

# FUNDAMENTAL SOLUTION TO THE CAUCHY PROBLEM FOR THE TIME-FRACTIONAL ADVECTION-DIFFUSION EQUATION

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**Abstract.** The one-dimensional time-fractional advection-diffusion equation with the Caputo time derivative is considered. The fundamental solution to the Cauchy problem is obtained using the integral transform technique. The numerical results are illustrated graphically.

## Introduction

In the last few decades, a considerable research effort has been expended to the study of fractional differential equations which have many applications in physics, geophysics, geology, chemistry, rheology, engineering and bioengineering (see, for example, [1-8] and references therein). The time-fractional diffusion-wave equation is a mathematical model of important physical phenomena ranging from amorphous, colloid, glassy and porous materials through fractals, random and disordered media to comb structures, dielectrics, semiconductors, polymers and biological systems. The time-fractional advection-diffusion equation can be interpreted in terms of diffusion or heat conduction with additional velocity field, transport processes in porous media and groundwater hydrology. In the case of the fractional advection-diffusion equation, as a rule, different numerical methods have been used to find the solution: the implicit difference method based on the shifted Grünwald approximation [9] and the explicit difference method [10], the fractional variational method [11], the finite volume method [12], etc. In [13, 14] the solution to one-dimensional time-fractional advection-diffusion equation was obtained in terms of the  $H$ -function.

In this paper, we get the fundamental solution to the Cauchy problem for the time-fractional advection-diffusion equation in terms of the Mittag-Leffler function. The Laplace transform with respect to time and the exponential Fourier transform with respect to the space coordinate are used. The results of numerical calculations are illustrated graphically.

## 1. Formulation of the problem

Consider the time-fractional advection-diffusion equation

$$\frac{\partial^\alpha T(x,t)}{\partial t^\alpha} = a \frac{\partial^2 T(x,t)}{\partial x^2} - v \frac{\partial T(x,t)}{\partial x}, \quad (1)$$

where:  $-\infty < x < \infty$ ,  $0 < t < \infty$ ,  $0 < \alpha \leq 1$ ,  $a > 0$ ,  $v > 0$ . In equation (1)  $\partial^\alpha T / \partial t^\alpha$  is the Caputo fractional derivative of the order  $\alpha$ :

$$\frac{\partial^\alpha T}{\partial t^\alpha} = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \frac{\partial^n T(\tau)}{\partial \tau^n} d\tau, \quad n-1 < \alpha < n. \quad (2)$$

Here  $\Gamma(x)$  is the gamma function.

In what follows we will study the fundamental solution to the Cauchy problem for the time-fractional advection-diffusion equation (1) with the initial condition

$$T(x,0) = p_0 \delta(x), \quad (3)$$

where  $\delta(x)$  is the Dirac delta function. In the initial condition (3) we have introduced the constant multiplier  $p_0$  to obtain the nondimensional quantity displayed in figures.

The zero condition at infinity is also assumed:

$$\lim_{x \rightarrow \pm\infty} T(x,t) = 0. \quad (4)$$

## 2. Method of solution

To solve the Cauchy problem under consideration we use the Laplace transform with respect to time  $t$ . For the function  $T(t)$ ,  $0 < t < \infty$ , this transform is defined as

$$L\{T(t)\} = T^*(s) = \int_0^\infty e^{-st} T(t) dt \quad (5)$$

with the inverse carrying out according to the Fourier-Mellin formula

$$L^{-1}\{T^*(s)\} = T(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} T^*(s) ds, \quad (6)$$

where  $c$  is a fixed positive number.

For the Laplace transform rule the Caputo fractional derivative requires the knowledge of the initial values of the function and its integer derivatives of the order  $k = 1, 2, \dots, n - 1$ :

$$L\left\{\frac{d^\alpha T(t)}{dt^\alpha}\right\} = s^\alpha T^*(s) - \sum_{k=0}^{n-1} s^{\alpha-1-k} T^{(k)}(0^+), \quad n-1 < \alpha \leq n. \quad (7)$$

Applying the Laplace transform to equation (1) while taking into account the initial condition (3) and the rule (7) gives

$$s^\alpha T^*(x, s) - s^{\alpha-1} p_0 \delta(x) = a \frac{\partial^2 T^*(x, s)}{\partial x^2} - v \frac{\partial T(x, s)}{\partial x}. \quad (8)$$

