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Models of *n*-th order linear time – varying systems

ANNA PIWOWAR, JANUSZ WALCZAK

Silesian University of Technology, Gliwice Poland
e-mail: anna.piwowar@polsl.pl, Janusz.walczak@polsl.pl

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Abstract. Effective algorithms of constructing models of generalized parametric sections of the *n*-th order consisting of cascade and parallel elementary LTV section connections are presented in this paper. Moreover, the methods of impulse responses determination have been shown.

Key words: LTV, parametric section, LTV connection, parallel connection, cascade connection

1. Introduction

In the synthesis of LTI (linear time invariant) systems, techniques consisting in the parallel or cascade connection of universal sections are often used. The section connections are used in designing higher order complex filters [3]. In practical applications, the first and second order filters are more often used. The rules of transforming systems composed of elementary SISO (single input single output) blocks of LTI class are well known in the control theory [3, 5]. In the continuous-time or discrete-time domain, the algebra of these transformations is based on operations (±, *) with impulse responses of elementary sections as kernels of these operations. In the domain of L-transform or F-transform, algebra is based on the classical definition of operations (±, ·) with arguments being the transfer functions or spectral functions of elementary blocks [5].

In case of LTV (linear time varying) sections with the coefficients variable in time (i.e. parametric sections), the classical definition of operational transfer functions has no sense [2], [6]. The method of determining differential equations, known from the literature [3], describing systems composed of the connection of LTI sections cannot be used for LTV systems [4].

For parametric systems, the relation between input $x(t)$ and output $y(t)$ signal [4, 9] is given by equation:

$$y(t) = \int_0^t h(t,\tau)x(\tau)d\tau, \tag{1}$$

where: $h(t, \tau)$ – impulse response of the system.

The impulse responses of the first and second order LTV systems have been determined in work [6].

In this article, two problems have been considered. The first of them concerns constructing the mathematical model of cascade and parallel connection of the first and second order elementary section in the time domain. This model is a linear differential equation with time-varying coefficients. The decomposition algorithm of the linear differential equation with variable coefficients into a linear system of the first order equations is known [4].

To the best knowledge of the authors, the reverse procedure, i.e. constructing the differential equation of n-th order based of known equations of the first and second order section has not been described in the literature and due to this fact, it is the original element of this work.

The second of the considered problems concerns the method of constructing the impulse response of parallel and cascade connections of first and second order parametric sections. Although this algorithm is known [4], it can be used only if the impulse responses of elementary sections are given in a closed form. The application of the described algorithm to determine the specific impulse response of complex LTV systems with exponentially variable coefficients is also a part of the original achievements.

It can be noticed that the synthesis of parametric systems in the form of cascade and parallel connections of first and second order elementary sections, similarly to the case of LTI systems, has several advantages:

- stability (in assumed sense) of the cascade and parallel connection if the elementary sections included in the system are stable,
- lack of parameter interaction of the sections composed of the substitute system,
- small, but dependent on the class of parametric function, sensitivity to the variability in value of parameters compared to the sensitivity of the LTV structure originally described by the n-th order equation.

The above advantages imply the usefulness of described in the article methods to create the alternative models of complex LTV systems composed of elementary sections in the LTV system synthesis. Such a synthesis was not the purpose of this work and thus it is not considered.

2. Generalized SISO parametric systems

The generalized n-th order parametric system (Fig. 1) is composed of the connection of elementary LTV first and second order sections. In this article two types of substitute systems have been considered. The former, model I, is composed of the parallel connection of elementary sections. The latter, model II, of the cascade connection of elementary sections.

2.1. Models of elementary LTV first and second order sections

The analysis of any parametric section of n-th order can be reduced to the analysis of the low-pass LTV section of n-th order [6]. Elementary LTV sections are described by differential equations with coefficients variable in time.

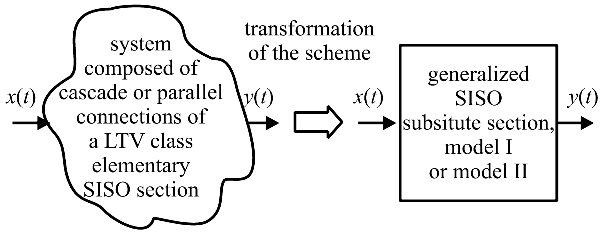

Fig. 1. Transformation of a complex LTV system into an equivalent system

The first order sections are described by a parametric differential equation of the first order (2):

$$y'(t) + \omega(t)y(t) = x(t), \tag{2}$$

whereas the second order LTV sections are described by a parametric differential equation of the second order:

$$y''(t) + 2\omega(t)\sigma(t)y'(t) + \omega^2(t) = x(t), \tag{3}$$

where: $\omega(t)$, $\sigma(t)$ – parametric function of $C^{(n-1)}[0,\infty)$ class. In this paper the exponentially variable parametric functions described by equations:

$$\omega(t) = \omega_g + C\mathrm{e}^{-\gamma t}, \quad \omega_g, \gamma \in \mathrm{R}^+, C \in \mathrm{R}, \tag{4}$$

$$\sigma(t) = \sigma_g + D\,\mathrm{e}^{-\gamma t}, \quad \sigma_g, \gamma \in \mathrm{R}^+, D \in \mathrm{R} \tag{5}$$

have been considered.

