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# Discretization schemes for the simulation of nonlinear propagation equations

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ABSTRACT

A technique is proposed for testing the stability and convergence of Finite Differences schemes for the simulation of the propagation of pulses in non linear media. As a first application, the propagation of a semi-infinite sine pulse in a medium with a periodic non linearity is analyzed.

## INTRODUCTION

Nonlinear problems have been widely analyzed in recent years, leading to the observation of several interesting effects, such as the generation of solitons, chaotic or singular solutions, subharmonics and/or higher harmonics [1], etc. Due to the formidable difficulties encountered in a general treatment of non linear equations, a theoretical analysis leading to closed form solution is usually not available. Even a numerical solution, based on sequential processing, is generally cumbersome and time consuming.

Moreover, several difficulties may arise in a numerical treatment, due to a lack of stability and convergence of the discrete version of the partial differential equation describing the physical model. Also, in non linear cases, the classical stability analysis, due to Von Neumann, is not always sufficient to guarantee the convergence of the numerical solution [2].

We propose here a simulation of the propagation of pulses in non linear media based on parallel processing, following a Local Interaction Simulation Approach (LISA), already succesfully applied in the linear case for 2-D and 3-D arbitrarily complex media [3,4,5]. The analysis of the convergence is confirmed empirically by means of a comparison between the results of two schemes with different orders of accuracy.

## THEORY

Let us consider the propagation of an ultrasonic wave in a 1-D linear medium with a non linear $x$-dependent forcing. We assume that a longitudinal pulse enters the specimen normal to its external surface. The wave equation can be written as

$$\rho\ddot{w}(x,t) = Sw''(x,t)\big(1+b(x)w(x,t)\big), \qquad (1)$$

where $\rho$ is the density of the specimen, $S$ its stiffness and $b(x)w(x,t)$ is an arbitrary perturbation.

In a general treatment, $b(x)$ should be a smooth function of $x$. However, the Sharp Interface Model (SIM) [3,4] allows us to treat also cases with sharp discontinuities of the perturbation. In this case, the iteration equation must be derived in correspondence of the discontinuities directly from physical principles and conservation laws. Due to space limitations, we restrict ourselves here to the assumption that $b(x)$ be a smooth function.

By discretizing space and time with steps $\epsilon$ and $\tau$ respectively, Eq.(1) yields the following basic Finite Difference (FD) schemes:

$$\begin{aligned} w_i^{t+1} &= 2w_i^t - w_i^{t-1} + \alpha\big(w_{i+1}^t \\ &\quad -2w_i^t + w_{i-1}^t\big)(1+bw_i^t) + O(\tau^4,\epsilon^4), \qquad (2) \\ w_i^{t+1} &= 2w_i^t - w_i^{t-1} - \frac{\alpha}{12}\big(w_{i+2}^t - 16w_{i+1}^t + 30w_i^t \\ &\quad -16w_{i-1}^t + w_{i-2}^t\big)(1+bw_i^t) + O(\tau^4,\epsilon^6), \qquad (3) \end{aligned}$$

where $\alpha = S\tau^2/\rho\epsilon^2$. Equations (2) and (3), if convergent, are expected to be completly equivalent, albeit with a different order of accuracy .

The error $E(\tau, \epsilon)$ in the discretization can be easily evaluated. If we consider separately each Fourier component of the solution

$$w(x,t) = \sum u_k(x,t) = \sum C_k e^{ik(x-vt)},$$

we have

$$\begin{aligned} E(\tau,\epsilon) &= \frac{\epsilon^2\tau^2}{12}(w^{IV}(1+bw) - C\ddot{w}) = \\ &= \frac{C((1+bw)-C)}{12n^4} \qquad (4) \\ E(\tau,\epsilon) &= -\frac{\epsilon^2\tau^2}{12}(C\ddot{w} + \frac{2}{15}\epsilon^2 w^{VI}(1+bw)) \\ &= \frac{C(\frac{2}{15n^2}(1+bw)-C)}{12n^4} \qquad (5) \end{aligned}$$

where $C = S\tau^2/(\rho\epsilon^2)$ and $n = 1/(k\epsilon)$ is the number of grid points describing the pulse. It can be easily seen that scheme (2) may be more accurate, due to the compensation between the errors due to the spatial and temporal derivatives. On the other side, in scheme (3) there is no compensation, but much smaller spatial errors, allowing a better convergence when errors due to the non linearity prevail.