Next, we use the exponential Fourier transform with respect to the spatial coordinate  $x$ :

$$F\{T(x)\} = \tilde{T}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\xi} T(x) dx, \quad (9)$$

$$F^{-1}\{\tilde{T}(\xi)\} = T(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi} \tilde{T}(\xi) d\xi. \quad (10)$$

The Fourier transform of the derivative of a function is defined by the following relation:

$$F\left\{\frac{d^n T(x)}{dx^n}\right\} = (-i\xi)^n \tilde{T}(\xi). \quad (11)$$

Application of the exponential Fourier transform (9) to equation (8) using (11) leads to

$$s^\alpha \tilde{T}^*(\xi, s) - s^{\alpha-1} \frac{p_0}{\sqrt{2\pi}} = -a\xi^2 \tilde{T}^*(\xi, s) + iv\xi \tilde{T}^*(\xi, s), \quad (12)$$

and in the transform domain we get

$$\tilde{T}^*(\xi, s) = \frac{p_0}{\sqrt{2\pi}} \frac{s^{\alpha-1}}{s^\alpha + a\xi^2 - iv\xi}. \quad (13)$$

Inversion of the Laplace and Fourier transforms results in the solution:

$$T(x, t) = \frac{p_0}{2\pi} \int_{-\infty}^{\infty} E_\alpha\left[-(a\xi^2 - iv\xi)t^\alpha\right] e^{-ix\xi} d\xi. \quad (14)$$

To obtain (14) the following formula [1]

$$L^{-1}\left\{\frac{s^{\alpha-1}}{s^\alpha + b}\right\} = E_\alpha(-bt^\alpha) \quad (15)$$

has been used, where  $E_\alpha(z)$  is the Mittag-Leffler function in one parameter  $\alpha$  having the series representation

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad \alpha > 0. \quad (16)$$

For negative real values of the argument the Mittag-Leffler function  $E_\alpha(-x)$  can be calculated as the integral [15]:

$$E_\alpha(-x) = \frac{\sin(\alpha\pi)}{\pi} \int_0^\infty e^{-x^{1/\alpha}u} \frac{u^{\alpha-1}}{u^{2\alpha} + 2u^\alpha \cos(\alpha\pi) + 1} du, \quad 0 < \alpha < 1. \quad (17)$$

Next, we recast the argument of the Mittag-Leffler function in the solution (14) as

$$T(x, t) = \frac{p_0}{2\pi} \int_{-\infty}^{\infty} E_\alpha \left\{ -a \left[ \left( \xi - i \frac{v}{2a} \right)^2 + \frac{v^2}{4a^2} \right] t^\alpha \right\} e^{-ix\xi} d\xi \quad (18)$$

and use the substitution

$$\eta = \xi - i \frac{v}{2a} \quad (19)$$

which allows us to rewrite the solution (18) in the form

$$T(x, t) = \frac{p_0}{2\pi} \int_{-\infty}^{\infty} E_\alpha \left[ -a \left( \eta^2 + \frac{v^2}{4a^2} \right) t^\alpha \right] \exp \left[ -ix \left( \eta + i \frac{v}{2a} \right) \right] d\eta \quad (20)$$

and consequently as

$$T(x, t) = \frac{p_0}{2\pi} \exp \left( \frac{vx}{2a} \right) \int_{-\infty}^{\infty} E_\alpha \left[ -a \left( \eta^2 + \frac{v^2}{4a^2} \right) t^\alpha \right] \cos(x\eta) d\eta. \quad (21)$$

In the particular case  $\alpha = 1$  the Mittag-Leffler function  $E_1(z) = e^z$ , hence for the classical advection-diffusion equation we have

$$T(x,t) = \frac{p_0}{2\pi} \exp\left(\frac{vx}{2a}\right) \int_{-\infty}^{\infty} \exp\left[-a\left(\eta^2 + \frac{v^2}{4a^2}\right)t\right] \cos(x\eta) d\eta \quad (22)$$

or

$$T(x,t) = \frac{p_0}{2\pi} \exp\left(\frac{vx}{2a}\right) \exp\left(-\frac{v^2t}{4a}\right) \int_{-\infty}^{\infty} e^{-a\eta^2} \cos(x\eta) d\eta. \quad (23)$$