These functions can be interpreted as a cut-off (or resonance) angular frequency $\omega(t)$ variable in time or the attenuation ratio $\sigma(t)$ of low-pass filter variable in time. The parametric functions are strictly positive due to the assumed variability of the coefficients. It is significant for the stability of parametric systems [1, 9]. It ought to be noticed that for considered parametric function classes (4), (5) the solutions to the Equations (2), (3) exist in a closed form [6].

2.2. Model of generalized section type I

The first of the considered models of a substitute parametric section is a generalized section type I (Fig. 2) composed of the parallel connection of the first and second order elementary sections.

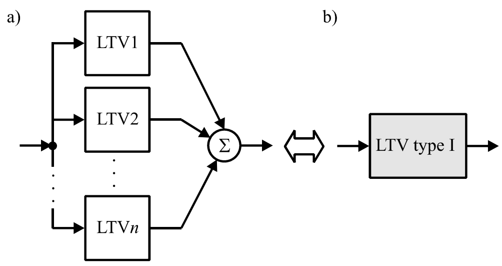

Fig. 2. Parallel connection of LTV sections: a) connection scheme, b) generalized LTV section type I

2.3. Model of generalized section type II

The cascade connection of elementary LTV sections, shown in Figure 3, is equivalent to the generalized parametric section type II.

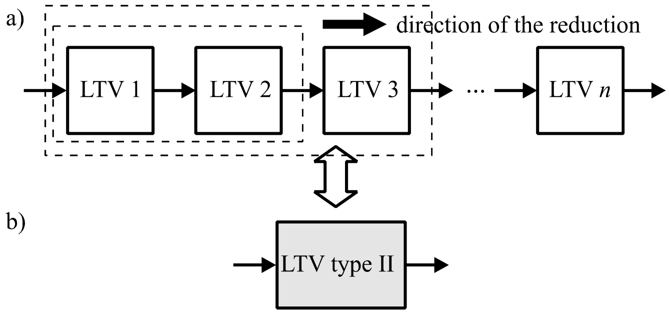

Fig. 3. Cascade connection of LTV sections: a) connection scheme, b) generalized section of type II

The order of the differential equation describing the substitute section type II is equal to the sum of orders of all sections included in the connection. On the contrary to cascade connections of LTI sections, the cascade connection of LTV sections is not commutative [7].

3. Method of construction of type I generalized section

The construction of generalized parametric section of I and II type consists in the determination of differential equations describing the systems and their impulse responses.

3.1. Model of section type I

The scheme of N-order system composed of the parallel connection of n first order LTV sections and m sections of the second order has been presented in Figure 5. The thorough analysis of the first order section connections has been carried out in work [8]. The method of constructing the differential equation describing the parallel connection of the first order LTV section presented there can be extended to connections of the second order sections. The generalization is a consequence of the fact that any second order section can be conventionally presented as a parallel connection of two first order sections, described by the components of the formula (6) (see Fig. 4)

$$y(t) = y^{(1)}(t) + y^{(2)}(t) = G(t)I_1(t) + H(t)I_2(t), \tag{6}$$

where: $G(t)$, $H(t)$ – the expression containing the composition of exponential functions, Bessel functions and hypergeometric functions [6], $I_1(t)$, $I_2(t)$ – the integrals of product exponential functions, Bessel functions, hypergeometric functions and the input signal of the section $x(t)$

[6].

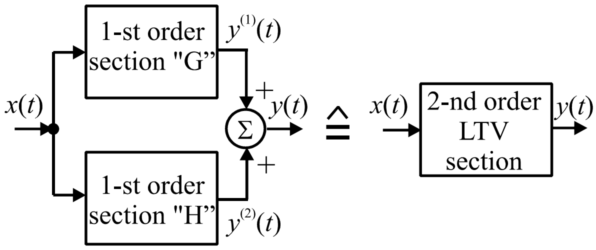

Fig. 4. Decomposition of the second order system

As a result, the parallel connection of n first order sections and m second order sections can be substituted by the connection of $n + 2m$ first order sections, described by the formula (7):

$$y(t) = \sum_{i=1}^{n} F_i(t) I_i(t) + \sum_{j=1}^{m} G_j(t) I_{j1}(t) + \sum_{j=1}^{m} H_j(t) I_{j2}(t). \tag{7}$$

The first component of the formula (7) describes the output signal of the n first order section parallel connection, whereas the remaining constituents determine the output signal of the parallel connection of m second order section (Fig. 4).

The algorithm for determining the differential equation of the first and second order LTV section parallel connection, modified from the previously presented (see work [7]), is composed of the following phases:

1) Firstly, the section included in the connection has to be classified into two groups (section of the first and second order) and ordered according to the formula (7).
2) The $(n + 2m - 1)$ times differentiation has to be done. As a result of the differentiation, a system of $(n + 2m)$ linear equations is obtained. The system includes the $n + 2m - 1$ expression resulting from the differentiation of the Equation (7) and the Equation (7). Functions (integrals) $I_i(t)$, $i = 1, 2, .., n$, $I_{j1}(t)$, $j = 1, 2, \ldots m$, $I_{j2}(t)$, $j = 1, 2, \ldots m$ are unknown in this system and should be determined.
3) After the last N-th (where: $N = n + 2m$) differentiation of Equation (7), one can obtain the differential equation of N-th order.
4) The coefficients of this order depend on the functions $I_i(t)$, $I_{j1}(t)$, $I_{j2}(t)$ determined in the II phase of this algorithm. After substituting those functions in the differential equation obtained during the III phase of the algorithm, the final differential equation of substitute N-th order system can be obtained.

