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THE PRACTICAL FEEDBACK STABILIZATION FOR EVOLUTION EQUATIONS IN BANACH SPACES

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Abstract: This paper investigates the notion of practical feedback stabilization of evolution equations satisfying some relaxed conditions in infinite-dimensional Banach spaces. Moreover, sufficient conditions are presented that guarantee practical stabilizability of uncertain systems based on Lyapunov functions. These results are applied to partial differential equations.

Key words: dynamical systems in control, linear operator, controllability, uncertain systems, practical stabilization, Banach spaces

1. INTRODUCTION

In the literature on control theory of time-varying dynamical systems, controllability and stabilizability are the qualitative control problems that play an important role in the systems and have attracted many researchers (Damak et al., 2016; Ikeda et al., 1972; Phat and Ha, 2008; Phat, 2001, 2002; Phat and Kiet, 2002). The theory was first introduced by Kalman et al. (1963) for the finite dimensional time-invariant systems. Furthermore, the theory which related to exponential stability was first introduced by Wonham (1967). Lyapunov function approach and the method based on spectral decomposition are the most widely used techniques for studying stabilizability of special classes of control systems, see for example Kobayashi (1989) and Tsinias (1991). In the infinite-dimensional control systems, the investigation of practical stabili-zation is more complicated and requires more sophisticated tech-niques. The practical stabilization is to find the state feedback candidate such that the solution of the closed-loop system is practically exponentially stable in the Lyapunov sense in which the origin is not necessary to be an equilibrium point. In this case, Damak et al. (2016) proved the practical feedback stabilization of the time-varying control systems in Hilbert spaces where the nominal system is a linear time-varying control system globally null controllable and the perturbation term satisfies some conditions. Kalman et al. (1963) and Wonham (1967) have shown that in the finite-dimensional autonomous control system, if the system is null controllable in finite time, then it is stabilizable. But it does not hold for the converse. Moreover, if the system is completely stabilizable, then it is null controllable in finite time. The results of stabilizability for the finite-dimensional systems can be generalized into infinite-dimensional systems. For time-invariant control systems in Banach spaces, Phat and Kiet (2002) defined an equivalence between solvability of the Lyapunov equation and exponential stability of linear system. Based on the Lyapunov theorem, a relationship between stabilizability and exact null controllability of linear time-invariant control systems is established. Moreover, they gave the exponential stabilizability of a class of nonlinear control systems. In recent years, non-autonomous differential equations on infinite-dimensional spaces have been studied by many researchers, see the references Damak and Hammami (2020), Damak (2021), Chen et al. (2020a, 2020b, 2020c, 2021) and Chen (2021) for more details. In the study by Chen et al. (2020b), sufficient conditions of existence of mild solutions and approximate controllability for the desired problem are given by introducing a new Green's function and constructing a control function involving Gramian controllability operator.

In this paper, we extend the results of Pazy (1983) and Phat and Kiet (2002) to discuss the problem of practical stabilization for evolution equations in Banach spaces. Based on the exact null controllability assumption of the linear control system, sufficient conditions for the stabilizability are established by solving a standard Lyapunov equation. Further, the nonlinear perturbation term is locally Lipschitz continuous and satisfies some appropriate growth conditions. A feedback controller that assures global practical uniform exponential stability of the closed-loop system has been proposed, that is, the solutions of the closed-loop system converge towards an arbitrary small neighbourhood of the origin.

The paper is organized as follows: Section 2 briefly introduces some notations and necessary preliminaries. Section 3 presents the required assumptions and the statement of the main results. Section 4 presents illustrative examples, which shows the importance of this study. Section 5 provides conclusion of this study.

2. PRELIMINARIES

Throughout this paper, we adopt the following notations R+ and X. R+ denotes the set of all non-negative real numbers and X denotes an infinite-dimensional Banach space with the norm ǁǁ. Let X^* be the topological dual space of X and U infinite-dimensional Banach space. Let $\langle y^*, x\rangle$ denote the value of y at x. L(X) (respectively L(X,Y)) denotes the Banach space of all linear bounded operators mapping X into X (respectively, X into Y) endowed with the norm $\|T\|=\sup_{x\in X}\frac{\|Tx\|}{\|x\|}$.

The domain, the image, the adjoint and the inverse operator of an operator A are denoted as D(A), ImA, A^* and A−1 respectively. Everywhere below A is a linear operator in X with domain D(A), generating a strongly continuous semigroup S(t), that is:

$$A=\lim_{h\to 0}\frac{S(h)-I}{h},$$

in the strong topology. L_2([t, s], X) denotes the set of all strongly measurable L_2 integrable and X valued functions on [t, s]. Let Q∈L(X,X^*) be a duality operator. We recall that the operator Q is positive definite in X if $\langle Qx,x\rangle \geq 0$ for arbitrary $x\in X$ and $\langle Qx,x\rangle >0$ for $x\neq 0$.

We will denote by LPD(X,X^*) and LSPD(X,X^*) the set of all linear bounded positive definite and strongly positive definite operators mapping X into X^*, respectively. Also, we define:

- $L^p(R_+,R_+)$ is the set of functions positive and integrable with pth power on R_+ where p ≥ 1 endowed with the norm $\|\phi\|_p=(\int_0^{+\infty}\phi^p)^{\frac{1}{p}}$ for $\phi\in L^p(R_+,R_+)$.
- $L^\infty(R_+,R_+)$ is the set of all measurable functions from R_+ to R_+ which are essentially bounded endowed with the norm
- $\|\phi\|_\infty=\sup_{t\in R}\phi(t)$ for $\phi\in L^\infty(R_+,R_+)$.
- $1_{[\vartheta,\zeta]}=\begin{cases}1, & \text{si } \vartheta\leq x\leq\zeta,\\ 0, & \text{elsewhere.}\end{cases}$

We consider the following system:

$$\begin{cases}\dot{x}=F(t,x,u), & t\geq t_0\geq 0,\\ x(t_0)=x_0,\end{cases} \tag{2.1}$$

where $x\in X$ is the system state, $u\in U$ is the control input and $F: R+\times X\times U\to R+$ is a given function.

Definition 2.1. System (2.1) is practically stabilizable if there exists a continuous feedback control u : X → U, such that system (2.1) with u(t) = u(x(t)) satisfies the following properties.

