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A REMARK ON THE INTERSECTIONS OF SUBANALYTIC LEAVES

Maciej P. Denkowski

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Abstract. We discuss a new sufficient condition – weaker than the usual transversality condition – for the intersection of two subanalytic leaves to be smooth. It involves the tangent cone of the intersection and, as typically non-transversal, it is of interest in analytic geometry or dynamical systems. We also prove an identity principle for real analytic manifolds and subanalytic functions.

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1. INTRODUCTION

The best known sufficient condition for the intersection of two submanifolds of $\mathbb{R}^n$ to be again a submanifold is the transversality condition (Transversal Intersection Theorem). Namely, if $M, N \subset \mathbb{R}^n$ are two $\mathcal{C}^1$-smooth submanifolds, $a \in M \cap N$ and the following condition on the tangent spaces holds (minimality of dimension):

$$(T) \qquad \dim(T_aM \cap T_aN) = \dim T_aM + \dim T_aN - n,$$

then the germ $(M \cap N)_a$ is $\mathcal{C}^1$-smooth.

Condition (T) is, of course, not a necessary condition. There are many situations when, for instance, both tangent spaces coincide (and so there is no transversality), but the intersection of the manifolds is nonetheless a manifold again. The need for investigating some other sufficient condition (especially the one this paper is dealing with) came to us from practical considerations — we needed to show smoothness in some peculiar examples in subanalytic geometry. But it is clearly of interest in analytic geometry as motivated by physical problems. Moreover, similar questions arise in dynamical systems where non-transversal intersections of stable and unstable manifolds were studied by F. Takens (see e.g. [8]) or J. Palis and W. de Melo ([6, 7]). Homoclinic and heteroclinic bifurcations occur at non-transversal intersections of stable and unstable manifolds.

Among the most natural conditions to be investigated there would be the following one introducing the (Peano) tangent cone of the intersection:

$$(C) \qquad C_a(M \cap N) = T_aM \cap T_aN.$$

We recall that for $a \in \overline{E}$, where $E \subset \mathbb{R}^n$,

$$C_a(E) = \{v \in \mathbb{R}^n \mid \exists (x_\nu) \subset E, x_\nu \to a \, \exists t_\nu > 0 \colon t_\nu(x_\nu - a) \to v\}.$$

In general, condition (C) is not sufficient for $(M \cap N)_a$ to be a $\mathcal{C}^1$ smooth germ:

Example 1.1. Consider in $\mathbb{R}^2$ the following two sets:

$$M = \mathbb{R} \times \{0\} \quad \text{and} \quad N = \{(x, x^4 \sin(1/x)) \mid x \neq 0\} \cup \{(0,0)\}$$

together with the point $a = (0,0)$. Since $T_aM = M = T_aN$, condition (T) is not satisfied at a. However, (C) is. Nevertheless, since $M \cap N$ is a numerable set accumulating at a, it is not a submanifold.

Remark 1.2. Clearly, (T) implies (C), but the converse is not true, as Example 1.1 above shows. Moreover, just as (T), condition (C) is not a necessary condition for $(M \cap N)_a$ to be smooth. To see this consider e.g. $M = \{y = x^2\}$ and $N = \{y = 0\}$ in $\mathbb{R}^2$.

In what follows we prove the sufficiency of condition (C) for the germ $(M \cap N)_a$ to be smooth in several instances: when M, N are subanalytic leaves and either one of them is one-dimensional (Theorem 2.6), or a hypersurface (Corollary 2.13), which gives the sufficiency of (C) in $\mathbb{R}^n$ for $n \leq 4$ (Corollary 2.15), but also in general, provided the dimensions of M, N do not exceed 2 (Theorem 2.16). Moreover, (C) is sufficient in the typically non-transversal case when we have inclusion of the tangent spaces (Theorem 2.7 and Corollary 2.9). Finally, the methods used allow us to prove a kind of identity principle for analytic submanifolds (Theorem 2.8) and subanalytic functions (Corollary 2.11).