The algorithm described above allows to obtain differential equations of parallel connections composed of the first and second LTV sections in a closed form. The effective usage of the algorithm for $N \gg 2$ requires the implementation in a symbolic language (i.e. Mathematica).

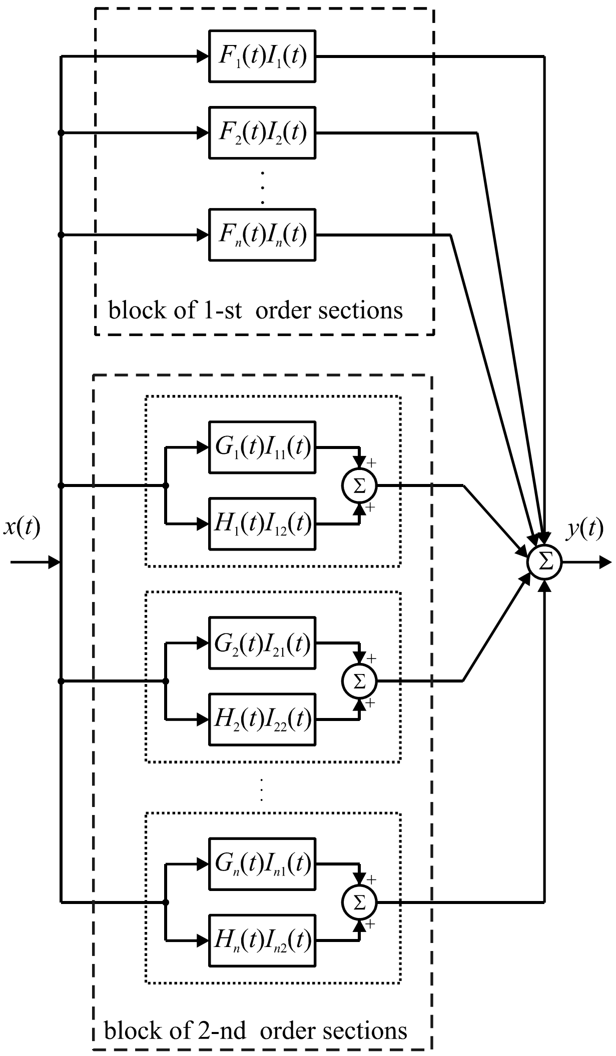

Fig. 5. *N*-th order LTV system

Example 1. The simplified illustration of the described algorithm is presented by the example below. The considered system (Fig. 6) is a connection of two parallel first order sections. These sections are described by the equations:

$$\text{LTV1: } y'(t) + \omega_1(t)y(t) = x(t), \tag{8}$$

$$\text{LTV2: } y'(t) + \omega_2(t)y(t) = x(t). \tag{9}$$

The variability of the parametric function is described by Equation (4) and presented in the Figure 6b.

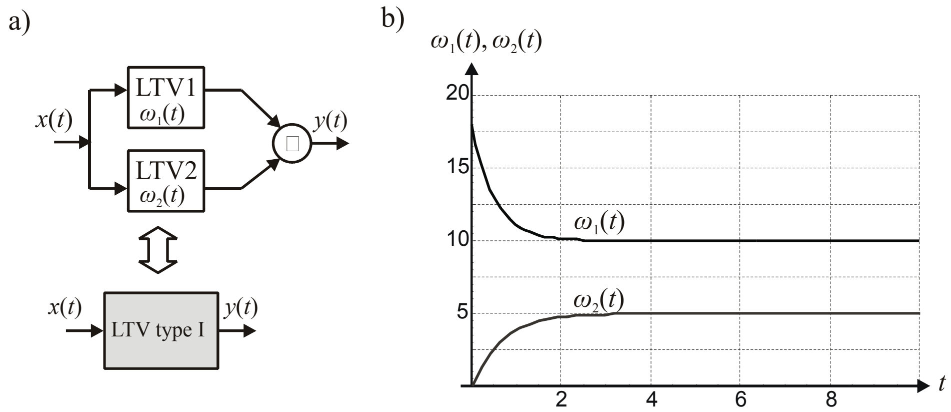

Fig. 6. Parallel connection of two first order LTV sections: a) connection scheme, b) waveforms of parametric functions

The output signal of the substitute system is a sum of output signals of elementary sections. Resulting from the first and second step of the algorithm, the system of the equations is obtained:

$$\begin{bmatrix} F_1(t) & \pm F_2(t) \\ F'_1(t) & \pm F'_2(t) \end{bmatrix} \cdot \begin{bmatrix} I_1(t) \\ I_2(t) \end{bmatrix} = \begin{bmatrix} y(t) \\ y'(t) - F_1(t)I'_1(t) \mp F_2(t)I'_2(t) \end{bmatrix}. \tag{10}$$

The solution to this system is expressed by equations:

$$I_1(t) = \frac{\mp F_2(t)\left[y'(t) - F_1(t)I'_1(t) \mp F_2(t)I'_2(t)\right] \pm F'_2(t)y(t)}{\pm F_1(t)F'_2(t) \mp F_2(t)F'_1(t)}, \tag{11}$$

$$I_2(t) = \frac{F_1(t)\left[y'(t) - F_1(t)I'_1(t) \mp F_2(t)I'_2(t)\right] - F'_1(t)y(t)}{\pm F_1(t)F'_2(t) \mp F_2(t)F'_1(t)}, \tag{12}$$

thus:

$$F'_i(t) = -F_i(t)\omega_i(t), \tag{13}$$

$$I'_i(t) = G_i(t)x(t), \tag{14}$$

$$G_i(t) = \mathrm{e}^{\alpha_i(t)}, \tag{15}$$

$$F''_i(t) = F_i(t)\left(\omega_i^2(t) + \omega'_i(t)\right), \tag{16}$$

$$I''_i(t) = G_i(t)\left[\omega_i(t)x(t) + x'(t)\right], \tag{17}$$

$$F_i(t)G_i(t) = \mathrm{e}^{-\alpha_i(t)}\mathrm{e}^{\alpha_i(t)} = 1. \tag{18}$$

