



**ANALYSIS OF THE POINTWISE COMPLETENESS AND THE POINTWISE
DEGENERACY OF THE STANDARD AND FRACTIONAL DESCRIPTOR
LINEAR SYSTEMS AND ELECTRICAL CIRCUITS**

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Abstract – The Drazin inverse of matrices is applied to analysis of the pointwise completeness and the pointwise degeneracy of the descriptor standard and fractional linear continuous-time and discrete-time systems. It is shown that: 1) The descriptor linear continuous-time system is pointwise complete if and only if the initial and final states belong to the same subspace. 2) The descriptor linear discrete-time system is not pointwise complete if its system matrix is singular. 3) System obtained by discretization of continuous-time system is always not pointwise complete. 4) The descriptor linear continuous-time system is not pointwise degenerated in any nonzero direction for all nonzero initial conditions. 5) The descriptor fractional system is pointwise complete if the matrix defined by (36) is invertible. 6) The descriptor fractional system is pointwise degenerated if and only if the condition (41) is satisfied. Considerations are illustrated by examples of descriptor linear electrical circuits.

Key words – descriptor, fractional, linear, electrical circuit, Drazin inverse.

1 INTRODUCTION

A dynamical system described by homogenous equation is called pointwise complete if every final state of the system can be reached by suitable choice of its initial state. A system, which is not pointwise complete is called pointwise degenerated. The pointwise completeness and pointwise degeneracy of linear continuous-time systems with delays have been investigated in [2, 3, 8, 10, 12], the pointwise completeness of linear discrete-time cone systems with delays in [13] and of fractional linear systems in [1, 6-8]. The pointwise completeness and pointwise degeneracy of standard and positive hybrid systems described by the general model have been analyzed in [4] and of positive linear systems with state-feedbacks in [5]. Mathematical fundamentals of the fractional calculus are given in the monographs [9, 11].

In this paper the Drazin inverse of matrices will be applied to analysis of the pointwise

completeness and the pointwise degeneracy of the descriptor linear continuous-time and discrete-time systems.

The paper is organized as follows. In section 2 the basic definitions and theorems concerning descriptor linear continuous-time and discrete-time systems and the Drazin inverse of matrices are recalled. The pointwise completeness of descriptor linear continuous-time and discrete-time systems is investigated in section 3 and the pointwise degeneracy in section 4. In section 5 the considerations have been extended to fractional descriptor linear continuous-time systems. Concluding remarks are given in section 6. The considerations are illustrated by examples of the linear electrical circuits.

The following notation will be used: \mathfrak{R} - the set of real numbers, $\mathfrak{R}^{n \times m}$ - the set of $n \times m$ real matrices, $\mathfrak{R}_+^{n \times m}$ - the set of $n \times m$ real matrices with nonnegative entries and $\mathfrak{R}_+^n = \mathfrak{R}_+^{n \times 1}$, I_n - the $n \times n$ identity matrix. $\text{Im}P$ is the image of the operator(matrix) P .

2 AUTONOMOUS DESCRIPTOR LINEAR SYSTEMS AND THEIR SOLUTIONS

Consider the autonomous descriptor continuous-time linear system

$$E\dot{x} = Ax, \quad \dot{x} = \frac{dx}{dt} \quad (1)$$

where $x = x(t) \in \mathfrak{R}^n$ is the state vector and $E, A \in \mathfrak{R}^{n \times n}$.

It is assumed that $\det E = 0$ but the pencil (E, A) is regular, i.e.

$$\det[Es - A] \neq 0 \text{ for some } s \in C \text{ (the field of complex numbers)}. \quad (2)$$

Assuming that for $c \in \mathfrak{R}$ $\det[Ec - A] \neq 0$ and premultiplying (1) by $[Ec - A]^{-1}$ we obtain

$$\bar{E}\dot{x} = \bar{A}x, \quad (3a)$$

where

$$\bar{E} = [Ec - A]^{-1}E, \quad \bar{A} = [Ec - A]^{-1}A. \quad (3b)$$

The equations (1) and (3a) have the same solution x .