As already metioned, a proper choice of lattice and time steps is fundamental to guarantee stability. The Von Neumann analysis leads to the following stability conditions for the above mentioned schemes (2 and 3, respectively):

$$\tau \leq \frac{\epsilon}{\sqrt{S/\rho(1+bu)}} \qquad (6)$$

and

$$\tau \leq \frac{\epsilon\sqrt{12}}{4\sqrt{S/\rho(1+bu)}}, \qquad (7)$$

where $u$ is a local value of the wave amplitude $w$ (i.e. its maximum to guarantee everywhere stability).

The same analysis allows an estimate of the convergence properties of the numerical solution. Considering the dispersion relations for both continuous and discrete equations, it is possible to write the ratio $Q(k)$ between the discrete and the continuous velocities ($v_D$ and $v_C$) of the wave as a function of the wave number $k$. Of course, maximal convergence is obtained when such ratio approaches to 1. For the schemes (2) and (3), we obtain respectively:

$$\begin{aligned} Q(k) &= \frac{arccos\left(1+\alpha\left(\cos k\epsilon - 1\right)\left(1+bu\right)\right)}{k\tau\sqrt{S/\rho(1+bu)}} \\ Q(k) &= \frac{arccos\left(1+\alpha\left(\cos 2k\epsilon - 16\cos k\epsilon + 15\right)\left(1+bu\right)\right)}{k\tau\sqrt{S/\rho(1+bu)}} \end{aligned}$$

Therefore, the choice

$$\tau = \frac{\epsilon}{\sqrt{S/\rho(1+bu)}} \qquad (8)$$

for scheme (2) leads to $Q$ equal one, confirming the convergence of the method. However, it is not possible to find an optimal value of $\tau$ for scheme (3). In this case, the shift between the velocities can only be minimized and some effects due to numerical dispersion are expected.

## NUMERICAL RESULTS

Several numerical calculations have been performed. In the following, unless otherwise specified, the results are referred to scheme (2), with $S/\rho = 1$. This assumption is not a restriction, but only a simplification, easily achieved by means of a proper choice of units. The source pulse has been chosen to be a semiinfinite sine pulse of the form

$$w(0,t) = \sin\omega t \quad , \quad t \geq 0, \qquad (9)$$

injected into the specimen from the left border ($x = 0$). Wherever possible, the period of the pulse has been kept constant ($\omega = 2\pi/100$) and the number of time steps $t$ in each figure has been chosen in such a way as to guarantee the same physical time $\tilde{t} = t\tau$ for all the plots.

Figure 1 shows that scheme (3) is slightly less convergent than scheme (2), as expected from previous considerations. It is, however, nore stable when the non linearity increases.

In Fig.2, the dependence of the convergence on the discretization of the initial pulse (gaussian) is investigated. The numerical results are independent from the number $n$ of points describing the initial pulse, provided that $n$ is sufficiently large ($n = 120, 60, 30$). Poorer convergence is obtained for small values of $n$ ($n = 6, 3$), in agreement with the predictions for the errors in Equations (4) and (5). Of course a large value of $n$ is equivalent to a