Two-times differentiation of Equation (7) (n = 2, m = 0) and substitution of the equations (11)-(18) (using the algorithm successively) results in the differential equation which corresponds to the analyzed connection from Figure 6:

$$y''(t) + A_1(t)y'(t) + A_0(t)y(t) = 2x'(t) + B_0(t)x(t), \tag{19}$$

where:

$$A_1(t) = \frac{\omega_1^2(t) - \omega_2^2(t) + \omega'_2(t) - \omega'_1(t)}{\omega_1(t) - \omega_2(t)}, \tag{20}$$

$$A_0(t) = \frac{\left(\omega_1^2(t) - \omega'_1(t)\right)\omega_2(t) - \left(\omega_2^2(t) - \omega'_2(t)\right)\omega_1(t)}{\omega_1(t) - \omega_2(t)}, \tag{21}$$

$$B_0(t) = \frac{\omega_1^2(t) - \omega_2^2(t) - 2\left(\omega_1'(t)\right)\omega_2'(t)\right)}{\omega_1(t) - \omega_2(t)}. \tag{22}$$

The time-varying coefficients of substitute systems, computed based on Equations (20)-(22), are shown in Figure 7.

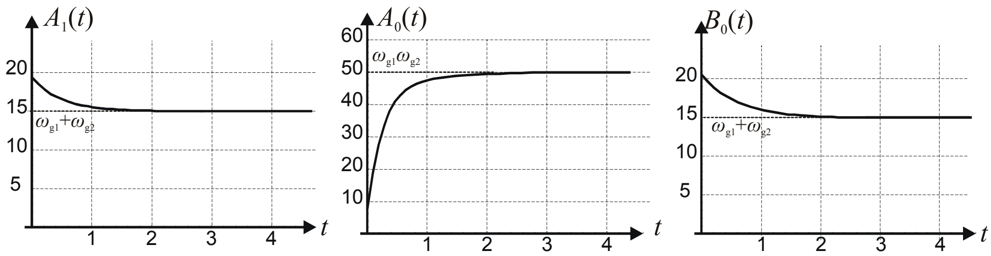

Fig. 7. Time-varying coefficients of substitute systems of a parallel connection

The substitute parametric section which is the effect of the connection of two first order sections is described by the differential equations with the time-varying coefficients. The waveforms of parametric functions of substitute system depend on the variability of elementary systems parameters.

In Figure 7 one can clearly observe that if time-varying coefficients are positive-definite functions, substitute systems are stable. Once the variable parameters reach set values, the substitute system becomes equal to the system of parallelly connected low-pass stationary sections with cut-off angular frequencies ω_{g1} and ω_{g2}.

3.2. The impulse response of section type I

The output signal of system in Figure 2 can be expressed as:

$$y(t) = \sum_{i=1}^{n} \int_0^t h_i(t,\tau)x(\tau)\,\mathrm{d}\,\tau. \tag{23}$$

Assuming that each section in the parallel connection is described by a known [6] impulse response $h_i(t, \tau)$, $i = 1, 2, \ldots, n$, based on the comparison of Equations (1) and (23), the following expression can be obtained:

$$h_z(t,\tau) = \sum_{i=1}^{n} h_i(t,\tau). \tag{24}$$

The substitute sections impulse response is the sum of the impulse responses of elementary sections.

Due to the fact that impulse responses of the second order elementary sections are very complex [6], the idea presented in this subchapter has not been supported with an example.

Example 2. The considered system (Fig. 6a) is a connection of two first order LTV sections. The parametric functions of elementary sections have been presented in fig. 6b. Impulse responses of elementary sections are known [6] and have been shown in Figure 8. The parametric function of a substitute system, computed based on the dependency (42) has been presented in Figure 9.

a)

b)