Definition 1. A matrix $E^D \in \mathfrak{R}^{q \times n}$ is called the Drazin inverse of E if it satisfies the conditions

$$\bar{E}\bar{E}^D = \bar{E}^D\bar{E}, \quad (4a)$$

$$\bar{E}^D\bar{E}\bar{E}^D = \bar{E}^D, \quad (4b)$$

$$\bar{E}^D\bar{E}^{q+1} = \bar{E}^q, \quad (4c)$$

where q is the index of E defined as the smallest nonnegative integer satisfying the condition

$$\text{rank}E^q = \text{rank}E^{q+1}. \tag{5}$$

Theorem 1. Let

$$P = \bar{E}\bar{E}^D, \tag{6a}$$

$$Q = \bar{A}\bar{E}^D \tag{6b}$$

Then

$$P^k = P \text{ for } k = 2,3,\dots \tag{7a}$$

$$PQ = QP = Q, \tag{7b}$$

$$P\bar{E}^D = \bar{E}^D, \tag{7c}$$

$$Px = x. \tag{7d}$$

Proof is given in [8].

Theorem 2. The solution of the equation (3a) has the form

$$x(t) = e^{\bar{A}\bar{E}t} \bar{E}\bar{E}^D w, \tag{8}$$

where $w \in \mathfrak{R}^n$ is any vector and $x(0) \in \text{Im}\bar{E}\bar{E}^D = \text{Im}P$.

Proof is given in [8].

Consider the autonomous descriptor discrete-time linear system

$$Ex_{i+1} = Ax_i, \quad i = 0,1,\dots \tag{9}$$

where $x_i \in \mathfrak{R}^n$ is the state vector and $E, A \in \mathfrak{R}^{n \times n}$.

It is assumed that $\det E = 0$ and

$$\det[Ez - A] \neq 0 \text{ for some } z \in C \tag{10}$$

Choosing $c \in \mathfrak{R}$ such that $\det[Ec - A] \neq 0$ and premultiplying (9) by $[Ec - A]^{-1}$ we obtain

$$\bar{E}x_{i+1} = \bar{A}x_i, \tag{11}$$

where \bar{E} and \bar{A} defined by (3b).

Theorem 1 is also valid for the discrete-time systems.

Using the Drazin inverse \bar{E}^D of the matrix E we may find the solution x_i of the equation (11) by the use of the following Theorem.

Theorem 3. The solution of the equation (11) has the form

$$x_i = [\bar{E}\bar{A}]^i \bar{E}^D E v = Q^i x_0, \quad i=1,2,\dots \quad (12)$$

where $v \in \mathfrak{R}^n$ is any vector and $x_0 \in \text{Im} \bar{E}\bar{E}^D = \text{Im} P$, the matrices P and Q are defined by (6).

Proof is given in [8].

3 POINTWISE COMPLETENESS OF DESCRIPTOR LINEAR SYSTEMS

In this section conditions for the pointwise completeness of descriptor continuous-time and discrete-time linear systems will be established.

3.1 CONTINUOUS-TIME SYSTEMS

Definition 2. The descriptor continuous-time linear system (1) is called pointwise complete for $t = t_f$ if for final state $x_f = x(t_f) \in \mathfrak{R}^n$ there exists an initial condition $x(0) \in \text{Im} P$ such that

$$x_f = x(t_f) \in \text{Im} P \quad (14)$$

where P is defined by (6a).

Theorem 4. The descriptor system (1) is pointwise complete for any $t = t_f$ and every $x_f \in \mathfrak{R}^n$ if and only if the condition (14) is satisfied.

Proof. Note that $\det e^{Qt} \neq 0$ and $[e^{Qt}]^{-1} = e^{-Qt}$ for any t . From (8) for $t = t_f$ we have

$$x(0) = e^{-Qt_f} x_f \quad (15)$$

Therefore, for every x_f there exists $x(0) \in \text{Im} P$ such that $x_f = x(t_f)$. \square

Example 1. Consider the descriptor linear electrical circuit shown in Figure 1 with given resistances R_1, R_2, R_3 , inductances L_1, L_2, L_3 and source voltages e_1, e_2 .