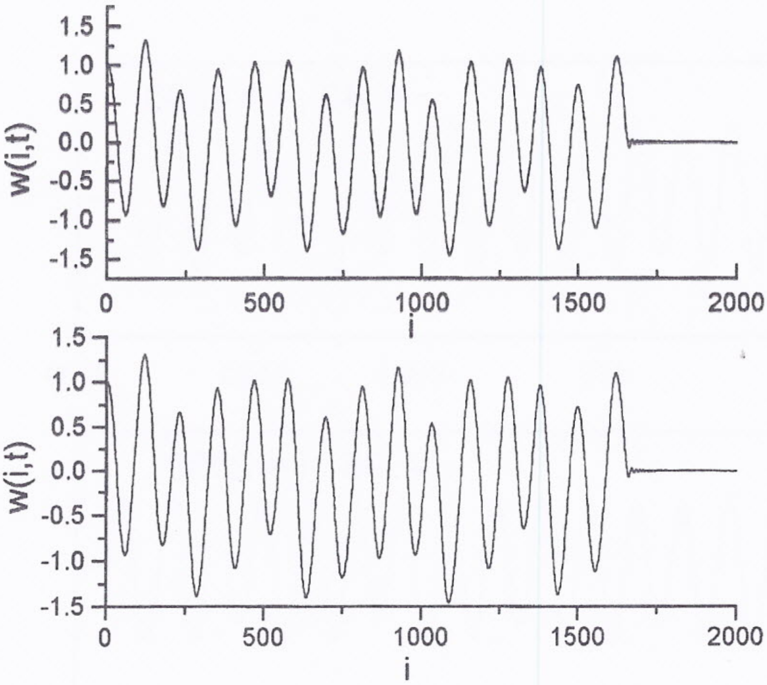


Figure 1: Wave amplitude $w_i^t$ vs. $i$ as obtained from the simulation using scheme (3) (upper plot) and scheme (4) (lower plot). The negligible difference between the results show the good convergence of the method, even though a few numerical errors appear in the bottom plot.

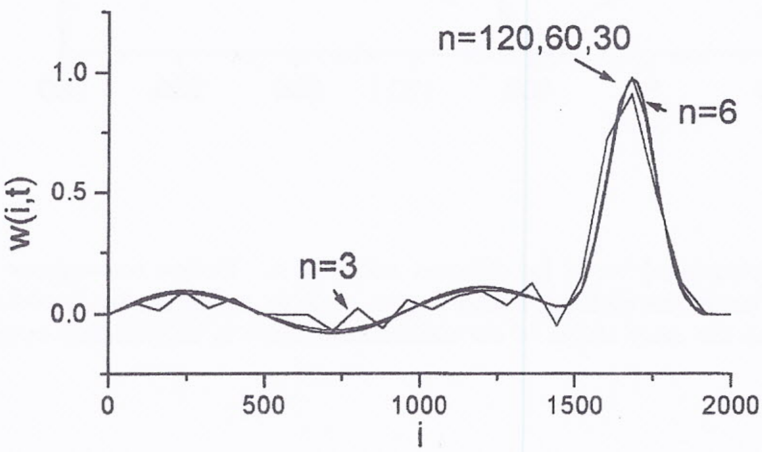


Figure 2: Wave amplitude $w_i^t$ vs. $i$ for different values of $n$. Schemes with $n = 120, 60, 30$ are perfectly convergent. Convegence is poorer and numerical errors are larger decreasing $n$ (i.e. increasing $\epsilon$).