2. CONDITION (C) FOR SUBANALYTIC LEAVES

Note that what plays an important role in Example 1.1 above is the fact that N oscillates which is a proscribed behaviour in the subanalytic case. That is why we will restrict ourselves to the case of *subanalytic leaves*, i.e. submanifolds that are subanalytic subsets of $\mathbb{R}^n$ (or of an open subset of it; anyway, we are actually dealing with germs). For subanalytic geometry we refer the reader to [3]). We assume that submanifolds are connected sets.

Example 2.1 (cf. [3])**.** In order to better understand the notion of a subanalytic leaf, observe that the graph of $f(x) = \sin(1/x)$ for $x > 0$ is an analytic submanifold of the plane, but it is not subanalytic in the whole of $\mathbb{R}^2$, hence it is not a subanalytic leaf in $\mathbb{R}^2$.

Remark 2.2. In general, a subanalytic leaf is required to be a *real analytic* submanifold. However, we may as well consider subanalytic sets that are only $\mathcal{C}^k$ submanifolds. We shall discern them by calling them $\mathcal{C}^k$ *subanalytic leaves.*

Example 2.3. Let us observe after [1] that the equation $y^3 = x^5$ defines a plane analytic curve that is only $\mathcal{C}^1$-smooth at the origin, hence only a $\mathcal{C}^1$ subanalytic leaf, though at the same time an analytic set.

Recall that the tangent cone of a subanalytic set at a given point is again subanalytic and its dimension does not exceed that of the set at that point (see [5]; it also follows directly from the convergence results of [2]). A simple application of the classical Curve Selecting Lemma shows that any vector from such a tangent cone can be obtained as the derivative of an analytic curve.

Before we go any further, we will state some obvious remarks concerning condition (C). First we note that the inclusion

$$C_a(M \cap N) \subset T_a M \cap T_a N$$

always holds, since $C_a(M \cap N) \subset C_a(M) = T_a M$.

Secondly, observe that if (C) is satisfied, then

$$\dim C_a(M \cap N) = \dim M + \dim N - \dim(T_a M + T_a N).$$

Even though in general we have only $\dim_a X \geq \dim C_a(X)$ [1)], in our case it turns out that the dimension of $M \cap N$ at a coincides with that of the tangent cone:

Proposition 2.4. *If (C) is satisfied for two $\mathcal{C}^1$ subanalytic leaves M, N, then*

$$\dim_a(M \cap N) = \dim C_a(M \cap N) = \dim M + \dim N - \dim(T_a M + T_a N).$$

Proof. It suffices to prove the inequality $\dim_a(M \cap N) \leq \dim C_a(M \cap N)$. From the dimension theory for subanalytic sets it follows that there is a subanalytic leaf $\Gamma \subset M \cap N$ such that $a \in \overline{\Gamma}$ and $\dim \Gamma = \dim_a M \cap N$. Take any sequence $\Gamma \ni x_\nu \to a$. By extracting a subsequence we may assume that $T_{x_\nu}\Gamma$ converge to some linear space L (see e.g. [4]). Then $L \subset T_a M \cap T_a N$ which is $C_a(M \cap N)$ by (C). But $\dim L = \dim \Gamma$ so that $\dim \Gamma \leq \dim C_a(M \cap N)$ as required. □

Remark 2.5. Unfortunately, the dimension at a point being only upper semi-continuous we cannot conclude that $M \cap N$ has constant dimension (which is a necessary condition for $M \cap N$ to be smooth at a). A possible idea how to prove that (C) implies smoothness would be (if a proof there should be) to take a Whitney stratification of a neighbourhood of a compatible with $M, N, M \cap N$ and show that along the leaves contained in $M \cap N$ the tangent spaces vary continuously, i.e. the limits coincide at any point belonging to the cluster points of two strata. This should be considered in the light of some results from [2] and [4].