- For any initial condition $x_0\in X$, there exists a unique mild solution $x(t,x_0)$ defined on R_+.
- There exist positive scalars ω, k, and r, such that the solution of the system (2.1) satisfies the following:

$$\|x(t)\|\leq k\|x_0\|e^{-\omega(t-t_0)}+r, \qquad \forall t\geq t_0\geq 0.$$

When (i) and (ii) are satisfied, we say that Eq. (2.1) with u(t) = u(x(t)) is globally practically uniformly exponentially stable.

Definition 2.2. (Diesel and Uhl Jr, 1977). A Banach X^* has the Radon-Nikodym property if:

$$L_2([0,T],X^*)=(L_2([0,T],X))^*.$$

In the proof of the mains results, we shall use the following lemmas.

Lemma 2.1. (Nonlinear generalization of Gronwall's inequality, Zhoo, 2017).

Let θ be a non-negative function on R+, that satisfies the following integral inequality:

$$\theta(t)\leq \nu+\int_{t_0}^{t}(\chi(s)\theta(s)+\sigma(s)\theta^{\alpha}(s))ds,\ v\geq 0,$$

$0\leq\alpha<1, t\geq t_0$, where χ and σ are non-negative continuous functions. Then:

$$\theta(t)\leq[\nu^{1-\alpha}e^{(1-\alpha)\int_{t_0}^{t}\chi(s)ds}+(1-\alpha)\int_{t_0}^{t}\sigma(s)e^{(1-\alpha)\int_s^t\chi(r)dr}]^{\frac{1}{1-\alpha}}$$

Lemma 2.2. (Generalized Gronwall-Bellman Inequality, Dragomir, 2002).

Let λ, ρ : $R_+\to R$ be continuous functions and $\phi: R_+\to R_+$ is a function, such that:

$$\dot{\phi}(t)\leq\lambda(t)\phi(t)+\rho(t),\ \forall t\geq t_0.$$

Then, we have the following inequality:

$$\phi(t)\leq\phi(t_0)e^{\int_{t_0}^{t}\lambda(v)dv}+\int_{t_0}^{t}e^{\int_s^t\lambda(v)dv}\rho(s)ds.$$

Lemma 2.3. Let $a,b\geq 0$ and $p\geq 1$. Then:

$$(a+b)^p\leq 2^{p-1}(a^p+b^p).$$

3. MAIN RESULTS

In this section, we shall state and prove our main results.

3.1 Practical stabilization of infinite-dimensional evolution equations

The purpose of this subsection is to establish the practical stabilization of evolution equations in Banach spaces. Based on the exact null-controllability in finite time of the nominal system whose origin is an equilibrium point, a stabilizing controller for the nonlinear system is then synthesized that guarantees the uniform exponential convergence to a neighborhood of the origin. This leads us to address the problem of practical stability of time-varying perturbed systems.

Consider infinite-dimensional evolution equations of the follwing form:

$$\begin{cases}\dot{x}=Ax+Bu+F(t,x),\\ x(t_0)=x_0,\end{cases} \tag{3.1}$$

where $x\in X$ is the system state, $u\in U$ is the control input, X is a Banach space, X^* has the Radon-Nikodym property and U is a Hilbert space. The operator $A:D(A)\subset X\to X$ is assumed to be the infinitesimal generator of the C_0-semigroup S(t) on X, B∈ $L(U,X)$ and the function $F: R_+\times X\to R_+$ is continuous in t and locally Lipschitz continuous in x, that is for every $t_1\geq 0$ and constant $c\geq 0$, there is a constant M(c, t_1), such that:

$$\|F(t,u)-F(t,v)\|\leq M(c,t_1)\|u-v\|$$

holds for all u, v∈ X, with $\|u\|\leq c$, $\|v\|\leq c$ and t∈ $[0,t_1]$.

This system is seen as a perturbation of the nominal system:

$$\begin{cases}\dot{x}=Ax+Bu,\\ x(0)=x_0,\end{cases} \tag{3.2}$$

Next, we are interested in suitable feedback of the for the following:

$$u(t)=Dx(t), \tag{3.3}$$

where D∈ $L(X,U)$.

Let $x(t,x_0,u)$ denote the state of a system (3.1) at moments $t\geq t_0\geq 0$ associated with an initial condition $x_0\in X$ at $t=t_0$ and input u∈ U.

Now, we recall the definition of the generator of an exponentially stable semi-group as well as that of the exponential stability (Curtain and Zwart, 1995).

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Definition 3.1. The operator A generates an exponentially stable semigroup S(t) if the initial value problem:

$$\dot{x}=Ax, \qquad t\geq 0, \quad x(0)=x_0 \tag{3.4}$$

has a unique solution $x(t)=S(t)x_0$ and $\|S(t)\| \leq Me^{-\alpha t}$ for all $t\geq 0$ with some positive numbers M and α.

Definition 3.2. The linear control system (3.4) is exponentially stable if there exist numbers $M> 0$ and $\alpha > 0$, such that:

$$\|x(t)\|\leq Me^{-\alpha t}\|x_0\|, \quad \forall\, t \geq 0.$$

Definition 3.3. The control system (3.2) is exactly null-controllable in finite time if for every $x_0 \in X$, there exist a number $T> 0$ and an admissible control $u(t) \in U = \{u(.) \in L_2([0,\infty), U)\}$, such that:

$$S(T)x_0 + \int_0^T S(T-s)Bu(s)ds = 0.$$

If we denote by C_T the set of null-controllable points in time T of system (3.2) defined by:

$$C_T = \left\{x_0 \in X; S(T)x_0 - \int_0^T S(T-s)Bu(s)ds;\ u(.) \in U\right\},$$

the system (3.2) is exactly null-controllable in time $T> 0$ if $C_T = X$.

If A is the generator of an analytic semigroup S(t) for $T > 0$, then we can define the operator $W_T \in L(U,X)$ by:

$$W_T=\int_0^T S^{-1}(s)Bu(s)ds, u(.) \in U,$$

and we have $C_T = \operatorname{Im} W_T$.

