} \tag{27}$$

The statement of the lemma follows from (25) and (27). The lemma is proved. □

Lemma 3 ([6, p. 47]) *Let $u(\cdot) \in L_\infty(T_*; \mathbb{R})$ and $v(\cdot) \in W(T_*; \mathbb{R})$, $T_* = [a, b]$, $-\infty < a < b < +\infty$,*

$$\left| \int_a^t u(\tau) d\tau \right| \leq \varepsilon, \quad |v(t)| \leq K \quad \forall t \in T_*.$$

Then, for all $t \in T_$, the inequality*

$$\left| \int_a^t u(\tau) v(\tau) d\tau \right| \leq \varepsilon (K + \text{var}(T_*; v(\cdot)))$$

is valid.

Here, the symbol $\text{var}(T_*; \nu(\cdot))$ means the variation of the function $\nu(\cdot)$ over the segment T_* , and the symbol $W(T_*; \mathbb{R})$ means the set of functions $y(\cdot) : T_* \rightarrow \mathbb{R}$ of bounded variation.

Note that from Condition 1a) it follows that there exists a number $E > 0$ such that

$$\text{var}(T; (k(t) + \lambda x_1(t))^{-1} x_2(t)) \leq E.$$

Let

$$U^h(\tau_i, \xi_{i-1}^h, \xi_i^h) = u_i^h = \arg \min\{|v| : v \in \tilde{U}^h(\tau_i, \xi_{i-1}^h, \xi_i^h)\} \text{ for } i \in [1 : m_h] \quad (28)$$

and

$$u^h(t) = u_i^h \text{ for } t \in \delta_{i-1, h}, \quad i \in [1 : m_h]. \quad (29)$$

Theorem 1 *Let Condition 1a) be fulfilled. Then, the following estimate is valid:*

$$|u^h(\cdot) - x_2(\cdot)|_{L_2(T; \mathbb{R})}^2 \leq (c_1 h + c_2 \delta + c_3 h \delta^{-1})(d_2 c^{-1} + E).$$

Proof. Using (28), we deduce for $i \in [1 : m_h]$ that

$$|u_i^h| \leq \delta^{-1} \int_{\tau_{i-1}}^{\tau_i} |x_2(t)| dt. \quad (30)$$

Therefore, from (29) and (30), for $i \in [1 : m_h]$, we also have

$$\int_{\tau_{i-1}}^{\tau_i} |u^h(t)|^2 dt \leq \int_{\tau_{i-1}}^{\tau_i} |x_2^h(t)|^2 dt.$$

Therefore, $|u^h(\cdot)|_{L_2(T; \mathbb{R})}^2 \leq |x_2(\cdot)|_{L_2(T; \mathbb{R})}^2$. From this inequality, it follows that

$$\begin{aligned} |u^h(\cdot) - x_1(\cdot)|_{L_2(T; \mathbb{R})}^2 &\leq 2|x_2(\cdot)|_{L_2(T; \mathbb{R})}^2 - 2(u^h(\cdot), x_2(\cdot))_{L_2(T; \mathbb{R})}^2 \\ &= 2(x_2(\cdot) - u^h(\cdot), x_2(\cdot))_{L_2(T; \mathbb{R})}. \end{aligned} \quad (31)$$

Thus, taking into account (31), we obtain

$$\begin{aligned} &|u^h(\cdot) - x_2(\cdot)|_{L_2(T; \mathbb{R})}^2 \\ &\leq 2 \int_0^{\vartheta} \{(k(t) + \lambda x_1(t))(u^h(\tau) - x_2(\tau))\} \{(k(t) + \lambda x_1(t))^{-1} x_2(t)\} d\tau. \end{aligned}$$

Then, in virtue of Condition 1a), the following relation

$$|(k(t) + \lambda x_1(t))^{-1} x_2(t)| \leq d_2 c^{-1}$$

holds. The statement of the theorem follows from last two inequalities and Lemmas 2 and 3. The theorem is proved. \square

On the base of Theorem 1, we conclude that the following theorem can be proved.

