On Rational Functions Related to Algorithms for a Computation of Roots. II

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Abstract
We discuss a nice composition properties related to algorithms for computation of N-roots. Our approach gives direct proof that a version of Newton's iterative algorithm is of order 2. A basic tool used in this note are properties of rational function $\Phi(w,z) = \frac{z-w}{z+w}$, which was used earlier in [1] in the case of algorithms for computations of square roots.

Key words: Iterative methods, Newton methods, rational functions

1. A composition property.

Consider $\Phi(w,z) = \frac{z-w}{z+w}$ and

$$R_N(w,z) = \frac{1}{N}\left((N-1)z + w^N/z^{N-1}\right) = \frac{1}{Nz^{N-1}}\left((N-1)z^N + w^N\right),\ N \geq 2.$$

Proposition 1.1. *If N is an even number then*

$$\Phi(w, R_N(w,z)) = \Phi(w,z)^2 \frac{\Psi_N(w,z)}{\Psi_N(-w,z)},$$

where

$$\Psi_N(w,z) = (N-1)z^{N-2} + (N-2)z^{N-3}w + \cdots + w^{N-2}$$
$$= \sum_{k=0}^{N-2}(N-1-k)z^{N-2-k}w^k,\ N \geq 2,$$

are homogeneous polynomials of degree $N-2$.

Proof. By the definition,

$$\Phi(w, R_N(w,z)) = \frac{(N-1)z^N - Nwz^{N-1} + w^N}{(N-1)z^N + Nwz^{N-1} + w^N} = \frac{p_w(z)}{q_w(z)}$$

with polynomials of degree N:

$p_w(z) = (N-1)z^N - Nwz^{N-1} + w^N,\ q_w(z) = (N-1)z^N + Nwz^{N-1} + w^N.$
We have $p_w(w) = p'_w(w) = 0$ and dividing $p_w(z)$ by $(z-w)^2$ we get $p_w(z) = (z-w)^2\Psi_N(w,z)$. If now N is an even number then $q_w(z) = p_{-w}(z)$, which implies $q_w(z) = (z+w)^2\Psi_N(-w,z)$. The proof is finished.

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□

Theorem 1.2. *If $N \geq 1$ is an integer number then*

$$\Phi(w, R_N(w,z)) = h_N(\Phi(w,z)),$$

where

(1.1)

$$h_N(u) = \frac{(N-1)(1+u)^N - N(1-u)(1+u)^{N-1} + (1-u)^N}{(N-1)(1+u)^N + N(1-u)(1+u)^{N-1} + (1-u)^N} = u^2 \frac{\eta_N(u)}{\chi_N(u)},$$

where $\chi_N(u) = (N-1)(1+u)^N + N(1-u)(1+u)^{N-1} + (1-u)^N, \ \chi_N(0) = 2N$ *and*

(1.2)

$$\eta_N(u) = 2N \sum_{j=0}^{N-2} \binom{N-1}{j+1} \left(1 - \frac{1}{j+2}\frac{1-(-1)^j}{2}\right) u^j, \ \ \eta_N(0) = 2N(N-1).$$

If N is an even number then

$$\Phi(w, R_N(w,z)) = h_N(\Phi(w,z)), \ \ N \geq 2,$$

with

$$h_N(u) = u^2 \frac{\Psi_N(1-u, 1+u)}{\Psi_N(u-1, 1+u)}.$$

In particular $h_2(u) = u^2, \ h_4(u) = u^2 \frac{u^2+2u+3}{3u^2+2u+1}$ *and* $h_3(u) = u^2 \frac{2u+6}{3+3u+3u^2-u^3}$.

Proof. We can write

$$\Phi(w, R_N(w,z)) = \frac{(N-1)z^N - Nwz^{N-1} + w^N}{(N-1)z^N + Nwz^{N-1} + w^N} = \frac{(w/z)^N - N(w/z) + N - 1}{(w/z)^N + N(w/z) + N - 1}.$$

Now

$$\Phi(w,z) = \frac{1 - w/z}{1 + w/z} = \frac{1-u}{1+u} = g(u)$$

with $u = w/z$. Since $g^{-1}(u) = g(u)$ we get

$$u = g(\Phi(w,z)) = \frac{1 - \Phi(w,z)}{1 + \Phi(w,z)}$$

and therefore $\Phi(w, R_N(w,z)) =$

$$\frac{((1-\Phi(w,z)/(1+\Phi(w,z)))^N - N((1-\Phi(w,z)/(1+\Phi(w,z))) + N - 1}{((1-\Phi(w,z)/(1+\Phi(w,z)))^N + N((1-\Phi(w,z)/(1+\Phi(w,z))) + N - 1} =$$

$$\frac{(N-1)(1+\Phi(w,z))^N - N(1-\Phi(w,z))(1+\Phi(w,z))^{N-1} + (1-\Phi(w,z))^N}{(N-1)(1+\Phi(w,z))^N + N(1-\Phi(w,z))(1+\Phi(w,z))^{N-1} + (1-\Phi(w,z))^N}$$

$= h_N(\Phi(w,z))$, where $h_N(u)$ is given by (1.1).

