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Analytical Solutions of Supersonic Flutter Problems for Laminated Multilayered Composite Plates and Cylindrical Shells

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Abstract

For laminated rectangular plates and cylindrical shells the analytical, closed form solution is found using classical and first order transverse shear formulations of kinematical hypothesis. The analysis is carried out for a specific boundary conditions dealing with two opposite edges being simply supported. The evaluated method of solution can be treated as the benchmark for numerical analysis since analytical results can be obtained directly with the use of the symbolic packages, such as Mathematica, Maple or Matlab.

Keywords: Flutter, Multilayered Laminated Structures, Plates, Cylindrical Shells

1. Introduction

The divergence and flutter of thin plates and shells in subsonic or supersonic gas flows belong to the group of problems directly connected with the instability phenomena. These phenomena are an important and significant problem encountered in the design of aircraft constructions [1, 2] or turbine blades [3]. Divergence/flutter is thus a major concern for the designers regarding both the safety and costs. In the divergence/flutter analysis the attention is mainly focused on the discussion of different problems that can affect the structural behaviour, i.e. the aerodynamic theories, the form of boundary conditions, the structural geometry (the analysis deals mainly with 2D beam, plate and shell structures), the material properties and the effects of aerothermoelastic coupling.

The historical background of the above class of dynamic instability problems is discussed by Muc, Flis [4]. A broad literature review dealing with divergence/flutter problems and their optimal design is presented by Muc, Flis, Augustyn [5] and we do not intend to repeat it herein. The flutter/divergence problems are described by the characteristics showing the distributions of frequencies versus aerodynamic pressures for different modes of vibrations. For composite structures the characteristics are mainly derived with the use of the finite element method (Ansys, Abaqus, NisaII/Aero). More information about numerical analysis can be found in Ref. [6].

The aim of the present paper is to demonstrate the method of the analytical solutions of supersonic flutter problems for multilayered composite flat plates and cylindrical shells described with the use of classical plate/shell (CP/ST) and the first order transverse shear deformation (FSDT) theories. The analytical solutions can be carried out for structures having two opposite edges simply-supported with the aid of the symbolic packages (Mathematica, Maple, Matlab).

2. Governing relations

Let us consider composite multilayered structures where the fluid flows along the x axis – Fig. 1 (y axis is along the length of panels/plates, x axis – width and z axis is along the thickness of panels/plates). For FSDT, using the linear piston theory and neglecting the effects of friction (the aerodynamic pressure acting on a curved surface area $\Delta p = -\Lambda\, \partial \mathrm{w}/\partial \mathrm{x}$) the governing set of differential equations can be represented as follows:

$$L_{ij} s_j = F_{ij} s_i \,, \text{ where } \quad s_i = [u, v, \psi_1, \psi_2, w], \; i, j = 1,2, \dots, 5. \tag{1}$$

The details of the derivation of the relations (1) from the Hamilton principle are shown in Ref. [6]. The explicit form of the linear differential operators L_{ij}, F_{ij} is given in the Appendix A. t denotes time, h is the shell thickness, R is the radius of the cylindrical shell, and u – the longitudinal (x) displacements, v – the circumferential (y) displacements and w – the normal (radial z) displacements, ψ_1, ψ_2 are rotations of the normal with respect to the shell midsurface, Λ denotes the aerodynamic pressure and ρ is the density of the composite material.

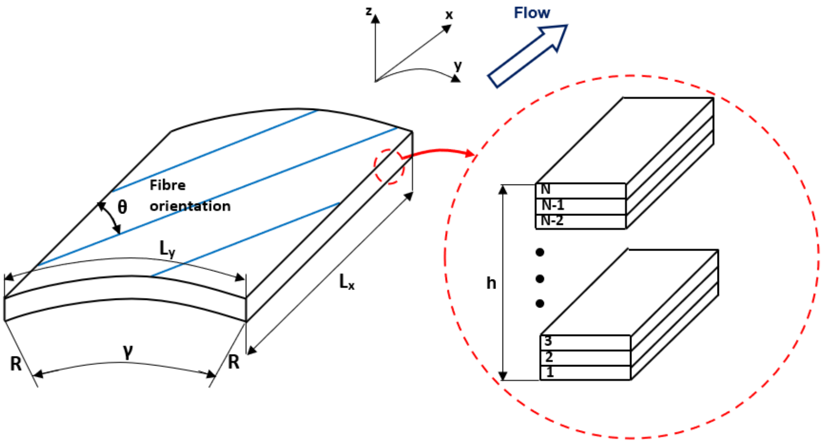

Figure 1. Geometry of shallow cylindrical panels

Assuming that:

$$\psi_1 = -\frac{\partial w}{\partial x} \;\; (\varepsilon_{\mathrm{xz}} = 0), \;\; \psi_2 = \frac{v}{R} - \frac{\partial w}{\partial y} \;\; (\varepsilon_{\mathrm{yz}} = 0). \tag{2}$$

The system of differential equations (1) is reduced to three equations with three unknown functions u, v, w. If the radius R tends to the infinity five relations (1) describe the deformations of the flat plates. However, the first two equations (u, v unknown functions) are separated from the last three (w, ψ_1, ψ_2). Since the linear piston theory is employed herein, the longitudinal effects associated with u and v variables can be neglected so that finally for plates the governing system of equations (1) is reduced to three for three unknown functions (w, ψ_1, ψ_2). Then, using equations (2) for CPT, the fundamental relation is reduced to one differential equation for the normal displacement w. Assuming that $F_{11} = F_{22} = F_{33} = F_{44} = 0$, it is well-known that the system of equations (1) can be reduced to one linear differential equation for one unknown function w both for FSDT and CST. However, the order of equations is different – see Table 1.

Table 1. Order of differential equation

	Cylindrical shells		**Flat plates**	
Order of differential equation (*K*) for the unknown function *w*	FSDT	CST	FSDT	CPT
	10	8	6	4

It is necessary to emphasize that the relations shown in the Appendix A are written for a specific form of laminates corresponding to the elimination of the terms $A_{16} = A_{26} = D_{16} = D_{26} = B_{ij} = 0$. The validity of such an assumption is discussed by Muc [7].

3. Solution of boundary value problem

Let us assume the shell structures are simply supported along the edges $x = 0$ and $x = L_x$ (Fig.1). The method of solution is discussed by Muc [8] and it can be easily applied in the problems dealing with local theories. The components of the midsurface deformations are approximated by the functions:

$$s_i = [u, v, \psi_1, \psi_2, w] = [A\cos(\alpha_m x), B\sin(\alpha_m x), C\cos(\alpha_m x), G\sin(\alpha_m x), H\sin(\alpha_m x)]\ \exp[ry]\sin(\omega t), \alpha_m = m\pi/L_x. \tag{3}$$

The solution of the equation (1) can be found with the use of the Cramer's rule. Let us insert the approximated solution (3) and then consider the system of the four equations that can be written as:

$$[L]_{4x4}\begin{bmatrix}u_1\\u_2\\\psi_1\\\psi_2\end{bmatrix} = -\begin{bmatrix}L_{15}u_3\\L_{25}u_3\\L_{35}u_3\\L_{45}u_3\end{bmatrix}. \tag{4}$$

Let us treat Eq. (4) as the system of linear algebraic equations with unknowns in the form of the vector $[u_1, u_2, \psi_1, \psi_2]^T$. The solution of the system (4) can be represented as:

$$x_i = \frac{\det(L_i)}{\det(L)}, \tag{5}$$

where L_i is the matrix formed by replacing the i-th column of the matrix L by the column vector of the right hand side of Eq. (4). Inserting the results (Eq. (5)) in the fifth row of Eq. (1) one can find that:

$$L_{5r}\frac{\det(L_r)}{\det(L)}u_3 + L_{55}u_3 = -\rho h\ddot{u}_3. \tag{6}$$

Finally Eq. (6) represents the polynomial relation for the unknown roots *r*:

$$r^K + \sum_{k=0}^{K-1} a_k r^k = 0. \tag{7}$$

The maximal order *K* of the algebraic equation is defined by the order of the differential equations demonstrated in Table 1. The explicit form of the coefficients a_k is presented for CPT in Ref. [6] ($K = 4$) and for CST in Ref. [8] ($K = 8$). The works of Abel (1826) and Galois (1832) have shown that the general polynomial equations of degree higher than the fourth cannot be solved in radicals. As we noticed in Ref. [6], the numerical computations of the roots are not convenient. Therefore, now we propose to solve the algebraic equations (7) in an analytical closed form proposed by Kulkarni [9-12] – see Appendix B.