We state the following well-known controllability criterion for the infinite-dimensional control system presented in Curtain and Pritchard (1978) for reflexive Banach spaces and then in Xuejiao and Zhenchao (1999) for non-reflexive Banach spaces having the Radon-Nikodym property.

Proposition 3.1. (Curtain and Pritchard, 1978 and Xuejiao and Zhenchao, 1999). Let X, U be Banach and S(t) the C_0-semigroup of A. Assume that X^*, U^* has the Radon-Nikodym property. The following conditions are equivalent.

- Control system (3.2) is exactly null-controllable in time $T> 0$.
- There exists $c> 0$, $\|W_T{}^*x^*\| \geq c\|x^*\|, \forall\, x^* \in X^*$.
- There exists $c> 0$, $\|B^*S^*(s)x^*\|^2 \geq \|S^*(T)x^*\|$, $\forall\, x^* \in X^*$.
- If U is a Hilbert space, the operator $W_T = \int_0^T S^{-1}(s)BB^*S^{*-1}(s)ds$ is strongly positive definite.

The operator $P\in L(X,X^*)$ is called a solution of the Lyapunov equation if the following condition hold:

$$\langle PAx,x\rangle + \langle Px,Ax\rangle = -\langle Qx,x\rangle, \forall\, x \in D(A). \tag{3.5}$$

Note that, if A is bounded, then the above Eq. (3.5) has the standard form:

$$A^*Px + PAx = -Qx, \qquad \forall\, x \in X.$$

Remark 3.1. Datko (1970) showed that if A is exponentially stable in Hilbert space, then the Lyapunov equation has a solution.

We present the equivalence between the solvability of the Lyapunov equation and the exponential stability of the linear system (3.4) in the following Proposition 3.2.

Proposition 3.2. (Phat and Kiet, 2002) If for some $Q\in LSPD(X,X^*)$, $P\in LPD(X,X^*)$, the Lyapunov equation holds, then the operator A is exponentially stable. Conversely, if the generator A is exponentially stable, then for any $Q \in LSPD(X,X^*)$, there is a solution $P\in LPD(X,X^*)$ of the Lyapunov equation:

$$A^*P + PA = -Q. \tag{3.6}$$

Definition 3.4. The linear control system (3.2) is completely stabilisable if for every $\alpha > 0$, there exists a linear bounded operator $D:X\to U$ and a number $M> 0$, such that the solution satisfies the following condition:

$$\|x(t)\|\leq Me^{-\alpha t}\|x_0\|, \quad \forall\, t \geq 0.$$

Note that, if the operator D and number M do not depend on α, then the complete stabilizability implies exponential stabilizability in usual Lyapunov sense (Zabczyk, 1992). It is known from that if the linear control system (3.2), where X and U are Hilbert spaces is completely stabilizable then it is exactly null-controllable in finite time (Megan, 1975). Also, Phat and Kiet (2002) improved this result in Banach spaces.

Proposition 3.3. If the linear control system (3.2) is completely stabilisable then it is exactly null-controllable in finite time.

In the sequel, Phat and Kiet (2002) proved that the linear control system (3.2) is exponentially stabilizable by linear feedback $D:X\to U$, if it is null-controllable in finite time.

Proposition 3.4. If the linear control system (3.2) is exactly null-controllable in finite time, then it is exponentially stabilizable.

We define the Lie derivative of a function V(x) along solutions of (3.1) as:

$$\dot{V}(x) = \lim_{t\to 0^+}\sup \frac{1}{t}(V(x(t,x,u))-V(x)).$$

Now, we suppose the following assumptions.

(H1) The linear control system (3.2) is exactly null-controllable in finite time and there exists a constant operator $D:X\to U$, such that a sufficient condition specially related to operator is presented in Phat and Kiet (2002) as the following: for any $Q\in LSPD(X,X^*)$: $\langle Qx,x\rangle \geq b_1 \|x\|^2$, $\forall\, x \in X$, there exists $P\in LPD(X,X^*)$, $b_2\|x\|^2 \leq \langle Px,x\rangle \leq \|P\|\|x\|^2$, $\forall\, x \in X$, where $b_1, b_2 > 0$, which satisfies:

$$A_D{}^*P + PA_D = -Q. \tag{3.7}$$

(H2) The perturbation F: $R+\times X \to R+$ verifies the following estimation:

$$\|F(t,x)\| \leq \varphi(t)\|x\| + \mu(t) + \eta, \forall\, t \geq 0, \forall x \in X, \eta \geq 0,$$

where φ and μ are non-negative continuous functions with $\varphi \in L^1(R+,R+)$ and $\mu \in L^p(R+,R+)$ for some $p\in [1,+\infty)$.

Next, sufficient conditions are presented to guarantee the global existence and uniqueness of solutions of systems (3.1). Further, we investigate the practical stabilizability of the evolution equation using generalised Gronwall-Bellman inequality and Lyapunov theory.

Theorem 3.1. Under assumptions (H1) and (H2) the closed-loop system (3.1)-(3.3) have a unique solution, which is globally defined for all $t\geq t_0$ and this system is globally practically uniformly exponentially stable.

Proof. We break up the proof into two steps.

Step 1: Since F is locally Lipschitz continuous in x, uniformly in t, it follows from Pazy (1983) that for every initial condition the closed-loop equation possesses a unique mild solution on some interval $[t_0, t_0+\delta]$ with $\delta > 0$. Indeed, integrating (3.1) , we obtain the following for $t\in[t_0, t_0+\delta]$:

$x(t)=S(t-t_0)x_0+\int_{t_0}^{t} S(t-s)\left[Bu(s)+F\left(s,x(s)\right)\right]ds.$

Since $B\in L(U,X)$, then:

$$\|x(t)\| \leq M_1\|x_0\| + M_1\left(\int_{t_0}^{t} \|B\|\|D\|\|x(s)\| + M_2\|x(s)\| + M_3 + \eta ds\right), \quad (3.8)$$

where: $M_1=\sup\{\|S(t-s)\|: 0 \leq t_0 \leq s \leq t \leq t+ \delta\}$, $M_2 = \sup_{t\in[t_{0,}t_0+\delta]} \|\varphi(t)\|$ and $M_3 = \sup_{t\in[t_{0,}t_0+\delta]} \|\mu(t)\|$.