To see the second part, we apply Proposition 1

$$\Phi(w, R_N(w,z)) = \Phi(w,z)^2 \frac{\Psi_N(w,z)}{\Psi_N(-w,z)} = \Phi(w,z)^2 \frac{\Psi_N(w/z, 1)}{\Psi_N(-w/z, 1)}$$

$$= \Phi(w,z)^2 \frac{\Psi_N(g(\Phi(w,z)),1)}{\Psi_N(-g(\Phi(w,z)),1)} = \Phi(w,z)^2 \frac{\Psi_N(1-\Phi(w,z),1+\Phi(w,z))}{\Psi_N(\Phi(w,z)-1,1+\Phi(w,z))}.$$

□

2. A computation of N-th roots.

Newton's algorithm for N-th root can be written as

$$z_N[n+1] = \frac{1}{N}\left((N-1)z_N[n] + a/z_N[n]^{N-1}\right),\ z_N[0]=1,\ N \geq 2.$$

It is well known that this is a special case of the recuurence $z[n+1] = z[n] - f(z[n])/f'(z[n])$ with $f(z) = z^N - a$.

Now fix N and put $\zeta_a[N,n+1] = R_N(a^{1/N}, \zeta_a[N,n])$. Then $\lim_{n\to\infty} \zeta_a[N,n] = a^{1/N}$ and $\lim_{n\to\infty} \Phi(a^{1/N}, \zeta_a[N,n]) = 0$.

If $E(z,w) = z - w$ then $\Phi(z,w) = \frac{E(z,w)}{E(z,w)+2w}$ and we can write

$$\frac{|\zeta_a[N,n+1] - a^{1/N}|}{\zeta_a[N,n+1] + a^{1/N}} = |h_N(\Phi(a^{1/N}, \zeta_a[N,n]))|$$

$$= \Phi(a^{1/N}, \zeta_a[N,n])^2 \frac{|\eta_N(\Phi(a^{1/N}, \zeta_a[N,n]))|}{|\chi_N(\Phi(a^{1/N}, \zeta_a[N,n]))|}. \tag{2.1}$$

Applying Theorem 1 and (2.1) we derive the following.

Theorem 2.1. *Fix a positive a. Let the sequence $\zeta_a[n]$ be defined by*

$$\zeta_a[N,n+1] = R_N(a^{1/N}, \zeta_a[N,n]),\ \zeta_a[N,0] = 1, \tag{2.2}$$

and put $\varepsilon_a[N,n] = |\zeta_a[N,n] - a^{1/N}|$. Then

$$\gamma(N,a) := \lim_{n\to\infty} \frac{\varepsilon_a[N,n+1]}{\varepsilon_a[N,n]^2} = \frac{N-1}{2a^{1/N}} \tag{2.3}$$

and $\gamma(N,a) < 1$ if $a > ((N-1)/2)^N$. In particular $\gamma(2,a) < 1$ for $a > 1/4$ and $\gamma(3,a) < 1$ in the case $a > 1$.

Corollary 2.2. *Let k be a positive integer. Then $\gamma(Nk, a^k) = \left(k + \frac{k-1}{N-1}\right)\gamma(N,a)$, which implies that algorithm for computation $a^{1/N}$ is asymptotically faster than algorithm for computation of $(a^k)^{1/Nk}$.*

Example 2.3. Let us consider $a = 2$ and $N = 2,\ k = 2$. We compare two sequences $\zeta_2[2,n]$ and $\zeta_4[4,n]$, $\sqrt{2} = 1.4142135623730950488016887\ldots$

n	0	1	2	3	4
$\zeta_2[2,n]$	1	1.5	**1.41**6666666666667	**1.41421**568627451	**1.41421356237**469
$\zeta_4[4,n]$	1	1.75	**1.4**99088921282799	**1.42**1153542275386	**1.4142**64232528399

n	5	6
$\zeta_2[2,n]$	**1.414213562373095**	**1.414213562373095**
$\zeta_4[4,n]$	**1.41421356**509614	**1.414213562373095**

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