Substituting the expressions (3) into the equations (5) results in the algebraic equations for unknown constants A_i, B_i, C_i and G_i. Finally, the ten linear equations describing the boundary conditions take the following form (it is presented for six not ten boundary conditions due to the lack of space):

$$[\Xi]\begin{bmatrix}H_1\\H_2\\H_3\\H_4\\H_5\\H_6\end{bmatrix}=\begin{bmatrix}1 & 1 & 1 & 1 & 1 & 1\\ bc_1(r_1,y=0) & bc_1(r_2,y=0) & . & . & . & .\\ bc_2(r_1,y=0) & bc_2(r_2,y=0) & . & . & . & .\\ e^{r_1L_y} & e^{r_2L_y} & e^{r_3L_y} & e^{r_4L_y} & e^{r_5L_y} & e^{r_6L_y}\\ bc_1(r_1,y=L_y) & bc_1(r_2,y=L_y) & . & . & . & .\\ bc_2(r_1,y=L_y) & bc_2(r_2,y=L_y) & . & . & . & .\end{bmatrix}\begin{bmatrix}H_1\\H_2\\H_3\\H_4\\H_5\\H_6\end{bmatrix}. \tag{8}$$

The first and the fourth row of the matrix $[\Xi]$ demonstrate the explicit form of the boundary conditions $w = 0$ at the edges. The solution of the boundary value problem (8) exists if the determinant:

$$\det[\Xi] = 0. \tag{9}$$

Let us note that for CPT the matrix Ξ has the dimension 4x4 since two boundary conditions are required at each of the edges only.

Using the procedures described above the aerodynamic characteristics representing the variations of the frequencies ω^2 with the aerodynamic pressures Λ can be drawn. The flutter phenomenon occurs for two coalescent modes n_1 and n_2 – see Figure 2.

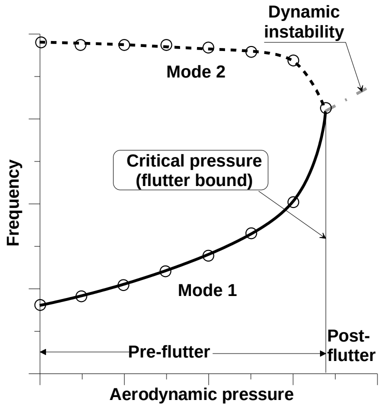

Figure 2. Variations of frequency vs. aerodynamic pressure for different modes

4. Results

The mechanical properties of the unidirectional layers used in computations are presented in Table 2.

Table 2. Properties of the unidirectional layer for carbon fibres

E_1 [GPa]	E_2 [GPa]	$G_{12} = G_{13} = G_{23}$ [GPa]	ν_{12}	ρ [kg/m^3]
138	8.28	6.29	0.33	1600

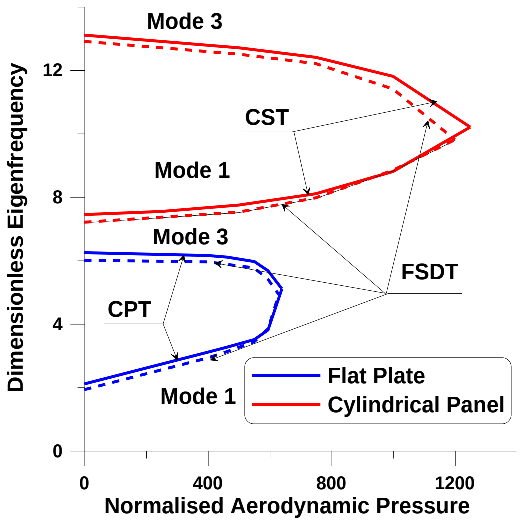

Figure 3. Flutter characteristics for plates and cylindrical panels ($L_x / L_y = 1$)