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1) Consider the surface X in $\mathbb{R}^3$ described by $x^3 = x^2 + y^2$; it is a horn, so that its tangent cone at zero is a semi-line.

There are two particular cases when it is rather easy to show that (C) implies smoothness.

Theorem 2.6. *Assume that M, N are $\mathcal{C}^1$ subanalytic leaves with M of dimension 1. Then condition (C) implies $(M \cap N)_a$ is smooth at a.*

Proof. Both sets being subanalytic, we know that $M \cap N$ is subanalytic as well. This implies that either $M \cap N$ is isolated at a in which case it is smooth, or $\dim_a M \cap N = 1$. If the latter occurs[2)], then we necessarily have $\dim_a C_a(M \cap N) = 1$ and since T_aM is one-dimensional, then $T_aM \cap T_aN = T_aM$, by (C). Now, applying the Curve Selecting Lemma, we see that we must have the inclusion of germs $(M)_a \subset (N)_a$ (M has two branches at a which yield the two semi-lines building up T_aM, see e.g. [1] where these observations are extensively used). This ends the proof. □

The assumptions of the theorem below should be considered in the light of Example 2.3. Recall that analytic germs are subanalytic as well.

Theorem 2.7. *Assume that M, N are $\mathcal{C}^1$ submanifolds that are also analytic sets in a neighbourhood of $a \in M \cap N$. Moreover, suppose that $T_aM \subset T_aN$ and that (C) holds. Then $(M \cap N)_a = M_a \subset N_a$.*

Proof. By Proposition 2.4 we have that $\dim_a(M \cap N) = \dim M$. Similarly as in the proof of this Proposition, we find a subanalytic leaf $\Gamma \subset M \cap N$ such that $a \in \overline{\Gamma}$. Note that due to the equality of dimensions, Γ has nonempty interior in M.

We may assume that we are working in a neighbourhood of a in which the representant of the germ M_a is connected. We can ask that in this same neighbourhood the analytic set $M \cap N$ be the zero set of an analytic function f. Since $f|_M$ vanishes on Γ, by the identity principle and the connectedness of M we conclude that $f|_M \equiv 0$. Therefore, $M \subset M \cap N$ which ends the proof. □

The proof suggests an identity principle for real analytic submanifolds similar to the one that holds in the complex case for analytic sets:

Theorem 2.8. *Let M, N be two connected analytic submanifolds that are closed subsets of an open set $\Omega \subset \mathbb{R}^n$ nad let $a \in M \cap N$. Assume that there is a neighbourhood U of a such that $U \cap M = U \cap N$. Then $M = N$.*

Proof. Let $M' := \{x \in M \mid \exists V$ – a neighbourhood of $x\colon V \cap M = V \cap N\}$. Then $a \in M'$ and M' is open in M.

Take a sequence $M' \ni x_\nu \to x_0 \in M$. By the definition of M', $x_\nu \in M \cap N$ and since M, N are closed in Ω. we conclude that $x_0 \in M \cap N$. Then, we find a neighbourhood W of x_0 such that:

(a) $W \cap M$ and $W \cap N$ are connected;
(b) $W \cap M = f^{-1}(0)$ and $W \cap N = g^{-1}(0)$ for two submersions $f\colon W \to \mathbb{R}^{n-p}$ and $g\colon W \to \mathbb{R}^{n-q}$.

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2) This means that $M \cap N$ at a consists of finitely many "half-branches".

Then for ν large enough, we have $x_\nu \in W$ and so we are able to find a neighbourhood $x_\nu \in V \subset W$ such that $V \cap M = V \cap N$. This implies that $f|_{V\cap N} \equiv 0$ and since $V \cap N$ is an open, nonempty subset of N, we conclude by the classical identity principle, that $f|_N \equiv 0$. Thence, $N \subset M$. Repeating this argument with g instead of f yields $M \subset N$. □

As a direct corollary to Theorem 2.7 we obtain another instance when (C) is a sufficient condition for smoothness in a typically non-transversal situation.