By applying Gronwall inequality (Tsinias, 1991, Lemma 2.7, p42) to inequality (3.8), any solution of this equation is uniformly bounded $\|x(t)\| \leq M_1(\|x_0\| + M_3 + \eta)$ on an arbitrary time interval $[t_0, t_0+\delta]$. Then, using Theorem 1.4 of Pazy (1983), we have $t_0+\delta = \infty$, and so we get global existence.

Step 2: Consider a Lyapunov function: $V(x)=\langle Px, x\rangle$. Let us compute the Lie derivative of V with respect to system (3.1) in a closed-loop with the controller (3.3). For $x\in D(A)$, we have $\dot{V}(x)=\langle P\dot{x}, x\rangle + \langle Px, \dot{x}\rangle$. From $\langle P(Ax), x\rangle = \langle Ax, Px\rangle$ and (3.7) with the help of Cauchy-Schawtz inequality, we obtain the following:

$$\begin{aligned}\dot{V}(x) &\leq -\langle Qx, x\rangle + 2\|P\|\|F(t,x)\|\|x\| \\ &\leq -b_1\|x\|^2 + 2\|P\|(\varphi(t)\|x\| + \mu(t) + \eta)\|x\| \\ &\leq \left(\frac{b_1}{\|P\|} + \frac{2\|P\|(\varphi(t)}{b_2}\right)V(x) + \frac{2\|P\|}{\sqrt{b_2}}(\mu(t)+\eta)\sqrt{V(x)}\,.\end{aligned}$$

Let $z(x)=\sqrt{V(x)}$, which implies that:

$$\dot{z}(x)\leq \left(-\frac{b_1}{2\|P\|}+\frac{2\|P\|\varphi(t)}{b_2}\right)z(x)+\frac{\|P\|}{\sqrt{b_2}}(\mu(t)+\eta). \quad (3.9)$$

These derivations hold for $x\in D(A)\subset X$. If $x\notin D(A)$, then the solution $x(t)\in D(A)$ and $t\to z(x(t))$ is a continuously differentiable function for all $t\geq t_0$ (for these properties of solutions, see Theorem 3.3.3 in Henry (1981)). Thus, by the mean value theorem we obtain that Eq.(3.9) holds for all $x\in X$. Using Lemma 2.2, we have for all $t \geq t_0$ $z(x)\leq z(x_0)\exp\left(\frac{\|P\|M_\varphi}{b_2}\right)\exp\left(-\frac{b_1}{2\|P\|}(t-t_0)\right)+ \exp\left(\frac{\|P\|M_\varphi}{b_2}\right)\int_{t_0}^{t}\frac{\|P\|}{\sqrt{b_2}}(\mu(s)+\eta)\exp\left(-\frac{b_1}{2\|P\|}(t-s)\right)ds$, where $M_\varphi = \int_0^\infty \varphi(s)ds$. We descriminate three cases:

1. If p=1, we get $\int_{t_0}^{t}(\mu(s)+\eta)\exp\left(-\frac{b_1}{2\|P\|}(t-s)\right)ds\leq \|\mu\|_1 + \frac{2\|P\|\eta}{b_1}$.
   Then, for all $t \geq t_0$, $\|x(t)\| \leq \sqrt{\frac{\|P\|}{b_2}}\exp\left(\frac{\|P\|M_\varphi}{b_2}\right)\|x_0\| \exp\left(-\frac{b_1}{2\|P\|}(t-t_0)\right)+ \frac{\|P\|}{b_2}\exp\left(\frac{\|P\|M_\varphi}{b_2}\right)\left(\|\mu\|_1 + \frac{2\|P\|\eta}{b_1}\right)$.
2. If $p\in(1,+\infty)$ and $q>0$, such that $\frac{1}{P}+\frac{1}{q}=1$, we have by applying Hölder inequality $\int_{t_0}^{t}(\mu(s)+\eta)\exp\left(-\frac{b_1}{2\|P\|}(t-s)\right)ds\leq \left(\frac{2\|P\|}{b_1 q}\right)^{\frac{1}{q}}\|\mu\|_p + \frac{2\|P\|\eta}{b_1}$.
   Therefore, for all $t \geq t_0$, the solution x(t) verifies the estimation $\|x(t)\| \leq \sqrt{\frac{\|P\|}{b_2}}\exp\left(\frac{\|P\|M_\varphi}{b_2}\right)\|x_0\| \exp\left(-\frac{b_1}{2\|P\|}(t-t_0)\right)+ \frac{\|P\|}{b_2}\exp\left(\frac{\|P\|M_\varphi}{b_2}\right)\left(\left(\frac{2\|P\|}{b_1 q}\right)^{\frac{1}{q}}\|\mu\|_p + \frac{2\|P\|\eta}{b_1}\right)$.
3. If $p=+\infty$. Then, we have $\int_{t_0}^{t}\mu(s)\exp\left(-\frac{b_1}{2\|P\|}(t-s)\right)ds\leq \frac{2\|P\|\eta}{b_1}\|\mu\|_\infty$.
   One can get, for all $t \geq t_0$ $\|x(t)\| \leq \sqrt{\frac{\|P\|}{b_2}}\exp\left(\frac{\|P\|M_\varphi}{b_2}\right)\|x_0\| \exp\left(-\frac{b_1}{2\|P\|}(t-t_0)\right)+ \frac{\|P\|}{b_2}\exp\left(\frac{\|P\|M_\varphi}{b_2}\right)\left(\frac{2\|P\|}{b_1}\|\mu\|_\infty + \frac{2\|P\|\eta}{b_1}\right)$.

We conclude that, the system (3.1) in closed-loop with the controller (3.3) is globally practically uniformly exponentially stable. This completes the proof. ■

As a consequence of Theorem 3.1, we have the following corollary.

Corollary 3.1. We consider the dynamical system (3.1). Assume that (H1) is fulfilled and there exists a continuous function μ : R+ → R+ with $\mu \in L^p$(R+, R+) for some $p\in[1,+\infty)$, such that $\|F(t,x)\| \leq \mu(t)$, $\forall\, t \geq 0, \forall x \in X$. Then, the system (3.1) in a closed-loop with the controller (3.3) is globally practically uniformly exponentially stable.