The dimensionless critical pressures $\bar{\Lambda}$ and frequencies $\bar{\kappa}$ are introduced in the following way:

$$\bar{\Lambda} = \Lambda \frac{L_x^3}{D_{11}(\theta = 0^0)}\ , \qquad \bar{\kappa} = \frac{\rho h L_x^4}{D_{11}(\theta = 0^0)} \omega^2. \tag{10}$$

The thickness-to-the length (h/L_x) ratio is assumed to be equal to 0.1 and fibres are oriented at $\theta = 90^0$. The numerical results are drawn in Figure 3. The dimensionless frequencies of modes 1 and 3 approach to each other and then merge into a same one after the flutter occurs. In general, the plot shows the identical results: the frequencies are lower for FSDT in comparison with the results for classical plate theory. The similar effects are observed for critical aerodynamic pressures.

5. Conclusions

In the present paper the method of the analytical solutions of supersonic flutter problems for multilayered composite flat plates and cylindrical shells are studied using classical plate/shell (CP/ST) and the first order transverse shear deformation (FSDT) theories. The analytical methods can be adopted for a specific boundary conditions - structures with two opposite simply-supported edges. The results show that frequencies and aerodynamic pressure are lower for FSDT. The evaluated method of solution can be treated as the benchmark for numerical analysis since analytical results can be obtained directly with the use of the symbolic packages, such as Mathematica, Maple or Matlab.

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Appendices

A. Form of differential operators

$$L_{11} = A_{11}\frac{\partial^2}{\partial x^2} + A_{66}\frac{\partial^2}{\partial y^2}\,, \qquad L_{12} = (A_{12} + A_{66})\frac{\partial^2}{\partial y \partial x}\,,$$
$$L_{15} = A_{12}\frac{1}{R}\frac{\partial}{\partial x}\,, \qquad L_{22} = A_{66}\frac{\partial^2}{\partial x^2} + A_{22}\frac{\partial^2}{\partial y^2} - \bar{A}_{55}\frac{1}{R^2}\,,$$
$$L_{24} = \bar{A}_{55}\frac{1}{R}\,, \qquad L_{25} = (A_{12} + \bar{A}_{55})\frac{1}{R}\frac{\partial}{\partial x}\,,$$
$$L_{33} = D_{11}\frac{\partial^2}{\partial x^2} + D_{66}\frac{\partial^2}{\partial y^2} - \bar{A}_{55}\,, \qquad L_{35} = -\bar{A}_{55}\frac{\partial}{\partial x}\,,$$
$$L_{44} = D_{66}\frac{\partial^2}{\partial x^2} + D_{22}\frac{\partial^2}{\partial y^2} - \bar{A}_{55}\,, \qquad L_{55} = \bar{A}_{55}\frac{\partial^2}{\partial x^2} - A_{22}\frac{1}{R^2}\,,$$
$$L_{51} = -L_{15}\,, \qquad L_{52} = -L_{25}\,, \qquad L_{53} = -L_{35}\,,$$
$$L_{13} = L_{14} = L_{23} = L_{32} = L_{34} = L_{45} = L_{54} = 0$$
$$F_{11} = F_{22} = \rho h\frac{\partial^2}{\partial t^2}\,, \qquad F_{33} = F_{44} = \frac{\rho h^3}{12}\frac{\partial^2}{\partial t^2}\,,$$
$$F_{55} = F_{11} - \Lambda\frac{\partial w}{\partial x}\,, \qquad F_{ij} = 0, \qquad i,j = 1,\ldots,5, \qquad i \neq j$$

B. Analytical form of the solutions of characteristic equations

The 5^{th} order polynomial principal equation [9]:

$$x^5 + ax^2 + bx + c = 0$$

Letting $x = u + f$ in above equation we obtain:

$$(u + f)^5 + a(u + f)^2 + b(u + f) + c = 0$$

Further expanding and rearranging gives a form:

$$u^5 + 5fu^4 + 10f^2u^3 + (10f^3 + a)u^2 + (5f^4 + 2af + b)u + f^5 + af^2 + bf + c = 0$$