Theorem 2 *The mapping U^h of form (28) solves Problem 1.*

Thus, Problem 1 is solved. Let us turn to the solution of the second problem. Note that the inequalities

$$|\xi_{1i}^h - x_1(\tau_i)| \leq h, \tag{32}$$

$$|\xi_{2i}^h - x_2(\tau_i)| \leq h, \tag{33}$$

follows from (2) and (4). Let

$$Q_T(\cdot) = \{v(\cdot) \in L_2(T; \mathbb{R}): |v(t)| \leq \gamma \text{ for a.a. } t \in T\}$$

and

$$V(x(\cdot)) = \left\{ v(\cdot) \in Q_T(\cdot): \right.$$

$$\left. \dot{x}_2(t) + k(t)x_2(t) + (\lambda x_1(t) + \mu)x_2(t) = v(t) \text{ for a.a. } t \in T \right\}.$$

Introduce the family of sets

$$\tilde{V}^h(\tau_i, \xi_{i-1}^h, \xi_i^h) = \left\{ v \in \mathbb{R}: \right. \tag{34}$$

$$\left. |v| \leq \gamma, |v - k(\tau_{i-1})\xi_{2i-1}^h - (\lambda\xi_{1i-1}^h + \mu)\xi_{2i-1}^h - (\xi_{2i}^h - \xi_{2i-1}^h)\delta^{-1}| \leq \sigma_h^{(2)} \right\},$$

$$V^h(\cdot) = \left\{ v(\cdot) \in Q_T(\cdot): \right.$$

$$\left. v(t) = v_i \text{ for a.a. } t \in [\tau_{i-1}, \tau_i], v_i \in \tilde{V}^h(\tau_i, \xi_{i-1}^h, \xi_i^h), i \in [1 : m] \right\},$$

where

$$K_3 = k_1 + \mu + \lambda(1 + d_1 + d_2), \quad K_4 = kd_2 + (k_1 + \mu)d_4 + \lambda(d_2d_3 + d_4 + d_1d_4)\delta,$$

$$\sigma_h^{(2)} = K_3h + K_4\delta + 2h\delta^{-1}.$$

Let, in addition,

$$V^h(\tau_i, \xi_{i-1}^h, \xi_i^h) = v_i^h = \arg \min\{|v|: |v| \in \tilde{V}^h(\tau_i, \xi_{i-1}^h, \xi_i^h)\}, \quad i \in [1 : m_h] \tag{35}$$

and

$$v^h(t) = v_{i-1}^h \quad \text{for } t \in \delta_{i-1,h}, \quad i \in [1 : m_h].$$

Lemma 4 *Let $v(\cdot) \in V(x(\cdot))$. Then, the inclusions*

$$\delta^{-1} \int_{\tau_{i-1}}^{\tau_i} v(t) dt \in \tilde{V}^h(\tau_i, \xi_{i-1}^h, \xi_i^h), \quad i \in [1 : m],$$

take place.

Proof. Let $f_2(t, x_1, x_2) = -k(t)x_2 - (\lambda x_1 + \mu)x_2$. Then, for all $t \in [\tau_{i-1}, \tau_i]$, the inequality

$$\Phi_i(t) \equiv |f_2(t, x_1(t), x_2(t)) - f_2(\tau_{i-1}, \xi_{1i-1}^h, \xi_{2i-1}^h)| \leq I_{i-1}^{(1)}(t) + I_{i-1}^{(2)}(t) + I_{i-1}^{(3)}(t)$$

is valid. Here,

$$\begin{aligned} I_{i-1}^{(1)}(t) &= |k(t) - k(\tau_{i-1})||x_2(t)|, \\ I_{i-1}^{(2)}(t) &= |k(\tau_{i-1})||x_2(t) - \xi_{2i-1}^h| + \mu|x_2(t) - \xi_{2i-1}^h|, \\ I_{i-1}^{(3)}(t) &= \lambda|x_1(t)x_2(t) - \xi_{1i-1}^h\xi_{2i-1}^h|. \end{aligned}$$

In virtue of (10) and (11), the following inequality

$$I_{i-1}^{(1)}(t) \leq kd_2\delta \tag{36}$$

holds. In turn, from (33) and (14), we derive for $t \in [\tau_{i-1}, \tau_i]$ the inequality

$$|x_2(t) - \xi_{2i-1}^h| \leq h + d_4\delta. \tag{37}$$

Therefore, from (12) and (37), we get the estimate

$$I_{i-1}^{(2)}(t) \leq (k_1 + \mu)(h + d_4\delta). \tag{38}$$

Note that

$$|\xi_{1i-1}^h| \leq 1 + d_2. \tag{39}$$

Using (10), (32), (37), and (39), we derive

$$\begin{aligned} I_{i-1}^{(3)}(t) &= \lambda|x_1(t)x_2(t) - \xi_{1i-1}^h\xi_{2i-1}^h| \\ &\leq \lambda|(x_1(t) - \xi_{1i-1}^h)x_2(t)| + \lambda|\xi_{1i-1}^h(x_2(t) - \xi_{2i-1}^h)| \\ &\leq \lambda d_2(h + d_3\delta) + \lambda(1 + d_1)(h + d_4\delta) = \lambda(1 + d_1 + d_2)h + \lambda(d_2d_3 + d_4 + d_1d_4)\delta. \end{aligned} \tag{40}$$