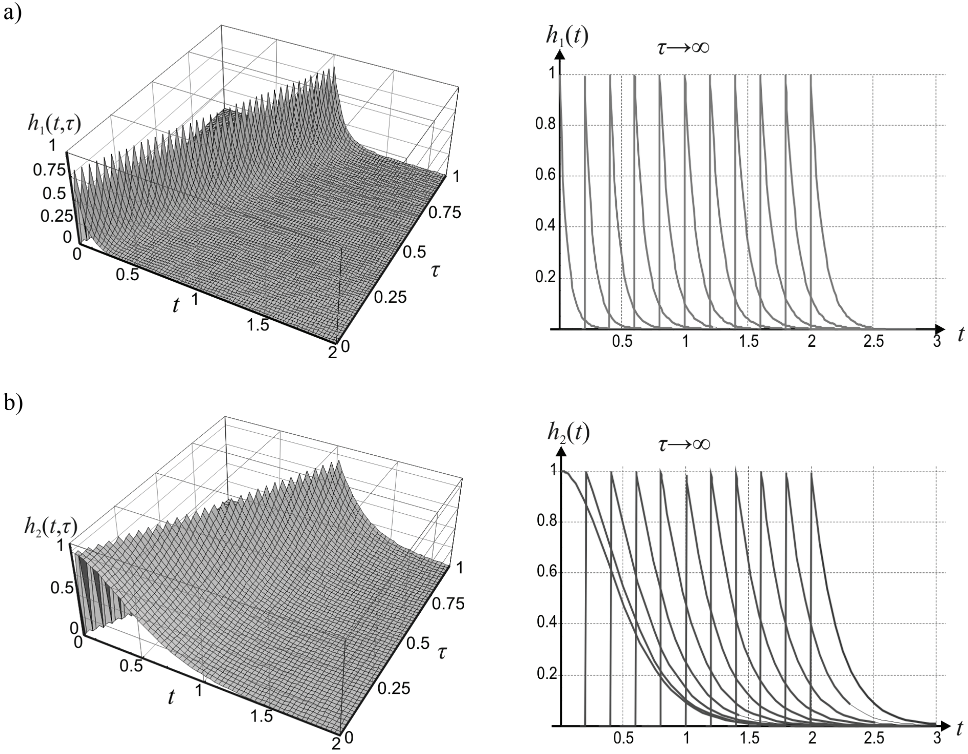

Fig. 8. Impulse responses of elementary sections: a) $h_1(t,\ \tau)$, b) $h_2(t,\ \tau)$

The impulse response of substitute section depends on both the time t and the moment of the application of excitation τ into the system input. It can be seen that the larger value of coefficients γ_1, γ_2 (responsible for the speed to limited value of parametric functions of elementary sections), the lower is the correlation between the response and the moment τ.

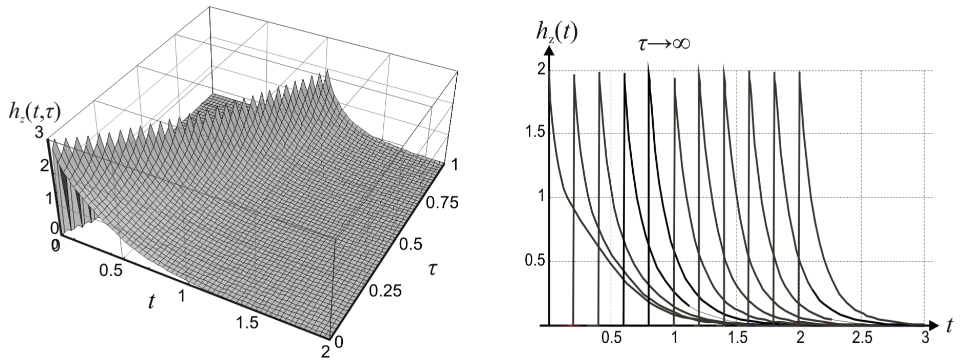

Fig. 9. The impulse response $h_z(t,\ \tau)$ of a substitute system

The mathematical formula of impulse response of generalized I type section composed of first and second order elementary sections is the differential equation containing the expressions consisting of the composition of exponential functions, Bessel and hypergeometric functions as well as the integrals of the products of exponential functions, Bessel functions, hypergeometric functions and an input signal [5].

4. Construction method of type II generalized section

4.2. Model construction of section type II

The connection shown in Figure 3 is not commutative [6]. Therefore, the number of the cascade connection structures of *n* sections is equal to the number of permutations of *n* elements (elementary sections). The reduction of the cascade connection can be obtained by the application of the successive method, by repeated blocking of two adjacent elementary sections [7]. Independently of the section cascade connection point and the direction of the reduction, the final reduction result, the substitute section (Fig. 3), must be described by an identical differential equation. In the article, the issues of the reduction of section connections have been considered for two cases:

- the connection of the first order LTV section with *n*-th order LTV section,
- the connection of the second order LTV section with *n*-th order LTV section.

Reduction of connection of the first order LTV section with n-th order LTV section.

The considered system shown in Figure 10 is expressed by the equations:

$$y'(t) + b(t)y(t) = x(t), \tag{25}$$

$$z^{(n)}(t) + a_{n+1}(t)z^{(n-1)}(t) + a_{n-2}(t)z^{(n-2)}(t) + \ldots + a_1 z'(t) + a_0(t)z(t) = y(t). \tag{26}$$

After the differentiation of the Equation (25) and the substitution of the Equations (25) and (26) to it, the equation of substitute section ($n + 1$)-th order is determined.

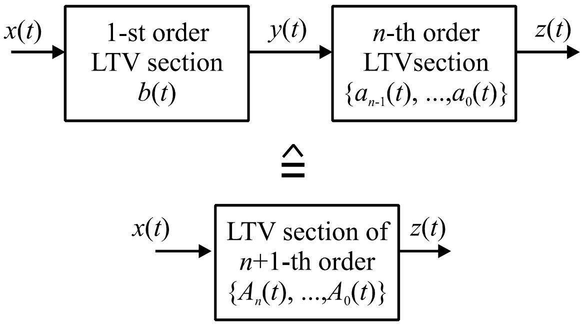

Fig. 10. Reduction of the connection cascade connections of the first order with *n*-th order LTV sections

$$\begin{gathered} z^{(n+1)}(t)+A_n(t)z^{(n)}(t)+A_{n-1}(t)z^{(n-1)}(t)+A_{n-2}(t)z^{(n-2)}(t)+\ldots \\ +A_1z'(t)+A_0(t)z(t)=x(t), \end{gathered} \tag{27}$$

where:

$$\begin{aligned} &A_n(t)=a_{n-1}(t)+b(t), \\ &A_{n-1}(t)=a'_{n-1}(t)+a_{n-2}(t)+a_{n-1}(t)b(t), \\ &A_{n-2}(t)=a'_{n-2}(t)+a_{n-3}(t)+a_{n-2}(t)b(t), \\ &\qquad\qquad\ldots \\ &A_1(t)=a'_1(t)+a_0(t)+a_1(t)b(t), \\ &A_0(t)=a'_0(t)+a_0(t)b(t). \end{aligned} \tag{28}$$

The observation of the way indicators in the formula (28) compose result in the basic rules of forming the varied coefficients (parametric functions) $A_n(t)$, $A_{n-1}(t)$, …, $A_1(t)$, $A_0(t)$ of the differential equation of substitute system.