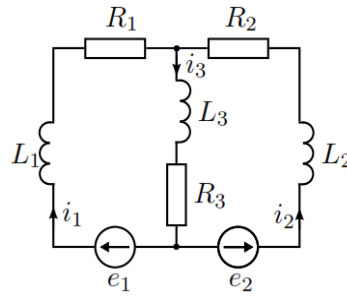


Fig. 1. Electrical circuit

Using Kirchoff's laws we may write the equations

$$\begin{aligned}
 e_1 &= R_1 i_1 + L_1 \frac{di_1}{dt} + R_3 i_3 + L_3 \frac{di_3}{dt} \\
 e_2 &= R_2 i_2 + L_2 \frac{di_2}{dt} + R_3 i_3 + L_3 \frac{di_3}{dt} \\
 i_1 + i_2 - i_3 &= 0
 \end{aligned}
 \tag{16}$$

which can be written in the form

$$E \frac{d}{dt} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = A \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} + B \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}
 \tag{17a}$$

where

$$E = \begin{bmatrix} L_1 & 0 & L_3 \\ 0 & L_2 & L_3 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -R_1 & 0 & -R_3 \\ 0 & -R_2 & -R_3 \\ 1 & 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}
 \tag{17b}$$

The assumption (2) for the electrical circuit is satisfied, since the matrix E is singular ($\det E = 0$) and

$$\begin{aligned}
 \det[Es - A] &= \begin{vmatrix} L_1 s + R_1 & 0 & L_3 s + R_3 \\ 0 & L_2 s + R_2 & L_3 s + R_3 \\ -1 & -1 & -1 \end{vmatrix} = [L_1(L_2 + L_3) + L_2 L_3] s^2 \\
 &+ [R_1(L_2 + L_3) + R_2(L_1 + L_3) + R_3(L_1 + L_2)] s + R_1(R_2 + R_3) + R_2 R_3
 \end{aligned}
 \tag{18}$$

Therefore, the electrical circuit is a descriptor linear continuous-time system.

Note that the matrix A defined by (17b) is nonsingular and we may choose in (2) $s = 0$. In this case we obtain

$$\bar{E} = [-A]^{-1} E = \frac{1}{R_1(R_2 + R_3) + R_2 R_3} \begin{bmatrix} L_1(R_2 + R_3) & -L_2 R_3 & L_3 R_2 \\ -L_1 R_3 & L_2(R_1 + R_2) & L_3 R_1 \\ L_1 R_2 & L_2 R_1 & L_3(R_1 + R_2) \end{bmatrix}
 \tag{19}$$

$$\bar{A} = [-A]^{-1} A = -I_3
 \tag{20}$$

and

$$\bar{E}^D = \begin{bmatrix} e_{d,11} & e_{d,1} & e_{d,13} \\ e_{d,21} & e_{d,22} & e_{d,23} \\ e_{d,31} & e_{d,32} & e_{d,33} \end{bmatrix} \frac{1}{\Delta_L^2}
 \tag{21}$$

$$P = \bar{E}\bar{E}^D = \begin{bmatrix} L_1(L_2 + L_3) & -L_2L_3 & L_2L_3 \\ -L_1L_3 & L_2(L_1 + L_3) & L_1L_3 \\ L_1L_2 & L_1L_2 & L_3(L_1 + L_2) \end{bmatrix} \frac{1}{\Delta_L^2} \quad (22)$$

$$Q = \bar{A}\bar{E}^D = -\bar{E}^D \quad (23)$$

where

$$\begin{aligned} e_{d,11} &= L_1(L_2^2R_1 + L_3^2R_1 + L_2^2R_3 + L_3^2R_2 + 2L_2L_3R_1), \\ e_{d,12} &= -L_2(L_3^2R_1 + L_3^2R_2 - L_1L_2R_3 + L_1L_3R_2 + L_2L_3R_1) \\ e_{d,13} &= L_3(L_2^2R_1 + L_2^2R_3 + L_1L_2R_3 - L_1L_3R_2 + L_2L_3R_1) \\ e_{d,21} &= -L_1(L_3^2R_1 + L_3^2R_2 - L_1L_2R_3 + L_1L_3R_2 + L_2L_3R_1) \\ e_{d,22} &= L_2(L_1^2R_2 + L_1^2R_3 + L_3^2R_1 + L_3^2R_2 + 2L_1L_3R_2) \\ e_{d,23} &= L_3(L_1^2R_2 + L_1^2R_3 + L_1L_2R_3 + L_1L_3R_2 - L_2L_3R_1) \\ e_{d,31} &= L_1(L_2^2R_1 + L_2^2R_3 + L_1L_2R_3 - L_1L_3R_2 + L_2L_3R_1) \\ e_{d,32} &= L_2(L_1^2R_2 + L_1^2R_3 + L_1L_2R_3 + L_1L_3R_2 - L_2L_3R_1) \\ e_{d,33} &= L_3(L_1^2R_2 + L_2^2R_1 + L_1^2R_3 + L_2^2R_3 + 2L_1L_3R_3) \\ \Delta_L &= L_1(L_2 + L_3) + L_2L_3. \end{aligned}$$