Combining (36), (38), and (40), we deduce that

$$\Phi_i(t) \leq K_4\delta + K_3h. \tag{41}$$

Also, from the equality

$$\delta^{-1} \int_{\tau_{i-1}}^{\tau_i} f_2(t, x_1(t), x_2(t)) dt = \delta^{-1}(x_2(\tau_i) - x_2(\tau_{i-1})), \tag{42}$$

we have

$$\left| \delta^{-1} \int_{\tau_{i-1}}^{\tau_i} f_2(t, x_1(t), x_2(t)) dt - \frac{\xi_{2i}^h - \xi_{2i-1}^h}{\delta} \right| \leq 2\frac{h}{\delta}.$$

In this case, from (41) and 42), it follows that the inequalities

$$\left| \delta^{-1} \int_{\tau_{i-1}}^{\tau_i} v(t) dt - k(\tau_{i-1})\xi_{2i-1}^h - (\lambda\xi_{i-1}^h + \mu)\xi_{2i-1}^h - (\xi_{2i}^h - \xi_{2i-1}^h)\delta^{-1} \right| \leq \sigma_h^{(2)},$$

$$i \in [1 : m],$$

are valid for any $v(\cdot) \in V(x(\cdot))$. The lemma is proved. □

Lemma 5 *Let sequences $\{h_j\}_{j=1}^\infty \subset \mathbb{R}$ and $\{v_j(\cdot)\}_{j=1}^\infty \subset Q_T(\cdot)$ with the properties*

$$h_j \delta^{-1}(h_j) \rightarrow 0 \quad \text{as } j \rightarrow \infty, \tag{43}$$

$$v_j(\cdot) \in V^{h_j}(\cdot), \quad v_j(\cdot) \rightarrow v_0(\cdot) \quad \text{weakly in } L_2(T; \mathbb{R}) \text{ as } j \rightarrow \infty \tag{44}$$

be given. Then, $v_0(\cdot) \in V(x(\cdot))$.

Proof. Assume the contrary, i.e., $v_0(\cdot) \notin V(x(\cdot))$. Then, there exist $t_1, t_2 \in T, t_1 < t_2$ such that

$$\left| \int_{t_1}^{t_2} (k(t) + \lambda x_1(t) + \mu)x_2(t) dt + x_2(t_2) - x_2(t_1) - \int_{t_1}^{t_2} v_0(t) dt \right| = b > 0. \tag{45}$$

Let j_1 be such that, for all $j \geq j_1$ and all $t_*, t^* \in T$ with the properties $0 \leq t^* - t_* \leq \delta(h_j)$, the inequalities

$$\left| \int_{t_*}^{t^*} (k(t) + \lambda x_1(t) + \mu)x_2(t) dt \right| \leq b/8, \quad \left| x_2(t_*) - x_2(t^*) - \int_{t_*}^{t^*} v_0(t) dt \right| \leq b/8 \tag{46}$$

take place. Denote $p_j = \min\{t \in \Delta_{h_j} : t \leq t_2\}$ and $q_j = \max\{t \in \Delta_{h_j} : t \leq t_2\}$. From (45) and (46), we have

$$\left| \int_{p_j}^{q_j} \{(k(t) + \lambda x_1(t) + \mu)x_2(t) + \dot{x}_2(t) - v_0(t)\} dt \right| \geq b/2 \quad \text{for } j \geq j_1.$$