Since [16]

$$\int_{-\infty}^{\infty} e^{-a\eta^2} \cos(x\eta) d\eta = \frac{\sqrt{\pi}}{\sqrt{at}} \exp\left(-\frac{x^2}{4at}\right), \quad (24)$$

then

$$T(x,t) = \frac{p_0}{2\pi} \exp\left(\frac{vx}{2a}\right) \exp\left(-\frac{v^2t}{4a}\right) \frac{\sqrt{\pi}}{\sqrt{at}} \exp\left(-\frac{x^2}{4at}\right), \quad (25)$$

and we get the known fundamental solution of the classical advection-diffusion equation

$$T(x,t) = \frac{p_0}{2\sqrt{\pi at}} \exp\left[-\frac{(x-vt)^2}{4at}\right]. \quad (26)$$

### 3. Numerical results

In numerical calculations we use the following nondimensional quantities:

$$\bar{x} = \frac{x}{\sqrt{a} t^{\alpha/2}}, \quad \bar{\eta} = \sqrt{a} t^{\alpha/2} \eta, \quad \bar{v} = \frac{v t^{\alpha/2}}{\sqrt{a}}, \quad \bar{T} = \frac{\sqrt{a} t^{\alpha/2} T}{p_0}. \quad (27)$$

Hence, the nondimensional solution takes the form

$$\bar{T}(\bar{x}) = \frac{1}{\pi} \exp\left(\frac{\bar{v}\bar{x}}{2}\right) \int_0^{\infty} E_{\alpha}\left(-\bar{\eta}^2 - \frac{\bar{v}^2}{4}\right) \cos(\bar{x}\bar{\eta}) d\bar{\eta}. \quad (28)$$

The results of numerical calculations for different values of the order  $\alpha$  of the fractional derivative and of the drift parameter  $\bar{v}$  are shown in Figures 1-4.

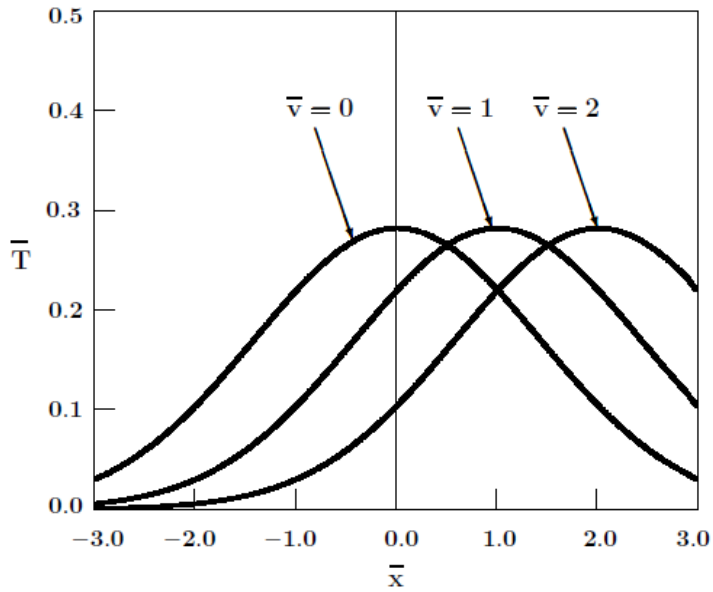


Fig. 1. Dependence of the fundamental solution to the Cauchy problem on distance;  $\alpha = 1$

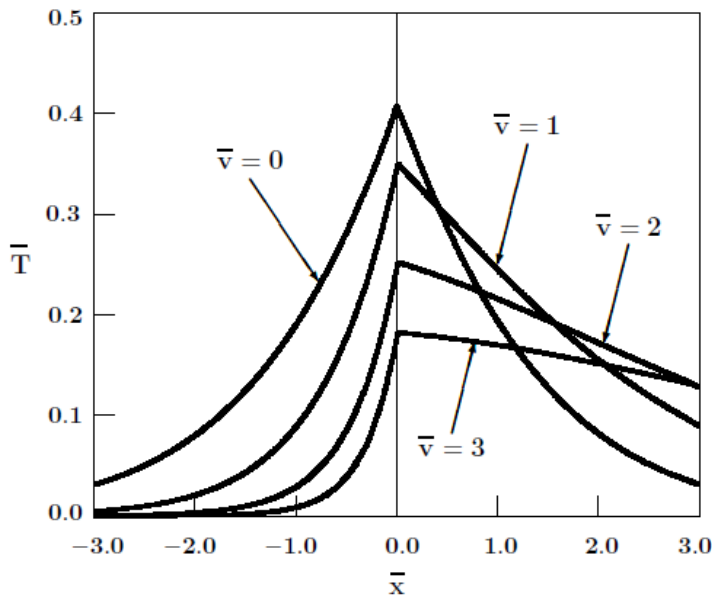


Fig. 2. Dependence of the fundamental solution to the Cauchy problem on distance;  $\alpha = 0.5$