The solution of the equation (17a) for $B = 0$ satisfies the condition

$$\begin{bmatrix} i_1(t_f) \\ i_2(t_f) \\ i_3(t_f) \end{bmatrix} = e^{Qt_f} \begin{bmatrix} i_1(0) \\ i_2(0) \\ i_3(0) \end{bmatrix} \in \text{Im } P \quad (24)$$

Therefore, the descriptor electrical circuit is pointwise complete.

From the above considerations we have the following conclusion.

Conclusion 1. In descriptor linear electrical circuit for $B = 0$ by suitable choice of initial conditions (currents in coils and voltages on capacitors) belonging to $\text{Im } P$ it is possible to obtain in a given time t_f the desired values of currents in coils and voltages on capacitors belonging also to $\text{Im } P$.

3.2 DISCRETE-TIME SYSTEMS

Definition 3. The descriptor discrete-time linear system (9) is called pointwise complete for $i = q$ if for every final state $x_q \in \mathfrak{R}^n$ there exists an initial condition $x_0 \in \text{Im } P$ such that $x_q = x_f \in \text{Im } P$

Theorem 5. The descriptor discrete-time linear system (9) is not pointwise complete for any $i = q$ and every x_f .

Proof. From (12) for $i = q$ we have $x_q = Q^q x_0$. Hence for given x_q it is possible to find x_0 if and only if $\det Q \neq 0$. Note that $\det Q = \det \bar{A} \det \bar{E}^D = 0$ since $\det \bar{E}^D = 0$ for any singular matrix E [8]. \square

Now we shall show that by Euler type discretization from pointwise complete continuous-time system (1) we obtain corresponding discrete-time system (9) which is not pointwise complete for any $i = q$.

Let $x_i = x(ih)$, $i = 0, 1, \dots$, $h > 0$ and

$$\dot{x}(t) \approx \frac{x_{i+1} - x_i}{h} \tag{25}$$

Then from (1) and (25) we have

$$E \frac{x_{i+1} - x_i}{h} = Ax_i, \quad i = 0, 1, \dots, \tag{26}$$

and

$$Ex_{i+1} = (E + hA)x_i, \quad i = 0, 1, \dots, \tag{27}$$

Note that

$$\det[Ec_1 - (E + hA)] = \det \left[h \left(E \frac{c_1 - 1}{h} - A \right) \right] = h^n \det[Ec_2 - A] \neq 0 \tag{28}$$

for $h > 0$ and $c_2 = \frac{c_1 - 1}{h}$

if and only if $\det[Ec_2 - A] \neq 0$.

Therefore, the pencil of the corresponding discrete-time system (27) is regular if and only if the pencil of the continuous-time system (1) is regular.

By Theorem 4 the descriptor continuous-time system (1) is pointwise complete and the corresponding discrete-time system (27) by Theorem 5 is not pointwise complete. Therefore, we have the following theorem.

Theorem 6. The system obtained by the discretization of continuous-time system is always not pointwise complete.

4 POINTWISE DEGENERACY OF DESCRIPTOR LINEAR SYSTEMS

In this section conditions for the pointwise degeneracy of descriptor continuous-time and discrete-time linear systems will be established.

4.1 CONTINUOUS-TIME SYSTEMS

Definition 3. The descriptor continuous-time linear system (1) is called pointwise degenerated in the direction $v \in \mathfrak{R}^n$ for $t = t_f$ if there exists nonzero vector v such that for all initial conditions $x(0) \in \text{Im} P$, the solution of (1) satisfies the condition

$$v^T x_f = 0, \tag{29}$$

where $x_f = x(t_f)$.

Theorem 7. The descriptor continuous-time linear system (1) is not pointwise degenerated in any nonzero direction $v \in \mathfrak{R}^n$ for all nonzero initial conditions $x(0) \in \text{Im} P$.