In this case, for $j \geq j_1$, we obtain

$$\sum_{q=1}^3 I_j^{(q)} \geq b/2, \tag{47}$$

where

$$I_j^{(1)} = \left| \int_{p_j}^{q_j} (v_j(t) - v_0(t)) dt \right|,$$

$$\begin{aligned}
 I_j^{(2)} &= \left| \int_{p_j}^{q_j} \{ (k(t) + \lambda x_1(t) + \mu) x_2(t) + \dot{x}_2(t) \} dt \right. \\
 &\quad \left. - \sum_{i=p_j}^{q_{j-1}} \int_{\tau_i}^{\tau_{i+1}} \{ k(\tau_{i-1}) \xi_{2i-1}^{h_j} + (\lambda \xi_{1i-1}^{h_j} + \mu) \xi_{2i-1}^{h_j} + (\xi_{2i}^{h_j} - \xi_{2i-1}^{h_j}) \delta^{-1}(h_j) \} dt \right|, \\
 I_j^{(3)} &= \left| \sum_{i=p_j}^{q_{j-1}} \int_{\tau_i}^{\tau_{i+1}} \{ k(\tau_{i-1}) \xi_{2i-1}^{h_j} + (\lambda \xi_{1i-1}^{h_j} + \mu) \xi_{2i-1}^{h_j} + (\xi_{2i}^{h_j} - \xi_{2i-1}^{h_j}) \delta^{-1}(h_j) \} dt \right. \\
 &\quad \left. - \int_{p_j}^{q_j} v_j(t) \} dt \right|.
 \end{aligned}$$

Then, relation (44) implies the convergence

$$I_j^{(1)} \rightarrow 0 \quad \text{as } j \rightarrow \infty. \tag{48}$$

In turn, using (41), (15), (33), and (43), we deduce that

$$I_j^{(2)} \rightarrow 0 \quad \text{as } j \rightarrow \infty. \tag{49}$$

Taking into account the inclusion $v_j(\cdot) \in V^{h_j}(\cdot)$, we obtain

$$v_j(t) = v_{ij} \in \tilde{V}^{h_j}(\tau_i, \xi_{i-1}^{h_j}, \xi_i^{h_j}) \quad \text{for } t \in [\tau_i, \tau_{i+1}).$$

Consequently,

$$|v_{ij} - \{ (k(\tau_{i-1}) + \lambda \xi_{1i-1}^{h_j} + \mu) \xi_{2i-1}^{h_j} + (\xi_{2i}^{h_j} - \xi_{2i-1}^{h_j}) \delta^{-1}(h_j) \}| \leq \sigma_{h_j}^{(2)}. \tag{50}$$

In this case, from (50), we deduce that

$$\begin{aligned}
 &\left| \int_{p_j}^{q_{j-1}} v_j(t) dt - \sum_{i=p_j}^{q_{j-1}} \{ k(\tau_{i-1}) \xi_{2i-1}^{h_j} + (\lambda \xi_{1i-1}^{h_j} + \mu) \xi_{2i-1}^{h_j} + (\xi_{2i}^{h_j} - \xi_{2i-1}^{h_j}) \delta^{-1} \} \right| \\
 &\leq (\vartheta - t_0) \sigma_{h_j}^{(2)}.
 \end{aligned} \tag{51}$$

Due to (51), we conclude that

$$I_j^{(3)} \rightarrow 0 \quad \text{as } j \rightarrow \infty. \tag{52}$$

Relations (48), (49), and (52) contradict (47). The lemma is proved. \square

Theorem 3 *Let $\delta(h) \rightarrow 0, h\delta^{-1}(h) \rightarrow 0$ as $h \rightarrow 0$. Then, the convergence*

$$v^h(\cdot) \rightarrow \gamma(\cdot) \quad \text{in } L_2(T; \mathbb{R}) \text{ as } h \rightarrow 0$$

takes place.

Proof. Let

$$\tilde{v}_h(t) = \begin{cases} v^h(t + \delta) & \text{for } t \in [0, \vartheta - \delta), \\ v_{m_h}^h & \text{for } t \in [\vartheta - \delta, \vartheta]. \end{cases}$$

First, we prove that $\tilde{v}_h(\cdot) \rightarrow \gamma(\cdot)$ as $h \rightarrow 0$ in $L_2(T; \mathbb{R})$. To do this, it is sufficient to prove that for any sequence $\{h_j\}_{j=1}^\infty, h_j > 0, h_j \rightarrow 0$ as $j \rightarrow \infty$, the convergence

$$\tilde{v}_{h_j}(\cdot) \rightarrow \gamma(\cdot) \quad \text{in } L_2(T; \mathbb{R}) \text{ as } j \rightarrow \infty$$

takes place. Assuming the contrary, we conclude that there exists a subsequence $\{\tilde{v}_{h_j}(\cdot)\}_{j=1}^\infty$ (for simplicity we denote it by the symbol $\{u_j(\cdot)\}_{j=1}^\infty$) such that

$$u_j(\cdot) \rightarrow u_0(\cdot) \neq \gamma(\cdot) \quad \text{weakly in } L_2(T; \mathbb{R}) \text{ as } j \rightarrow \infty. \tag{53}$$