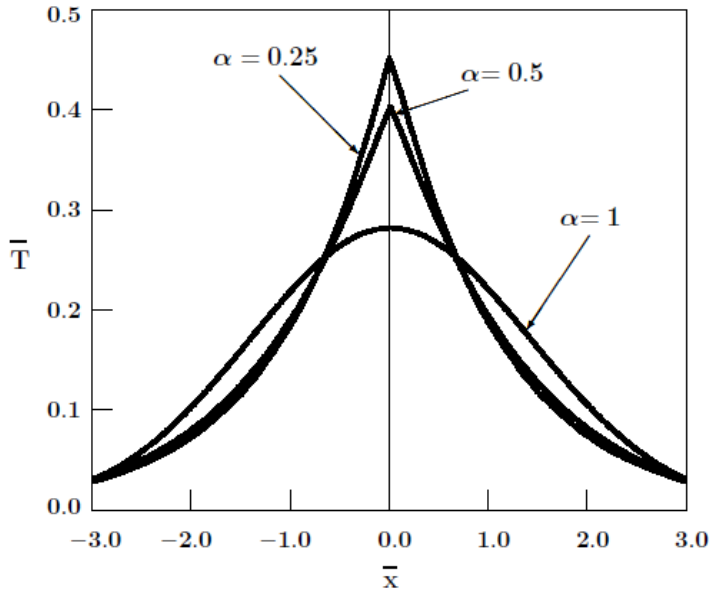


Fig. 3. Dependence of the fundamental solution to the Cauchy problem on distance;  $\bar{v} = 0$

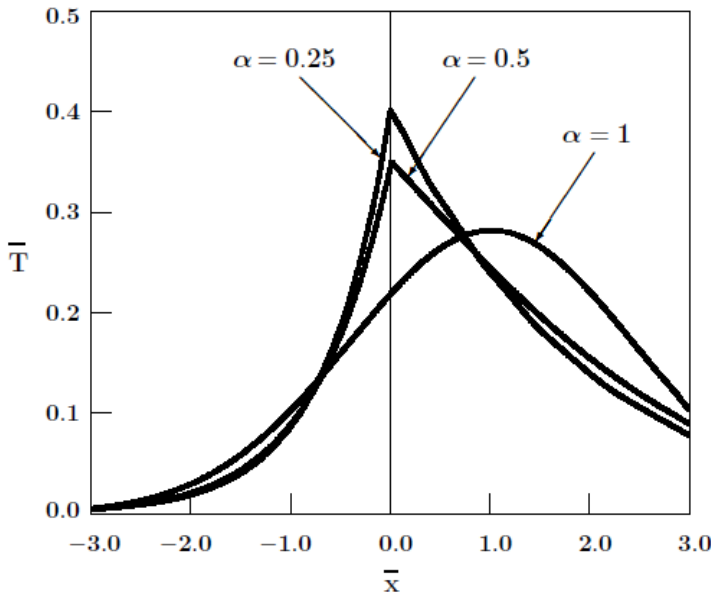


Fig. 4. Dependence of the fundamental solution to the Cauchy problem on distance;  $\bar{v} = 1$

## Conclusions

We have considered the time-fractional advection-diffusion equation with the Caputo fractional derivative in the case of one spatial coordinate. The fundamental solution to the Cauchy problem has been obtained using the integral transform technique. The numerical results show the significant influence of the order of fractional derivative on the solution. In the case of the standard advection-diffusion equation ( $\alpha = 1$ ) the quantity  $v$  only causes a drift of the maximum value of the solution in the  $x$ -direction ( $x - vt$  in the solution (26), see Fig. 1). In the case of time-fractional advection-diffusion equation with  $0 < \alpha < 1$  drift leads to the decreasing of the maximum value of the fundamental solution (see Fig. 2).

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