Corollary 2.9. *Let M, N be two $\mathcal{C}^1$ submanifolds that are also analytic sets of the same dimension d, $a \in M \cap N$ and assume that $T_aM = T_aN$. Then (C) implies $(M \cap N)_a = M_a = N_a$.*

Remark 2.10. One of the reasons why this kind of results are of interest is that in analytic geometry we often deal with a set Z described as the zeroes $f^{-1}(0)$ of some analytic function $f\colon (\mathbb{R}^n, a) \to (\mathbb{R}, 0)$ with $d_af = 0$. This degenerated differential does not exclude the possibility of Z being smooth at a. Note that Z corresponds to the intersection of the graph M of f with the domain $N = \mathbb{R}^n \times \{0\}$ and the condition on the differential means that $T_aM = T_aN$.

In view of this remark it is easy to obtain now a kind of identity principle for subanalytic functions[3)]:

Corollary 2.11. *Let $f\colon (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be a $\mathcal{C}^1$ subanalytic germ with analytic graph and such that $C_0(f^{-1}(0)) = \mathbb{R}^n$. Then $f = 0$.*

Proof. The zero-set $f^{-1}(0)$ corresponds to the intersection of the graph Γ_f (an n-dimensional $\mathcal{C}^1$ submanifold and an analytic set) with the domain $\mathbb{R}^n \times \{0\}$. We necessarily have $d_0f = 0$ (otherwise $(f^{-1}(0))_0$ would be a smooth hypersurface, contradicting the assumption on the tangent cone) which means that $T_0\Gamma_f \cap T_0(\mathbb{R}^n \times \{0\}) = \mathbb{R}^n \times \{0\} = C_0(\Gamma_f \cap (\mathbb{R}^n \times \{0\}))$, i.e. (C) is satisfied. Therefore, the previous Corollary implies that $(\Gamma_f \cap (\mathbb{R}^n \times \{0\}))_0 = (\mathbb{R}^n \times \{0\})_0 = (\Gamma_f)_0$ which ends the proof. □

In particular, Corollary 2.9 also implies the following result.

Corollary 2.12. *In $\mathbb{R}^3$ condition (C) is sufficient for $(M \cap N)_a$ to be smooth, regardless of the dimensions of the $\mathcal{C}^1$ subanalytic leaves M, N, both analytic at a.*

Proof. Clearly, there is nothing to do, if one of the leaves is open in $\mathbb{R}^3$, or a point. If one of them has dimension 1, we use Theorem 2.6. The remaining case is when T_aM, T_aN are two planes. If they intersect transversally, we use the Transversal Intersection Theorem. If they do not, then we are in the situation of the previous corollary. □

If one of the leaves is a *hypersurface*, we obtain the following corollary.

Corollary 2.13. *If the $\mathcal{C}^1$ subanalytic and analytic leaf N has codimension 1, then condition (C) is sufficient for $(M \cap N)_a$ to be smooth, for any $\mathcal{C}^1$ subanalytic leaf M in $\mathbb{R}^n$, analytic at a.*

[3)] The function $f(x) = x^{5/3}$ appearing in Example 2.3 is typically a non-analytic, $\mathcal{C}^1$-smooth function with analytic graph.

Proof. There are only two possibilities for $\dim T_aM \cap T_aN$: either this dimension is equal to $\dim M - 1$, or to $\dim M$. In the first case, we are dealing, actually, with condition (T) (cf. $\dim N = n - 1$) and the result follows. In the second case, we necessarily have $T_aM \subset T_aN$ and we can invoke Theorem 2.7 in order to finish the proof. □

From these results we are also able to infer the sufficiency of (C) for subanalytic leaves in $\mathbb{R}^4$:

Proposition 2.14. *In $\mathbb{R}^4$ condition (C) is sufficient for $(M \cap N)_a$ to be smooth, regardless of the dimensions of the subanalytic leaves M, N.*