We can state other assumptions to obtain the global existence, uniqueness and the practical stabilizability for the evolution Eq. (3.1) under a restriction about the perturbed term bounded by the sum of Hölder continuous function and a Lipschitz function.

(H3) There exists a non-negative constant $0<\alpha<1$, such that the perturbation term F: R+$\times X \to$ R+ satisfies the following condition: $\|F(t,x)\| \leq \varphi(t)\|x\|^\alpha + \sigma(t)\|x\|, \forall\, t \geq 0, \forall x \in X$, where φ and μ are non-negative continuous functions with $\sigma \in L^1$(R+, R+) and $\varphi \in L^p$(R+, R+) for some $p\in[1,+\infty)$.

Then, one has the following theorem.

Theorem 3.2. If assumptions (H1) and (H3) are fulfilled, then the closed-loop system (3.1)-(3.3) have a unique solution, which is globally defined for all $t\geq t_0$ and this system is globally practically uniformly exponentially stable.

Proof. We break up the proof into two steps.

Step 1: Since F is locally Lipschitz continuous in x, uniformly in t, it follows from Pazy (1983) that for every initial condition the closed-loop equation possesses a unique mild solution on some interval $[t_0, t_0+\delta]$ with $\delta > 0$. Indeed, integrating Eq.(3.1), we obtain the following for $t\in[t_0, t_0+\delta]$:

$x(t)=S(t-t_0)x_0+\int_{t_0}^{t} S(t-s)\left[Bu(s)+F\left(s,x(s)\right)\right]ds.$

Since $B\in L(U,X)$, then:

$$\|x(t)\| \leq M_1\|x_0\| + M_1\left(\int_{t_0}^{t} \|B\|\|D\|\|x(s)\| + M_2\|x(s)\|^\alpha + M_3\|x(s)\|ds\right), \quad (3.10)$$

where: $M_1 = \{\|S(t-s)\|: 0 \leq t_0 \leq s \leq t \leq t+ \delta\}$, $M_2 = \sup_{t\in[t_{0,}t_0+\delta]} \|\varphi(t)\|$ and $M_3 = \sup_{t\in[t_{0,}t_0+\delta]} \|\sigma(t)\|$.

By applying Lemmas 2.1 and 2.3 to inequality (3.10), any solution of this equation is unifomly bounded: $\|x(t)\| \leq 2^{\frac{\alpha}{1-\alpha}}e^{M_1\delta(\|B\|\|D\|+M_3)}\left(M_1\|x_0\| + \left(M_1M_2\delta(1-\alpha)\right)^{\frac{1}{1-\alpha}}\right)$, on an arbitrary time interval $[t_0, t_0+\delta]$. Applying Theorem 1.4 of Pazy (1983), we have $t_0+\delta = \infty$, and so we obtain global existence.

Step 2: Define the function V:D(A)→ R + by $V(x)=\langle Px, x\rangle$. Then, the Lie derivative of V in t along the solution of the system (3.1) in a closed-loop system with the controller (3.3) leads to $V(x)=\langle P\dot{x}, x\rangle + \langle Px, \dot{x}\rangle \leq -\langle Qx, x\rangle + 2\|P\|\|F(t,x)\|\|x\|$. Using assumptions ($H_1$) and ($H_3$) we get the following estimation:

$$\dot{V}(x)\leq \left(-\frac{b_1}{\|P\|}+\frac{2\|P\|\sigma(t)}{b_2}\right)V(x)+\frac{2\|P\|\varphi(t)}{\sqrt{b_2}^{\alpha+1}}V(x)^{\frac{\alpha+1}{2}}.$$

Let $\vartheta(x) = V(x)^{\frac{1-\alpha}{2}}$, which implies that:

$$\dot{\vartheta}(x) \leq -\frac{(1-\alpha)}{2}\left(\frac{b_1}{\|P\|} - \frac{2\|P\|\sigma(t)}{b_2}\right)\vartheta(x) + \frac{\|P\|(1-\alpha)\varphi(t)}{\sqrt{b_2}^{\alpha+1}},$$

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Using Lemma 2.2 , we get:

$$\vartheta(x) \le e^{\frac{\|P\|(1-\alpha)N_\sigma}{b_2}}(\vartheta(x_0)\exp(-\frac{b_1(1-\alpha)}{2\|P\|}(t-t_0) + \frac{\|P\|(1-\alpha)}{\sqrt{b_2}^{\alpha+1}}\int_{t_0}^{t}\exp\left(-\frac{b_1(1-\alpha)}{2\|P\|}(t-s)\right)\varphi(s)ds),$$

where $N_\sigma = \int_0^\infty \sigma(s)ds$. We discriminate three cases:

1. If p=1, we get:

$$\vartheta(x) \le e^{\frac{\|P\|(1-\alpha)N_\sigma}{b_2}}(\vartheta(x_0)\exp(-\frac{b_1(1-\alpha)}{2\|P\|}(t-t_0) + \frac{\|P\|(1-\alpha)}{\sqrt{b_2}^{\alpha+1}}\|\varphi\|_1).$$

From Lemma 2.3, it follows that:

$$\|x(t)\| \le 2^{\frac{\alpha}{1-\alpha}}e^{\frac{\|P\|N_\sigma}{b_2}}\left(\sqrt{\frac{\|P\|}{b_2}}\|x_0\|\exp\left(-\frac{b_1}{2\|P\|}(t-t_0)\right) + \frac{1}{\sqrt{b_2}}\left(\frac{\|P\|(1-\alpha)}{\sqrt{b_2}^{\alpha+1}}\|\varphi\|_1\right)^{\frac{1}{1-\alpha}}\right).$$

2. If $p \in (1,+\infty)$ and $q > 0$, such that $\frac{1}{p}+\frac{1}{q}=1$, we have by applying / inequality:

$$\vartheta(x) \le e^{\frac{\|P\|(1-\alpha)N_\sigma}{b_2}}(\vartheta(x_0)\exp\left(-\frac{b_1(1-\alpha)}{2\|P\|}(t-t_0)\right) + \frac{\|P\|(1-\alpha)}{\sqrt{b_2}^{\alpha+1}}\|\varphi\|_p(\frac{2\|P\|}{q(1-\alpha)b_1})^{\frac{1}{q}}).$$