Proof. Note that $\det e^{Qt_f} \neq 0$ for any matrix $Q = \overline{A} \overline{E}^D$ and all t_f . Substitution of $x_f = e^{Qt_f} x(0)$ into $v^T x_f$ yields

$$v^T x_f = v^T e^{Qt_f} x(0) \neq 0 \quad (30)$$

for all nonzero initial conditions $x(0) \in \text{Im} P$. \square

Example 1. (Continuation of Example 1).

Consider the descriptor linear electrical circuit shown in Figure 1 with given the resistances, inductances and source voltages. The electrical circuit is described by the equation (17).

The matrix (23) of the descriptor electrical circuit is singular since

$$\det Q = 0 \quad (31)$$

but the matrix e^{Qt_f} is nonsingular.

Therefore, by Theorem 7 the descriptor electrical circuit is not pointwise degenerated in any nonzero direction $v \in \mathfrak{R}^3$ for all nonzero initial conditions.

4.2 DISCRETE-TIME SYSTEMS

Definition 4. The descriptor discrete-time linear system (9) is called pointwise degenerated in the direction $v \in \mathfrak{R}^n$ for $i = q_f$ if there exists nonzero vector v such that for all initial conditions $x_0 \in \text{Im} P$, the solution of (9) satisfies the condition

$$v^T x_q = 0. \quad (32)$$

Theorem 8. The descriptor discrete-time linear system (9) is pointwise degenerated in the direction $v \in \mathfrak{R}^n$ if and only if

$$\det Q = 0. \quad (33)$$

Proof. Note that $v^T Q = 0$ if and only if (33) holds. In this case

$$v^T x_q = v^T Q^q x_0 = 0 \quad \text{for all } x_0 \in P. \quad (34)$$

Therefore, the descriptor system (9) is pointwise degenerated in the direction $v \in \mathfrak{R}^n$ if and only if the condition (33) is satisfied. \square

5 FRACTIONAL DESCRIPTOR LINEAR SYSTEMS AND ELECTRICAL CIRCUITS

Consider the fractional descriptor continuous-time linear system

$$E \frac{d^\alpha x(t)}{dt^\alpha} = Ax(t), \quad 0 < \alpha < 1, \quad (35a)$$

where $x(t) \in \mathfrak{R}^n$, $u(t) \in \mathfrak{R}^m$ are the state and input and $E, A \in \mathfrak{R}^{n \times n}$,

$$\frac{d^\alpha x(t)}{dt^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\dot{x}(\tau)}{(t-\tau)^\alpha} d\tau, \dot{x}(\tau) = \frac{dx(\tau)}{d\tau} \tag{35b}$$

is the Caputo fractional derivative and

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \operatorname{Re}(z) > 0 \tag{35c}$$

is the gamma function [6, 8, 9].

It is assumed that $\det E = 0$ and the pencil (E, A) is regular, i.e. the condition (2) is satisfied.

In a similar way as for standard systems we may defined for the fractional system (35) the matrices (3b), the Drazin inverse matrix \bar{E}^D satisfying (4) and the matrices (6).

By Theorem 1 the matrices P and Q satisfy the conditions (7).

Theorem 9. The solution of the equation (35a) has the form

$$x(t) = \phi_0(t)x(0) \tag{36a}$$

where

$$\phi_0(t) = \sum_{k=0}^\infty \frac{\hat{A}^k t^{k\alpha}}{\Gamma[(k+1)\alpha]}, \hat{A} = \bar{E}^D \bar{A}, x(0) \in \operatorname{Im} P \tag{36b}$$

Proof. Premultiplying the equation

$$\bar{E} \frac{d^\alpha x}{dt^\alpha} = \bar{A} x \tag{37}$$

by the matrix \bar{E}^D and taking into account (7d) we obtain

$$\frac{d^\alpha x}{dt^\alpha} = Qx \tag{38a}$$

where

$$Q = \bar{E}^D \bar{A} \tag{38b}$$

Therefore, the solution of the equation (38a) has the form (36a). □

Definition 5. The descriptor fractional linear system (35) is called pointwise complete for $t = t_f$

if for final state $x_f = x(t_f) \in \mathfrak{R}^n$ there exists an initial condition $x(0) \in \operatorname{Im} P$ such that

$$x_f = x(t_f) \in \operatorname{Im} P \tag{39}$$

where P is defined by (6a).