Let $\delta_j = \delta(h_j)$,

$$\bar{u}_j(t) \equiv u_{j,i} = \delta_j^{-1} \int_{\tau_{i,h_j}}^{\tau_{i+1,h_j}} \gamma(t) dt \quad \text{for } t \in [\tau_{i,h_j}, \tau_{i+1,h_j}). \tag{54}$$

In view of Lemma 4, we derive the inclusion $u_{j,i} \in \tilde{V}^{h_j}(\tau_i, \xi_{i-1}^{h_j}, \xi_i^{h_j})$. Consequently, $\bar{u}_j(\cdot) \in V^{h_j}(\cdot)$. Due to the rule of choosing the value v_i^h , we have

$$|u_j(\cdot)|_{L_2(T; \mathbb{R})} \leq |\bar{u}_j(\cdot)|_{L_2(T; \mathbb{R})}. \tag{55}$$

Using (54), we deduce that

$$\begin{aligned} |\bar{u}_j(\cdot)|_{L_2([\tau_i, \tau_{i+1}]; \mathbb{R})}^2 &= \delta_j |u_{j,i}|^2 \leq \delta_j^{-1} \left(\int_{\tau_i}^{\tau_{i+1}} |\gamma(t)| dt \right)^2 \\ &\leq \int_{\tau_i}^{\tau_{i+1}} |\gamma(t)|^2 dt = |\gamma(\cdot)|_{L_2([\tau_i, \tau_{i+1}]; \mathbb{R})}^2. \end{aligned}$$

Therefore,

$$|\bar{u}_j(\cdot)|_{L_2(T; \mathbb{R})} \leq |\gamma(\cdot)|_{L_2(T; \mathbb{R})}. \tag{56}$$

Combining (55) and (56), we conclude that

$$|u_j(\cdot)|_{L_2(T;\mathbb{R})} \leq |\gamma(\cdot)|_{L_2(T;\mathbb{R})}.$$

Consequently,

$$\limsup_{j \rightarrow \infty} |u_j(\cdot)|_{L_2(T;\mathbb{R})} \leq |\gamma(\cdot)|_{L_2(T;\mathbb{R})}. \quad (57)$$

As well, in virtue of the known property of the weak limit, we obtain

$$\liminf_{j \rightarrow \infty} |u_j(\cdot)|_{L_2(T;\mathbb{R})} \geq |u_0(\cdot)|_{L_2(T;\mathbb{R})}.$$

This and (57) implies

$$|u_0(\cdot)|_{L_2(T;\mathbb{R})} \leq |\gamma(\cdot)|_{L_2(T;\mathbb{R})}.$$

Then, in virtue of Lemma 5, we have $u_0(\cdot) \in V(x(\cdot))$. Therefore, $u_0(\cdot) = \gamma(\cdot)$. This equality contradicts (53). So, $\tilde{v}_h(\cdot) \rightarrow \gamma(\cdot)$ as $h \rightarrow 0$ in $L_2(T;\mathbb{R})$. Further, we prove the second statement of the theorem. We have

$$\begin{aligned} & \int_0^{\vartheta} |v^h(t) - \gamma(t)|^2 dt \\ & \leq \int_0^{\delta} |\gamma(t) - v_0^h|^2 dt + 2 \int_{\delta}^{\vartheta} |\tilde{v}_h(t - \delta) - \gamma(t - \delta)|^2 dt + 2 \int_{\delta}^{\vartheta} |\gamma(t - \delta) - \gamma(t)|^2 dt. \end{aligned}$$

Using the relations $\gamma(\cdot) \in L_2(T;\mathbb{R})$ and $|v_0^h| \leq \gamma$, we deduce that the second and third terms in the right-hand part of last inequality tend to zero as $\delta \rightarrow 0$. The second term does not exceed the value $2|\tilde{v}_h(\cdot) - \gamma(\cdot)|_{L_2(T;\mathbb{R})}^2$, that is infinitely small as $h \rightarrow 0$, in virtue of the proved convergence. The theorem is proved. \square

From Theorem 3, we conclude that the following theorem takes place.

Theorem 4 *The mapping V^h of form (35) solves Problem 2.*

3. Conclusions

In the paper, two algorithms of stable reconstruction of unknown characteristics of dynamical system described by second order nonlinear systems of ordinary differential equations are specified. These algorithms are stable with respect to informational noises and computational errors. The algorithms are based on constructions of the theory of dynamic inversion.

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