Then, using Lemma 2.3, we have:

$$\|x(t)\| \le 2^{\frac{\alpha}{1-\alpha}}e^{\frac{\|P\|N_\sigma}{b_2}}\sqrt{\frac{\|P\|}{b_2}}\|x_0\|\exp\left(-\frac{b_1}{2\|P\|}(t-t_0)\right) + \frac{2^{\frac{\alpha}{1-\alpha}}e^{\frac{\|P\|N_\sigma}{b_2}}}{\sqrt{b_2}}(\frac{\|P\|(1-\alpha)\|\varphi\|_p}{\sqrt{b_2}^{\alpha+1}})^{\frac{1}{1-\alpha}}\left(\frac{2\|P\|}{q(1-\alpha)b_1}\right)^{\frac{1}{q(1-\alpha)}}.$$

3. If p=+∞. Then, one has the following estimate:

$$\|x(t)\| \le 2^{\frac{\alpha}{1-\alpha}}e^{\frac{\|P\|N_\sigma}{b_2}}(\sqrt{\frac{\|P\|}{b_2}}\|x_0\|\exp\left(-\frac{b_1}{2\|P\|}(t-t_0)\right) + \frac{1}{\sqrt{b_2}}(\frac{2\|P\|^2}{b_1\sqrt{b_2}^{\alpha+1}}\|\varphi\|_\infty)^{\frac{1}{1-\alpha}}).$$

We deduce that, the system (3.1) in a closed-loop with the controller (3.3) is globally practically uniformly exponentially stable. This ends the proof. ■

For perturbed time-varying systems (3.1) in finite-dimensional spaces, we also have the following consequence.

Corollary 3.2. (Ellouze, 2019) Assume that $X=R^n$, $U=R^m$ and the assumptions (H1) and (H3) are satisfied, then the system (3.1) with the controller (3.3) is globally practically uniformly exponentially stable.

3.2 Feedback control of uncertain systems

Let X be a Banach space, X^* has the Radon-Nikodym property and U is a Hilbert space.

We consider the uncertain dynamical system:

$$\begin{cases}\dot{x} = Ax + Bu + G(t,x,u), \ t \ge t_0, \\ x(t_0) = x_0,\end{cases} \quad (3.11)$$

where $x \in X$ is the system state, $u \in U$ is the control input, A is the infinitesimal generator of the C_0-semigroup S(t) on a Banach space X, $B \in L(U,X)$ and $G: R+ \times X \times U \to R+$ is continuous in t and locally Lipschitz continuous in x uniformly in t on bounded intervals, that is, for every $t_1 \ge 0$ and constant $c \ge 0$, there is a constant $M(c,t_1)$, such that for all x, $y \in X$: $\|x\| \le c$, $\|y\| \le c$ and for all $t \in [0,t_1]$, $u \in U$ with $\|u\| \le c$ it holds that:

$$\|G(t,x,u) - G(t,y,u)\| \le M(c,t_1)\|x-y\|.$$

Let x(t,x_0,u) denote the state of a system (3.11) at moment $t \ge t_0$ associated with an initial condition $x_0 \in X$ at $t=t_0$ and input $u \in U$.

We suppose the following assumption relating to system (3.11).

(H4) The perturbation term $G: R+ \times X \times U \to R+$ satisfies the following condition:

$$\exists a,b > 0, \ \|G(t,x,u)\| \le a\|x\| + b\|u\| + d(t) + \varepsilon, \forall t \ge 0, \forall x \in X, \epsilon \ge 0. \quad (3.12)$$

where d is a non-negative continuous function with $d \in L^p$(R+, R+) for some $p \in [1,+\infty)$.

The following lemma proved sufficient conditions for the global existence and uniqueness of solutions of system (3.11).

Lemma 3.1. Under assumption (H4), the closed-loop system (3.3)- (3.11) have a unique solution which is globally defined for all $t \ge t_0$.

Proof. As G is locally Lipschitz continuous in x, uniformly in t, it follows from Pazy (1983) that for every initial condition the closed-loop equation possesses a unique mild solution on some interval $[t_0, t_0+\delta]$ with $\delta > 0$. Indeed, integrating (3.11), we obtain the following for $t \in [t_0, t_0+\delta]$:

$$x(t)=S(t-t_0)x_0+\int_{t_0}^{t} S(t-s)[Bu(s) + G(s,x(s),u(s))]ds.$$

Since $B \in L(U,X)$, then by applying Gronwall inequality (Teschl, 2012, Lemma 2.7, p42), we have the following:

$$\|x(t)\| \le M_1(\|x_0\| + M_2\,\delta + \varepsilon)e^{M_1\delta(\|B\|\,\|D\|+a+b\|D\|)},$$

where: $M_1 = \sup\{\|S(t-s)\|: 0 \le t_0 \le s \le t \le t+\delta\}$ and $M_2 = \sup_{t\in[t_0,t_0+\delta]}\|d(t)\|$ on an arbitrary time interval $[t_0, t_0+\delta]$. Now, Pazy (1983, Theorem 1.4) gives that and so we have global existence. The proof is completed. ■

The next theorem shows the practical stabilization of the system (3.11) using the Lyapunov indirect method and Gronwall-Bellman inequality.

Theorem 3.3. Assume that A is exponentially stable and the assumption (H4) is satisfied. Let P, $Q \in LPD(X,X^*)$ be the operators satisfying the Lyapunov Eq. (3.6) where $P=P^*$ and $\langle Qx,x\rangle \ge \lambda\|x\|^2$ for all $x \in X$, $\lambda > 0$.Then, the nonlinear control system is practically stabilizable by the feedback control $u(t)=-\rho B^*Px(t)$ if:

$$\rho < \frac{\lambda - 2a\|P\|}{2b\|B\|\|P\|^2}. \quad (3.13)$$

Proof. Let P∈ $LPD(X, X^*)$ be an operator which is a solution of the Lyapunov Eq. (3.6). Define the Lyapunov function V:D(A)→ $R+$ by V(x)=$\langle Px, x\rangle$. Noting that, there exists $\alpha > 0$ such that:

$$\alpha\|x\|^2 \leq V(x) \leq \|P\|\|x\|^2.$$

Then, the derivative of V in t along the trajectories of system (3.11) and using the chosen feedback control and the Lyapunov equation is given as follows

$$\dot{V}(x) = \langle P\dot{x}, x\rangle + \langle Px, \dot{x}\rangle = -\langle Qx, x\rangle - \rho\langle PBB^*Px, x\rangle - \rho\langle Px, BB^*Px\rangle + \langle PG(t, x, u), x\rangle + \langle Px, G(t, x, u)\rangle.$$

Since P is self-adjoint, by assumption (H4) and condition (3.13), we have for all t≥ t_0:

$$\dot{V}(x) \leq -kV(x) + \frac{2\|P\|}{\sqrt{\alpha}}(d(t) + \epsilon)\sqrt{V(x)},$$

where $k = \frac{\lambda - 2b\rho\|B\|\|P\|^2 - 2a\|P\|}{\|P\|} > 0$.