Reduction of the connection of the second order LTV section with n-th order LTV section

The considered system presented in Figure 11 is described by the following equatoions:

$$y''(t)+c(t)y'(t)+d(t)y(t)=x(t), \tag{29}$$

$$\begin{gathered} z^{(n)}(t)+a_{n+1}(t)z^{(n-1)}(t)+a_{n-2}(t)z^{(n-2)}(t)+ \\ \ldots+a_1z'(t)+a_0(t)z(t)=y(t). \end{gathered} \tag{30}$$

Once the differentiation of the Equation (30) is performed twice and the Equations (29), (30) are substituted to it, the equation of the substitute section $(n+2)$-th order is determined.

$$\begin{gathered} z^{(n+2)}(t)+A_{n+1}(t)z^{(n+1)}(t)+A_n(t)z^{(n)}(t)+ \\ +A_{n-1}(t)z^{(n-1)}(t)+\ldots+A_1z'(t)+A_0(t)z(t)=x(t), \end{gathered} \tag{31}$$

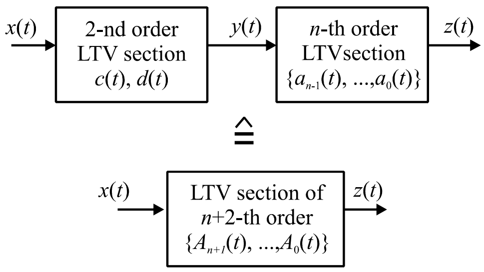

Fig. 11. Reduction of the connection cascade connections of the second order with *n*-th order LTV sections

where:

$$A_{n+1}(t) = a_{n-1}(t) + c(t),$$
$$A_n(t) = 2a'_{n-1}(t) + a_{n-2}(t) + a_{n-1}(t)c(t) + d(t),$$
$$A_{n-1}(t) = a''_{n-1}(t) + 2a'_{n-1}(t) +$$
$$+(a'_{n-1}(t) + a_{n-2}(t))c(t) + a_{n-1}(t)d(t), \quad (32)$$
$$\ldots$$
$$A_1(t) = a''_1(t) + 2a'_0(t) + (a_1'(t) + a_0)c(t) + a_1(t)d(t),$$
$$A_0(t) = a''_0(t) + a'_0(t)c(t) + a_0(t)d(t).$$

Similarly to the previous case, the observation of the way indicators in the formula (32) compose, one can generate the basic rules of forming varied coefficients (parametric functions) $A_n(t)$, $A_{n\text{-}1}(t)$, …, $A_1(t)$, $A_0(t)$ of the differential Equation (31) substitute system.

Example 3. The two variants of cascade connection have been considered. Variant A (Fig. 12a) consist of the cascade connection of first and second order sections, whereas variant B (Fig. 12b) is the cascade connection of second and first order sections.

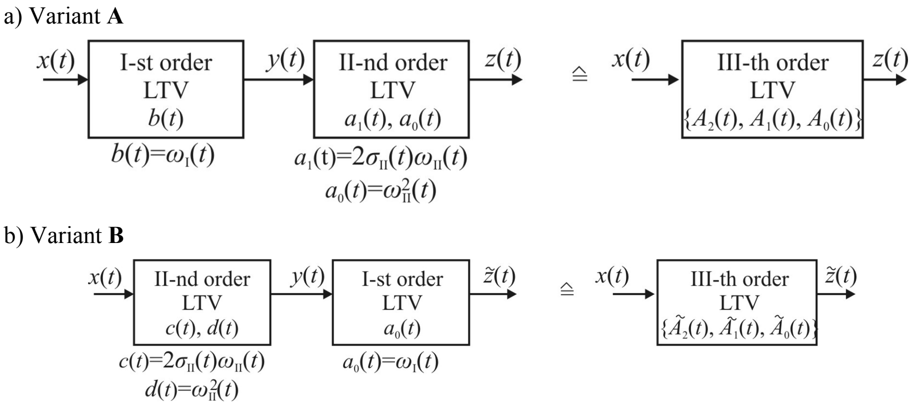

Fig. 12. The cascade connection of LTV sections a) variant **A** – connection of I-st and II-nd order sections b) variant **B** – connection of II-nd and I-st order sections

According to the presented algorithms of connection reduction, in both cases one can obtain the third order section:

- in case of variant A

The substitute LTV section is described by the equation:

$$z'''(t) + A_2(t)z''(t) + A_1(t)z'(t) + A_0(t)z(t) = x(t), \tag{33}$$

where variable coefficients are expressed by:

$$\begin{aligned} A_2(t) &= a_1(t) + b(t), \\ A_1(t) &= a'_1(t) + a_0(t) + a_1(t)b(t), \\ A_0(t) &= a'_0(t) + a_0(t)b(t), \end{aligned} \tag{34}$$