Proof. We may discard the cases when one of the leaves is open or a point. From the previous results, we need not bother about the cases when one of the leaves is one-dimensional or three-dimensional. Thence, it remains to consider the case of two surfaces M, N, i.e. $\dim M = \dim N = 2$. But then $C_a(M \cap N) = T_aM \cap T_aN$ can be of dimension

1. 0 – this means that $(M \cap N)_a$ is reduced to the point $\{a\}$ which is smooth (this is the transversal case when (T) is fulfilled);
2. 1 – in this case $\dim_a(M \cap N) = 1$ and it requires further discussion below;
3. 2 – this happens iff $T_aM = T_aN$, but then we have Corollary 2.9 to conclude.

Let us look at case (2). Using an analytic diffeomorphism in a neighbourhood of a, we may assume that $a = 0$, $M = T_0M = \mathbb{R}^2 \times \{0\}^2$ and N is the graph of an analytic function germ $f\colon (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ with the differential d_0f of rank 1, say $\mathrm{Ker} d_0 f = \mathbb{R} \times \{0\}$. Then $M \cap N$ corresponds to the zero-set $X := f^{-1}(0)$ and we necessarily have $C_0(X) \subset \mathrm{Ker} d_0 f$.

Now, analytic diffeomorphisms do not alter condition (C), hence there must be $C_0(X) = \mathrm{Ker} d_0 f$. Of course, $\dim_0 X = 1$, thus we are dealing with a one-dimensional analytic set X in a neighbourhood of $0 \in \mathbb{R}^2$ and whose tangent cone is $\mathbb{R} \times \{0\}$. It is a classical fact [4)] that $X \setminus \{0\}$ consists of an even number of branches, i.e. images $\gamma_i((0, \varepsilon))$ of some injective analytic curves $\gamma_i\colon (-\varepsilon, \varepsilon) \to \mathbb{R}^2$ with $\gamma_i(0) = 0$. Note the derivatives $\gamma_i'(0)$ give the two semi-lines making up the tangent cone.

Write $f = (f_1, f_2)$ and let (x, y) be the variables in $\mathbb{R}^2$. The assumptions made so far yield in particular:

$$\frac{\partial f_i}{\partial x}(0) = 0 \quad \text{for} \quad i = 1, 2, \quad \text{and} \quad \left(\frac{\partial f_1}{\partial y}(0)\right)^2 + \left(\frac{\partial f_2}{\partial y}(0)\right)^2 \neq 0.$$

To fix the attention, let us say that $\frac{\partial f_1}{\partial y}(0) \neq 0$. Then by the (Analytic) Implicit Function Theorem, 0 is a regular point of $f_1^{-1}(0)$, i.e. this set is an analytic submanifold in a neighbourhood of zero, of dimension 1. Actually, it is an analytic graph $x \mapsto y(x)$. But then the set $X \subset f_1^{-1}(0)$, one-dimensional and analytic itself and sharing the same two tangent semi-lines (for $x > 0$ and $x < 0$), must coincide with $f_1^{-1}(0)$ in a neighbourhood of zero. Therefore, X is smooth at the origin. □

[4)] It can be proved e.g. using the Weierstrass Preparation Theorem and the the Curve Selecting Lemma does the rest.

Summing up the preceding results we get the following corollary.

Corollary 2.15. *In $\mathbb{R}^n$, for $n \leq 4$, condition (C) is sufficient for $(M \cap N)_a$ to be smooth for any two subanalytic leaves M, N.*

The last proof can be repeated in order to obtain the following theorem.