Let $\vartheta(x) = \sqrt{V(x)}$. Then, $\dot{\vartheta}(x) \leq -\frac{k}{2}\vartheta(x) + \frac{\|P\|}{\sqrt{\alpha}}(d(t) + \varepsilon), \forall\, x \in X, \forall\, t \geq t_0$. Applying Lemma 2.2, we obtain the following:

$$\vartheta(x) \leq \vartheta(x_0)e^{-\frac{k}{2}(t-t_0)} + \frac{\|P\|}{\sqrt{\alpha}}\int_{t_0}^{t} \exp\left(\frac{k}{2}(s-t)\right)(d(s) + \varepsilon)\, ds, \quad \forall\, t \geq t_0.$$

We distinguish three cases:

1. If p=1, we get:

$$\|x(t)\| \leq \sqrt{\frac{\|P\|}{\alpha}}\|x_0\|e^{-\frac{k}{2}(t-t_0)} + \frac{\|P\|}{\alpha}\left(\|d\|_1 + \frac{2\varepsilon}{k}\right), \forall\, t \geq t_0$$

2. If p∈ $(1, +\infty)$ and q> 0, such that $\frac{1}{p} + \frac{1}{q} = 1$, we have by applying Hölder inequality:

$$\|x(t)\| \leq \sqrt{\frac{\|P\|}{\alpha}\|x_0\|e^{-\frac{k}{2}(t-t_0)} + \frac{\|P\|}{\alpha}\left(\left(\frac{2}{qk}\right)^{\frac{1}{q}}\|d\|_p + \frac{2\varepsilon}{k}\right)}, \forall\, t \geq t_0.$$

3. 3. If p=+∞. Then, we obtain the following:

$$\|x(t)\| \leq \sqrt{\frac{\|P\|}{\alpha}\|x_0\|e^{-\frac{k}{2}(t-t_0)} + \frac{2\|P\|}{\alpha k}\left(\|d\|_\infty + \frac{2\varepsilon}{k}\right)}, \forall\, t \geq t_0$$

We deduce that, the system (3.11) is practically stabilizable. This ends the proof. ■

In the following, we derive some sufficient conditions that guarantee practical stabilizability of system (3.11) in the case A is not exponentially stable and it is a generator of bounded C_0-semigroup, but the associated linear control system (3.2) is exactly null-controllable in finite time and the nonlinear perturbation satisfies a condition.

Theorem 3.4. Assume that the linear control system (3.2) is exactly null-controllable in finite time, then the system (3.11) is practically stabilizable for some appropriate numbers a, b satisfying the condition (3.12).

Proof. The linear control system is exactly null-controllable in finite time, then from Proposition 3.4 there is an operator D∈ $L(X, U)$, such the operator W_L=A+BD is exponentially stable. Let $P, Q \in LPD(X, X^*)$ be the operators satisfying the Lyapunov Eq. (3.6) where P=$P^*$and $\langle Qx, x\rangle \geq \lambda\|x\|^2$for all x∈ X and $\lambda > 0$. Consider the Lyaunov function V(x)=$\langle Px, x\rangle$. We have:

$$\alpha\|x\|^2 \leq V(x) \leq \|P\|\|x\|^2, \alpha > 0.$$

The Lie derivative of V along the trajectories of system (3.11) is given as follows:

$$\dot{V}(x) \leq -\lambda\|x\|^2 + 2\langle PG(t, x, Dx), x\rangle \leq -\eta\|x\|^2 + 2\|P\|(d(t) + \varepsilon),$$

where $\eta = \lambda - 2(a\|P\| + b\|D\|)$. We take a, b> 0, such that $\eta > 0$, that is, $a\|P\| + b\|D\| < \frac{\lambda}{2}$.

Let $\vartheta(x) = \sqrt{V(x)}$. Then:

$$\dot{\vartheta}(x) \leq -\frac{\eta}{2\|P\|}\vartheta(x) + \frac{\|P\|}{\sqrt{\alpha}}(d(t) + \varepsilon), \forall\, x \in X, \forall\, t \geq t_0.$$

Using Lemma 2.2, we have:

$$\vartheta(x) \leq \vartheta(x_0)e^{-\frac{\eta}{2\|P\|}(t-t_0)} + \frac{\|P\|}{\sqrt{\alpha}}\int_{t_0}^{t} \exp\left(\frac{\eta}{2\|P\|}(s-t)\right)(d(s) + \varepsilon)\, ds, \forall\, t \geq t_0.$$

We distinguish three cases:

1. If p=1, we have for all t≥ t_0:

$$\|x(t)\| \leq \sqrt{\frac{\|P\|}{\alpha}\|x_0\|\, e^{--\frac{\eta}{2\|P\|}(t-t_0)} + \frac{\|P\|}{\alpha}\left(\|d\|_1 + \frac{2\|P\|\varepsilon}{\eta}\right)}, \forall\, t \geq t_0.$$

2. If p∈ $(1, +\infty)$ and q> 0, such that $\frac{1}{p} + \frac{1}{q} = 1$, we obtain by applying Hölder inequality:

$$\|x(t)\| \leq \sqrt{\frac{\|P\|}{\alpha}\|x_0\|\, e^{-\frac{\eta}{2\|P\|}(t-t_0)} + \frac{\|P\|}{\alpha}\left(\left(\frac{2\|P\|}{q\eta}\right)^{\frac{1}{q}}\|d\|_p + \frac{2\|P\|\varepsilon}{\eta}\right)}, \forall\, t \geq t_0.$$

3. If p=+∞. Then, we have:

$$\|x(t)\| \leq \sqrt{\frac{\|P\|}{\alpha}\|x_0\|\, e^{-\frac{\eta}{2\|P\|}(t-t_0)} + \frac{2\|P\|^2}{\alpha\eta}(\|d\|_\infty + \varepsilon)}, \forall\, t \geq t_0.$$

We deduce that, the system (3.11) is practically stabilizable. This finishes the proof. ■

Remark 3.2. The above results generalise theorems of stabilizability in Phat and Kiet (2002) with d(t)=ε =0.