- in case of variant B

The substitute LTV section is described by the equation:

$$\tilde{z}'''(t) + \tilde{A}_2(t)\tilde{z}''(t) + \tilde{A}_1(t)\tilde{z}'(t) + \tilde{A}_0(t)\tilde{z}(t) = x(t), \tag{35}$$

where variable coefficients are expressed by:

$$\begin{aligned} \tilde{A}_2(t) &= a_0(t) + c(t), \\ \tilde{A}_1(t) &= 2a'_0(t) + a_0(t)c(t) + d(t), \\ \tilde{A}_0(t) &= a''_0(t) + a'_0(t)c(t) + a_0(t)d(t). \end{aligned} \tag{36}$$

Assuming the following parameters of elementary sections:
$\omega_I(t)$ – time varying cut-off angular frequency of first order section, Fig. 13a
$\omega_{II}(t)$ – time varying resonance frequency of second order section, Fig. 13a
$\sigma_{II}(t)$ – time varying attenuation ratio of second order section, Fig. 13b

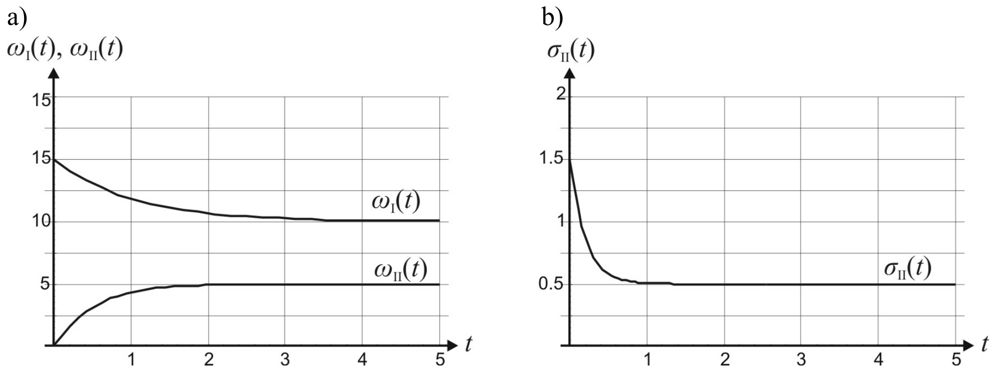

Fig. 13. The waveforms of parametric functions of elementary section of cascade connection

the coefficients of differential Equations (33) and (35) can be written as:

- for variant A (33), (34) the coefficients of elementary sections are:

$$b(t) = \omega_{\mathrm{I}}(t),$$
$$a_1(t) = 2\omega_{\mathrm{II}}(t)\sigma_{\mathrm{II}}(t), \tag{37}$$
$$a_0(t) = \omega_{\mathrm{II}}^2(t),$$

consequently, the parameters of the substitute section (34) are expressed by equations:

$$A_2(t) = 2\omega_{\mathrm{II}}(t)\sigma_{\mathrm{II}}(t) + \omega_{\mathrm{I}}(t),$$
$$A_1(t) = \frac{\mathrm{d}}{\mathrm{d}t}\big(2\omega_{\mathrm{II}}(t)\sigma_{\mathrm{II}}(t)\big) + \omega_{\mathrm{II}}^2(t) + 2\omega_{\mathrm{II}}(t)\sigma_{\mathrm{II}}(t)\omega_{\mathrm{I}}(t), \tag{38}$$
$$A_0(t) = \frac{\mathrm{d}\omega_{\mathrm{II}}^2(\mathrm{t})}{\mathrm{d}t} + \omega_{\mathrm{II}}^2(t)\omega_{\mathrm{I}}(t),$$

- for variant B (35), the coefficients of elementary sections are:

$$a_0(t) = \omega_{\mathrm{I}}(t),$$
$$c(t) = 2\omega_{\mathrm{II}}(t)\sigma_{\mathrm{II}}(t), \tag{39}$$
$$d(t) = \omega_{\mathrm{II}}^2(t),$$

consequently, equivalent section coefficients (33), (35) are determined by relations:

$$\widetilde{A}_2(t) = \omega_{\mathrm{I}}(t) + 2\omega_{\mathrm{II}}(t)\sigma_{\mathrm{II}}(t),$$
$$\widetilde{A}_1(t) = 2\frac{\mathrm{d}\omega_{\mathrm{I}}(t)}{\mathrm{d}t} + 2\omega_{\mathrm{II}}(t)\sigma_{\mathrm{II}}(t)\omega_{\mathrm{I}}(t) + \omega_{\mathrm{II}}^2(t), \tag{40}$$
$$\widetilde{A}_0(t) = \frac{\mathrm{d}^2\omega_{\mathrm{I}}(t)}{\mathrm{d}t^2} + \frac{\mathrm{d}\,\omega_{\mathrm{I}}(t)}{\mathrm{d}t}2\omega_{\mathrm{II}}(t)\sigma_{\mathrm{II}}(t) + \omega_{\mathrm{I}}(t)\omega_{\mathrm{II}}^2(t).$$

The graphic illustration of waveforms of substitute section parameters is presented in Figure 14. Solid line is used to plot the variation of parameters of the section being the result of type A connection. Dot line is used to plot the variation of parameters of the section obtained in connection type B.