Inserting a root, $-g$, into 5^{th}order equation and rearranging it in descending powers of u results in a 6^{th} order equation:

$$u^6 + (5f+g)u^5 + (10f^2+5fg)u^4 + (10f^3+a+10f^2g)u^3 + [5f^4+2af+b+(10f^3+a)g]u^2 + [f^5+af^2+bf+c+(5f^4+2af+b)g]u + (f^5+af^2+bf+c)g = 0$$

The 6^{th} order polynomial equation [10]:

$$x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0$$

the equivalent form:

$$[x^3 + b_2x^2 + (b_1 - c_1)x + b_0 - c_0][x^3 + b_2x^2 + (b_1 + c_1)x + b_0 + c_0] = 0$$

where:

$2b_2 = a_5$, $b_2^2 + 2b_1 = a_4$, $2(b_0 + b_1b_2) = a_3$, $b_1^2 - 2b_0b_2 - c_1^2 = a_2$,
$2(b_0b_1 - c_0c_1) = a_1$, $b_0^2 - c_0^2 = a_0$

The 8^{th} order polynomial equation [11]:

$$x^8 + a_7x^7 + a_6x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0$$

the equivalent form:

$$\{[(x^4 + b_3x^3 + b_2x^2 + b_1x + b_0) - p(x^4 + c_3x^3 + c_2x^2 + c_1x + c_0)]/(1-p)\}$$
$$\{[(x^4 + b_3x^3 + b_2x^2 + b_1x + b_0) + p(x^4 + c_3x^3 + c_2x^2 + c_1x + c_0)]/(1+p)\} = 0$$

where:

$[2\,(b_3 - c_3p^2)/(1-p^2)] = a_7, \{[(b_3^2 + 2b_2) - (c_3^2 + 2c_2)p^2]/(1-p^2)\} = a_6,$
$\{2\,[(b_1 + b_2b_3) - (c_1 + c_2c_3)p^2]/(1-p^2)\} = a_5,$
$\{[(b_2^2 + 2b_0 + 2b_1b_3) - (c_2^2 + 2c_0 + 2c_1c_3)p^2]/(1-p^2)\} = a_4,$
$\{2\,[(b_0b_3 + b_1b_2) - (c_0c_3 + c_1c_2)p^2]/(1-p^2)\} = a_3,$
$\{[(b_1^2 + 2b_0b_2) - (c_1^2 + 2c_0c_2)p^2]/(1-p^2)\} = a_2,$
$[2(b_0b_1 - c_0c_1p^2)/(1-p^2)] = a_1, [(\,b_0^2 - c_0^2p^2)/(1-p^2)] = a_0$

The 10^{th} order polynomial equation [12]:

$$x^{10} + a_8x^9 + a_7x^8 + a_6x^7 + a_5x^6 + a_4x^5 + a_3x^4 + a_2x^3 + a_1x^2 + a_0x = 0$$

the equivalent form:

$$[(x^5 + b_4x^4 + b_3x^3 + b_2x^2 + b_1x + b_0) - (c_4x^4 + c_3x^3 + c_2x^2 + c_1x + c_0)]$$
$$[(x^5 + b_4x^4 + b_3x^3 + b_2x^2 + b_1x + b_0) + (c_4x^4 + c_3x^3 + c_2x^2 + c_1x + c_0)] = 0$$

where:

$2b_4 = a_8,\ b_4^2 + 2b_3 - c_4^2 = a_7, 2(b_2 + b_3b_4 - c_3c_4) = a_6,$
$b_3^2 + 2b_1 + 2b_2b_4 - c_3^2 - 2c_2c_4 = a_5, 2(b_0 + b_1b_4 + b_2b_3 - c_1c_4 - c_2c_3) = a_4,$
$b_2^2 + 2b_0b_4 + 2b_1b_3 - c_2^2 - 2c_0c_4 - 2c_1c_3 = a_3,$
$2(b_0b_3 + b_1b_2 - c_0c_3 - c_1c_2) = a_2$, $b_1^2 + 2b_0b_2 - c_1^2 - 2c_0c_2 = a_1,$
$2(b_0b_1 - c_0c_1) = a_0$, $b_0^2 - c_0^2 = 0$