Theorem 10. The descriptor system (35) is pointwise complete for $t = t_f$ and every $x_f \in \mathfrak{R}^n$ if and only if the matrix $\phi_0(t)$ is invertible for t_f .

Proof. If the matrix $\phi_0(t)$ is invertible then from (36a) for $t = t_f$ we have

$$x(0) = \phi_0^{-1}(t_f)x_f. \tag{40}$$

Therefore, for every x_f there exists $x(0) \in \operatorname{Im} P$ such that $x_f = x(t_f)$ if and only if the matrix

is invertible for $t = t_f$. \square

Definition 6. The descriptor fractional linear system (35) is called pointwise degenerated in the direction $v \in \mathfrak{R}^n$ for $t = t_f$ if there exists a non zero vector $v \in \mathfrak{R}^n$ such that for all initial conditions $x(0) \in \text{Im}P$ for $t = t_f$ satisfies the condition

$$v^T x_f = 0 \quad (41)$$

where $x_f = x(t_f)$.

Theorem 11. The descriptor fractional linear system (35) is pointwise degenerated in the direction $v \in \mathfrak{R}^n$ for all nonzero initial conditions $x(0) \in \text{Im}P$ if and only if

$$\det \phi_0(t_f) = 0, \quad (42)$$

where $\phi_0(t)$ is defined by (36b).

Proof. Using (36a) for $t = t_f$ we obtain

$$v^T x_f = v^T \phi_0(t_f) x(0) = 0 \quad (43)$$

for nonzero $x(0) \in \text{Im}P$ if and only if the condition (41) is satisfied. \square

Example 3. Consider the descriptor linear electrical circuit shown in Figure 2 with given resistances R_1, R_2, R_3 , inductances L_1, L_2, L_3 , capacitance C and source voltages e_1, e_2 .

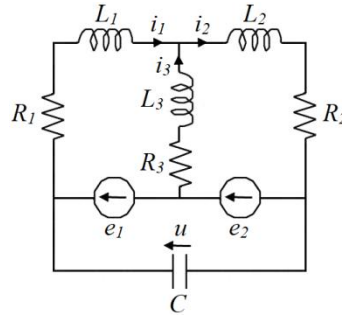


Fig. 2. Electrical circuit

Using Kirchoff's laws we may write the equations

$$\begin{aligned} e_1 &= R_1 i_1 + L_1 \frac{d^\alpha i_1}{dt^\alpha} - R_3 i_3 - L_3 \frac{d^\alpha i_3}{dt^\alpha}, \\ e_2 &= R_2 i_2 + L_2 \frac{d^\alpha i_2}{dt^\alpha} + R_3 i_3 + L_3 \frac{d^\alpha i_3}{dt^\alpha}, \end{aligned} \quad (44)$$

$$e_1 + e_2 = u,$$

$$i_2 = i_1 + i_3.$$

The equations (44) can be written in the form

$$E \frac{d^\alpha}{dt^\alpha} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ u \end{bmatrix} = A \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ u \end{bmatrix} + B \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \tag{45a}$$

where

$$E = \begin{bmatrix} L_1 & 0 & -L_3 & 0 \\ 0 & L_2 & L_3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{45b}$$

$$A = \begin{bmatrix} -R_1 & 0 & R_3 & 0 \\ 0 & -R_2 & -R_3 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}.$$

The condition (2) is satisfied since $\det E = 0$ and

$$\det[Es^\alpha - A] = \begin{vmatrix} s^\alpha L_1 + R_1 & 0 & -s^\alpha L_3 - R_3 & 0 \\ 0 & s^\alpha L_2 + R_2 & s^\alpha L_3 + R_3 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \tag{46}$$

$$= -\{[L_1(L_2 + L_3) + L_2L_3]s^{2\alpha} + [R_1(L_2 + L_3) + R_2(L_1 + L_3) + R_3(L_1 + L_2)]s^\alpha + R_1(R_2 + R_3) + R_2R_3\}.$$