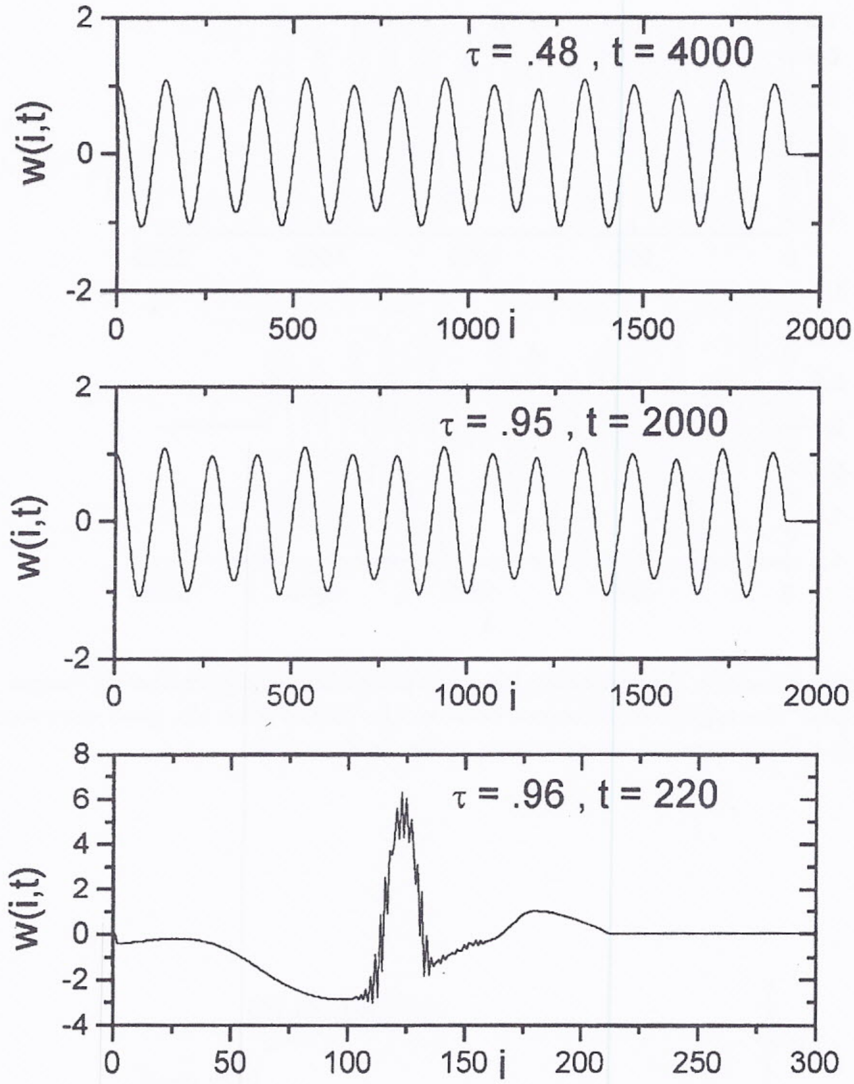


Figure 3: Wave amplitude $w_i^t$ vs. $i$ for different values of $\tau$. Perfect convergence and stability is obtained when $\tau$ is inside the stability region (upper plot) or on its border (middle plot). A local instability develops at the early stages of the simulation when $\tau$ is immediately outside the stability region (lower plot).