Theorem 2.16. *Assume that M, N are subanalytic leaves in $\mathbb{R}^n$ satisfying $\max\{\dim M, \dim N\} \leq 2$. Then condition (C) is sufficient for $(M \cap N)_a$ to be smooth.*

Proof. In view of the preceding results we may assume that $n > 4$. If one of the leaves has dimension 0 or 1, there is nothing to do. Therefore, we turn to the situation when we are dealing with two surfaces M, N, i.e. both dimensions are 2. As earlier, $\dim(T_aM \cap T_aN) \in \{0, 1, 2\}$. This time, however, the zero-dimensional case may not involve (T) (if $n \neq 4$). But it follows from the Curve Selecting Lemma that any vector from the tangent cone $C_a(M \cap N)$ can be obtained as the derivative of an injective analytic curve through a and whose half-branch lives in $M \cap N$. Therefore, the dimension of the tangent cone is equal to zero iff $M \cap N$ is isolated at a and then it is smooth.

Next, $\dim(T_aM \cap T_aN) = 2$ iff $T_aM = T_aN$ and we use Corollary 2.9. Finally, if $\dim(T_aM \cap T_aN) = 1$, we may repeat the argument from the proof of Proposition 2.14: in a neighbourhood of a that we may assume to be the origin, we flatten M to represent it as $\mathbb{R}^2 \times \{0\}^{n-2}$ and consider N as the graph of an analytic function germ $(\mathbb{R}^2, 0) \to (\mathbb{R}^{n-2}, 0)$ with differential of rank 1. The rest of the proof can be repeated unaltered. □

Example 2.17. Such results as that of Proposition 2.14 may be used, for example, the negative way: the analytic set $\{(x, y) \in \mathbb{R}^2 \mid |y| = x^2\}$ in $\mathbb{R}^2 \times \{0\}^2$ cannot be obtained (as a germ at the origin) from the intersection of two analytic, $\mathcal{C}^1$-smooth surfaces in $\mathbb{R}^4$ with their tangent planes intersecting along $\mathbb{R} \times \{0\}^3$.

On the other hand, a naïve example of a direct application of Proposition 2.14 could be the following: consider in $\mathbb{R}^4$ with the coordinates (x, y, z, t) the two sets M, N defined respectively by

$$M\colon (x^2 + y^2 + xy + (1/2)x - y)^2 + (t - x^2 - y^2)^2 = 0$$

and

$$N\colon (x^2 - y^2 - z^2 + z)^2 + (t - x^2 - y^2)^2 = 0.$$

It is clear at once that both sets are two-dimensional submanifolds with

$$T_0M \cap T_0N = \{(x, y, z, t) \in \mathbb{R}^4 \mid 2y = x, z = t = 0\}.$$

This contains the tangent cone $C_0(M \cap N)$. It suffices now to show that the vectors $\pm(1, 1/2, 0, 0)$ belong to this cone. It is easy to see what sequences $M \cap N \ni a_\nu \to 0$,

$\lambda_\nu > 0$ will give this, e.g. $(1, 1/2, 0, 0)$ can be obtained as the limit $\lim_\nu \lambda_\nu a_\nu$ for $\lambda_\nu = \nu$ and $a_\nu = (x_\nu, y_\nu, z_\nu, t_\nu)$ where

$$\begin{aligned}
&x_\nu = 1/\nu, \quad y_\nu = \left[(1-(1/\nu)) - \sqrt{1-3(1/\nu)^2 - 4(1/\nu)}\right]/2,\\
&z_\nu = \left[1 - \sqrt{(6(1/\nu^2) + 6(1/\nu) + 2(1-1/\nu)\sqrt{1-3(1/\nu^2)-4(1/\nu)}}\right]/2,\\
&t_\nu = x_\nu^2 + y_\nu^2.
\end{aligned}$$

Therefore, $(M \cap N)_0$ is smooth. All this is a mere computation. Of course, in this precise case it is possible to argue differently by reducing the problem to $\mathbb{R}^3$ where we actually are dealing with a transversal intersection. But this example has the advantage of being simple enough to be a – hopefully – good illustration of how (C) is supposed to work.

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Maciej Denkowski
maciej.denkowski@uj.edu.pl

Jagiellonian University
Faculty of Mathematics and Computer Science
Institute of Mathematics
Łojasiewicza 6, 30-348 Kraków, Poland

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