Note that the matrix A defined by (45b) is nonsingular and we may choose in (2) $s = 0$. In this case we have

$$\bar{E} = [-A]^{-1}E = \frac{1}{R_1(R_2 + R_3) + R_2R_3} \begin{bmatrix} L_1(R_2 + R_3) & L_2R_3 & -L_3R_2 & 0 \\ L_1R_3 & L_2(R_1 + R_3) & L_3R_1 & 0 \\ -L_1R_2 & L_2R_1 & L_3(R_1 + R_2) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{47}$$

$$\bar{A} = [-A]^{-1}A = -I_4$$

and

$$\bar{E}^D = \begin{bmatrix} e_{d,11} & e_{d,12} & e_{d,13} & 0 \\ e_{d,21} & e_{d,22} & e_{d,23} & 0 \\ e_{d,31} & e_{d,32} & e_{d,33} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \frac{1}{\Delta_L^2}, \tag{48}$$

$$P = \bar{E}\bar{E}^D = \begin{bmatrix} L_1(L_2 + L_3) & L_2L_3 & -L_2L_3 & 0 \\ L_1L_3 & L_2(L_1 + L_3) & L_1L_3 & 0 \\ -L_1L_2 & L_1L_2 & L_3(L_1 + L_2) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \frac{1}{\Delta_L}, \quad (49)$$

$$Q = \bar{A}\bar{E}^D = -\bar{E}^D, \quad (50)$$

where

$$\begin{aligned} e_{d,11} &= L_1(L_2^2R_1 + L_3^2R_1 + L_2^2R_3 + L_3^2R_2 + 2L_2L_3R_1), \\ e_{d,12} &= L_2(L_3^2R_1 + L_3^2R_2 - L_1L_2R_3 + L_1L_3R_2 + L_2L_3R_1), \\ e_{d,13} &= -L_3(L_2^2R_1 + L_2^2R_3 + L_1L_2R_3 - L_1L_3R_2 + L_2L_3R_1), \\ e_{d,21} &= L_1(L_3^2R_1 + L_3^2R_2 - L_1L_2R_3 + L_1L_3R_2 + L_2L_3R_1), \\ e_{d,22} &= L_2(L_1^2R_2 + L_1^2R_3 + L_3^2R_1 + L_3^2R_2 + 2L_1L_3R_2), \\ e_{d,23} &= L_3(L_1^2R_2 + L_1^2R_3 + L_1L_2R_3 + L_1L_3R_2 - L_2L_3R_1), \\ e_{d,31} &= -L_1(L_2^2R_1 + L_2^2R_3 + L_1L_2R_3 - L_1L_3R_2 + L_2L_3R_1), \\ e_{d,32} &= L_2(L_1^2R_2 + L_1^2R_3 + L_1L_2R_3 + L_1L_3R_2 - L_2L_3R_1), \\ e_{d,33} &= L_3(L_1^2R_2 + L_1^2R_3 + L_2^2R_3 + 2L_1L_2R_3). \end{aligned}$$

The solution of the equation (45a) satisfies the conditions of Theorem 10 and

$$\begin{bmatrix} i_1(t_f) \\ i_2(t_f) \\ i_3(t_f) \\ u(t_f) \end{bmatrix} = \sum_{k=0}^{\infty} \frac{Q^k t_f^{k\alpha}}{\Gamma[(k+1)\alpha]} \begin{bmatrix} i_1(0) \\ i_2(0) \\ i_3(0) \\ u(0) \end{bmatrix} \in \text{Im } P \quad (51)$$

Therefore, by Theorem 11 the fractional descriptor electrical circuit is pointwise complete.

6 CONCLUDING REMARKS

The Drazin inverse of matrices has been applied to analysis of the pointwise completeness and the pointwise degeneracy of the descriptor linear continuous-time and discrete-time systems. It has been shown that: 1) The descriptor linear continuous-time system is pointwise complete if and only if the initial and final states belong to the same subspace (Theorem 4). 2) The descriptor linear discrete-time system is not pointwise complete if its system matrix is singular (Theorem 5). 3) The system obtained by discretization of continuous-time system is always not pointwise complete (Theorem 6). 4) The descriptor linear continuous-time system is not degenerated in any nonzero direction for all nonzero initial conditions (Theorem 7). 5) The descriptor fractional system is pointwise complete if the matrix defined by (36) is invertible (Theorem 10). 6) The descriptor fractional system is pointwise degenerated if and only if the condition (43) is satisfied (Theorem 11). Considerations have been illustrated by examples of descriptor linear electrical circuits. The considerations can be easily extended to fractional descriptor discrete-time linear systems.

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