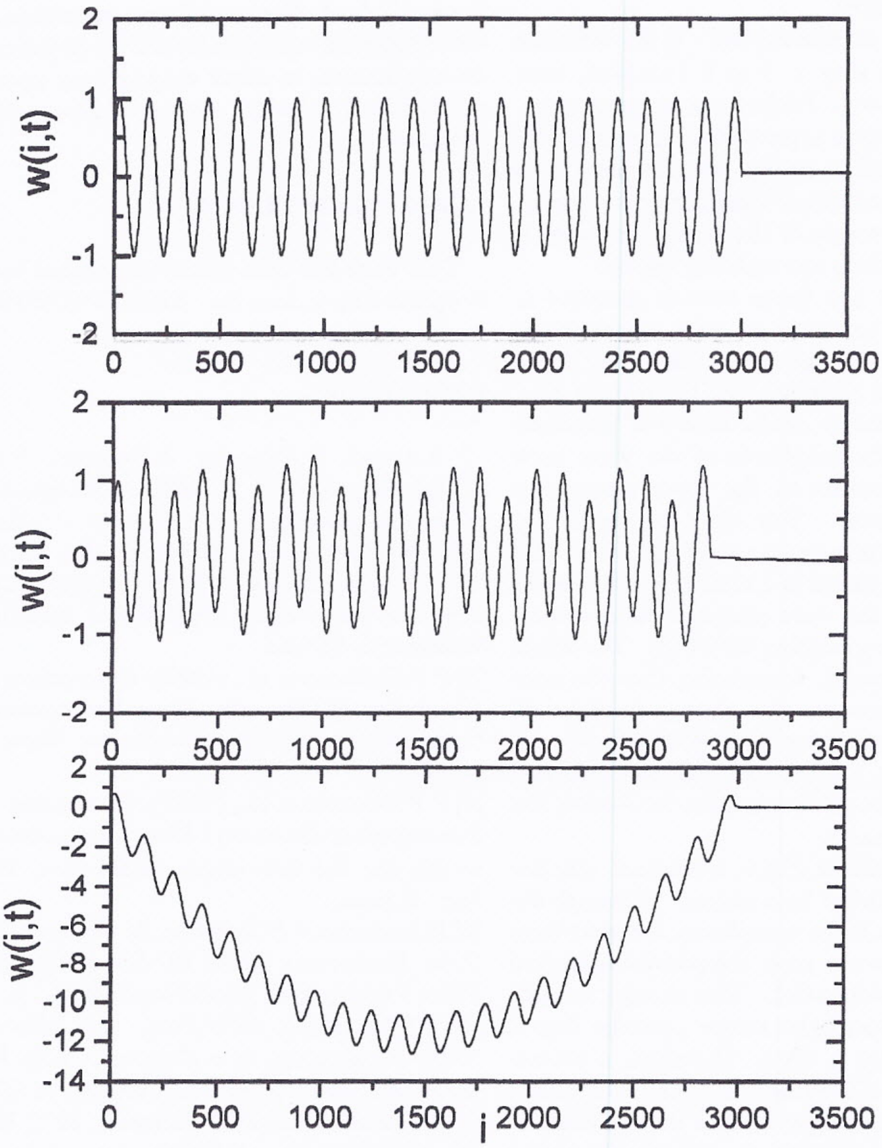


Figure 4: Wave amplitude $w_i^t$ vs. $i$ for different kind of non linearities. The first plot represents the linear case $b = 0$. The second plot a periodic non linearity $b = 0.1\cos\omega i$ and the third plot a constant non linearity $b = .01$.

small value of $\epsilon$. Similar results have been found for other pulse sources.

In Fig. 3, $w_i^t$ is plotted vs. $i$ for different values of the time step $\tau$, $b = 0.1\cos(\theta x)$, with $\theta = 2\pi/200$,and $\epsilon = 1$. Perfect convergence is obtained when $\tau$ is inside (upper plot) or even at the border of the stability region (middle plot). As expected, a local numerical instability develops already at the early stages of the simulation when $\tau$ is even slightly outside the stability region.

The effect of the non linear term is analyzed in Fig.4, where $w_i^t$ is reported vs. $i$ for several kinds of non linearities. In the upper plot, the linear case is reported for reference. In the middle plot, a periodic non linearity of the kind $b = 0.1\cos\theta x$ has been used. The amplitude of the wave both increases and decreases as the wave propagates through the material. The effect is completely analogous to that of acoustic beatings. It is also similar to an effect found in a model recently developed for studying the wave propagation in a composite with wavy reinforcing fibers [6]. The effect can be easily explained, considering that the nonlinear term is equivalent to a change in the stiffness of the material (amplified by the factor $w$). Therefore different amounts of material deformation are needed along the specimen for storing the same amount of energy.

In the bottom plot of Fig.4, a constant non linearity with $b = 0.01$ has been chosen. Although the behaviour appears to be completely different from the one in the previous case, the previous physical interpretation is still valid. The change in stiffness of the specimen is no longer periodic, but it increases constantly in time. Therefore, a permanent deformation of the material is needed for storing the additional energy. Such a deformation, of course, increases constantly in time in the region already reached by the wave and vanishes where the forcing of the injection keeps the deformation fixed.

## CONCLUSIONS

In general the discretization of non linear problems may introduce severe errors due to lack of convergence. Since the usual stability analysis is not always satisfactory to test the convergence when the continuous solution of the equation is not known, we have proposed a method based on a comparison between FD schemes with different orders of accuracy. The results, applied to a particular non linear propagation equation, are in good agreement with the ones which can be obtained from a von Neumann stability analysis.

The method has been applied succesfully also to the analysis of other non linear equations, such as the Boussinesq equation for soliton propagation [7]. An application to other propagation equations of physical and practical relevance is currently under progress.

## ACKNOWLEDGMENTS

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