4. EXAMPLES

In this section, we give some examples to illustrate the effectiveness of the results obtained in the present paper.

Example 4.1. We consider the controlled metal bar:

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$$\begin{cases} \frac{\partial x(\zeta,t)}{\partial t} = \frac{\partial^2 x(\zeta,t)}{\partial^2 \zeta} + \mathbf{1}_{[\frac{1}{2},1]} u(t) + \frac{1}{1+t^2} x(\zeta,t) + \frac{1+t}{(1+t^2)(1+\|x(\zeta,t)\|)}, \\ \frac{\partial x}{\partial \zeta}(0,t) = 0 = \frac{\partial x}{\partial \zeta}(1,t),\ x(\zeta,0) = x_0(\zeta),\ t \geq 0, \end{cases} \quad (4.1.)$$

where $x(\zeta,t)$ represents the temperature at position ζ at time t and $x_0(\zeta)$ represents the initial temperature profile and u(t) represents the addition of heat along the bar. The two boundary conditions state that there is no heat flow at the boundary, and thus the bar is insulated. Let X=$L^2(0,1)$ and U=C. Equation (4.1) can be rewritten as (3.1), where A=$\frac{\partial^2 x(\zeta,t)}{\partial^2 \zeta}$ with D(A)={h∈ $L^2(0,1)$, h, $\frac{\partial h}{\partial \zeta}$ is absolutely continuous, $\frac{\partial^2 h}{\partial^2 \zeta} \in L^2(0,1)$ and $\frac{\partial h}{\partial \zeta}(0) = 0 = \frac{\partial h}{\partial \zeta}(1)$}, B=$\mathbf{1}_{[\frac{1}{2},1]}$ and F(t,x)= $\frac{1}{1+t^2} x(\zeta,t) + \frac{1+t}{(1+t^2)(1+\|x(\zeta,t)\|)}$.

A possesses an orthonormal basis of eigenvector $\varphi_0(\zeta) = 1$ and $\varphi_n(\zeta) = \sqrt{2}$ cos (n$\pi\zeta$), $n \geq 1$. Furthermore, the semigroup $(S(t))_{t \geq t_0}$ generated by A is given by:

S(t)x= $\sum_{n=0}^{\infty} e^{-n^2\pi^2 t} \langle x, \varphi_n \rangle \varphi_n$.

Using Proposition 3.1, it is easy to see that the nominal system of (4.1) is exactly null-controllable in finite time. Moreover, the assumption (H2) is satisfied with η=0 and $\varphi(t) = \frac{1}{1+t^2}$ and $\mu(t) = \frac{1+t}{1+t^2}$ are non-negative functions with $\varphi \in L^1(R+, R+)$ and $\mu \in L^p(R+, R+)$ for some p∈ $[1, +\infty)$. Then, all hypotheses of Theorem 3.1 are satisfied and the controlled heat Eq. (4.1) is practically stabilizable.

Example 4.2. We consider the controlled perturbed heat equation:

$$\begin{cases} \frac{\partial x(\zeta,t)}{\partial t} = \frac{\partial^2 x(\zeta,t)}{\partial^2 \zeta} + \frac{2+t^2}{1+t^2} u(t) + x(\zeta,t) + e^{-t}\mathbf{1}_{[0,\frac{\pi}{2}]} \\ x(0,t) = 0 = x(\pi,t),\ x(\zeta,0) = x_0(\zeta),\ t \geq 0, \end{cases} \quad (4.2)$$

where $x(\zeta,t)$ represents the temperature at position $\zeta \in [0,\pi]$ time t and $x_0(\zeta)$ represents the initial temperature profile.
Let X=$L^2(0,\pi)$ and U=C. It is useful to formulate the equation (4.2) as an abstract differential equation of the form (3.11), where A=$\frac{\partial^2 x(\zeta,t)}{\partial^2 \zeta}$ with D(A) ={ $h \in L^2(0,\pi)$, $\frac{\partial h}{\partial \zeta}$ is absolutely continuous $\frac{\partial^2 h}{\partial^2 \zeta} \in L^2(0,\pi)$ and $h(0) = 0 = h(\pi)$}, B=I and G(t, $x(\zeta,t)$u)=$x(\zeta,t) + \frac{1}{1+t^2}$u(t)+ $e^{-t}\mathbf{1}_{[0,\frac{\pi}{2}]}$.

A possesses an orthonormal basis of eigenvector $\varphi_n(\zeta) = \sqrt{\frac{2}{\pi}}$sin(nζ), n≥ 0. Furthermore, the semigroup $(S(t))_{t \geq t_0}$ generated by A is given by:

S(t)=$\sum_{n=1}^{\infty} e^{-n^2 t} \langle x, \varphi_n \rangle \varphi_n$.

Obviously, S(t) is exponentially stable. Therefore, A is exponentially stable. Moreover, G satisfies the assumption (H4), just take a=1, b=1, $\varepsilon = 0$ and d (t) =$\frac{\pi}{2} e^{-t}$, is a non-negative continuous function, with d∈ $L^p(R+, R+)$ for some p∈ $[1, +\infty)$. Consequently, by applying Theorem 3.3, the controlled heat Eq. (4.2) is practically stabilizable.

5. CONCLUSION

Practical stabilization of infinite-dimensional evolution equations in Banach spaces has been investigated. Moreover, sufficient conditions have been derived to guarantee the practical stabilization of a class of uncertain systems in Banach spaces. Illustrative examples are given to indicate significant improvements and the application of the results.

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