The cascade connection of first and second order LTV systems results in an equivalent system described by the parametric third order differential equation. However, it should be noticed that the coefficients of this equation are various and depend on the order of connecting the elementary systems. In the stationary state, the complex system of cascade connection of LTV sections becomes equivalent to the cascade connection of stationary sections.

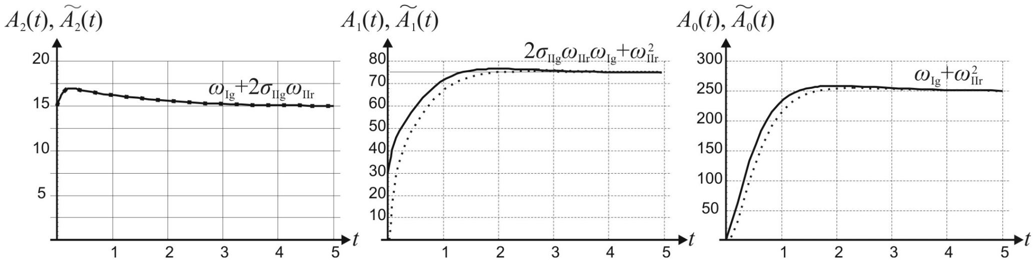

Fig. 14. Results of reducing the cascade section connection – waveforms of substitute system parameters (variant A – solid line, variant B – dot line)

4.2. The impulse response of section type II

Assuming that each section in the considered connection is described by an impulse response $h_i(t, \tau)$ [6], it can be proved that for the cascade connection of two sections numbered i, $i + 1$, [4]:

$$y(t) = \int_0^t \int_\tau^t h_{i+1}(t,\tau_1) h_i(\tau_1,\tau) x(\tau) \mathrm{d}\tau_1 \mathrm{d}\tau = \int_0^t h_z(t,\tau) x(\tau) \mathrm{d}\tau, \tag{41}$$

where:

$$h_z(t,\tau) = \int_\tau^t h_{i+1}(t,\tau_1) h_i(\tau_1,\tau) \mathrm{d}\tau_1, \tag{42}$$

is the impulse response of substitute system.

Using the formula (42) for the connection of two successive sections from Figure 3, the reduction of the cascade connection of n section can be reduced to a single substitute section.

Example 4. The systems created by the connection of two elementary low-pass first order sections (Fig. 15) have been considered. The first system, which is the parallel connection, has been considered in example 1, whereas the second type is the cascade connection of the same elementary sections.

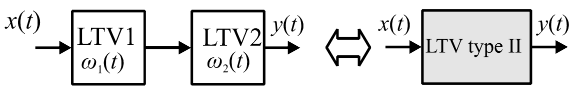

Fig. 15. Cascade of two elementary first order sections

In Figure 6b, the waveforms of parametric functions of elementary sections have been presented. The impulse responses of elementary sections are shown in Figure 8. In Figure 16 the impulse response of substitute system has been presented, computed based on the equation (42).

On the contrary to classical stationary sections, the impulse responses of parametric section are functions of two variables – time t, and excitation application moment τ to the input of the system.

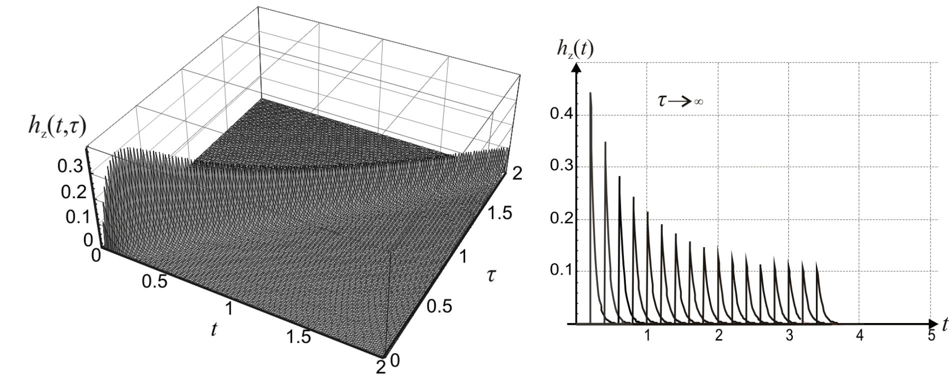

Fig. 16. Impulse response of substitute system type II

5. Conclusions

In this article, the effective algorithms for constructing the differential equations of generalized n-th order substitute, consisting of cascade and parallel connections of the first and second order LTV sections, have been described.

Cascade and parallel connections of parametric elementary sections enable the construction of higher order parametric systems. The coefficients of differential equations describing the substitute sections are complex functions of parameters of elementary sections varying in time, which considerably affects various system properties. The significant property of LTV systems is the fact that the sequence in cascade connection is crucial.

In addition, the article provides methods of calculating substitute system impulse responses, which can be used to determinate the time –frequency responses of considered LTV connection systems.

Connections of parametric sections asymptotically approach the substitute sections consisting of LTI systems. It results from the assumptions concerning the parametric functions. Variability of parameters has an impact on the shape and parameters of the impulse response. In transient states of parameter variability, the connection of non-stationary systems allows to modify their properties.

The advantages of LTV section connections include three aspects. Firstly, stability (in the specified sense) of the field-cascade and parallel connection, providing the elementary sections are stable. Secondly, lack of interaction of the parameters' sections making up the substitute system. Finally, low sensitivity, obviously depending on the class of parameterizing functions, to changes in numeric parameters, when compared to the sensitivity of the structure originally described by the LTV equation of n-th order.

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