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ARCHIVES of FOUNDRY ENGINEERING

ISSN (2299-2944)
Volume 19
Issue 3/2019

38 – 42

10.24425/afe.2019.127136

7/3

Published quarterly as the organ of the Foundry Commission of the Polish Academy of Sciences

Identification of the Thermal Conductivity Coefficient in the Heat Conduction Model with Fractional Derivative

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Received 15.05.2019; accepted in revised form 05.06.2019

Abstract

Main goal of the paper is to present the algorithm serving to solve the heat conduction inverse problem. Authors consider the heat conduction equation with the Riemann-Liouville fractional derivative and with the second and third kind boundary conditions. This type of model with fractional derivative can be used for modelling the heat conduction in porous media. Authors deal with the heat conduction inverse problem, which, in this case, consists in identifying an unknown thermal conductivity coefficient. Measurements of temperature, in selected point of the region, are the input data for investigated inverse problem. Basing on this information, a functional describing the error of approximate solution is created. Minimizing of this functional is necessary to solve the inverse problem. In the presented approach the Ant Colony Optimization (ACO) algorithm is used for minimization.

Keywords: Heat conduction equation, Inverse problem, Fractional derivative, Thermal conductivity

1. Introduction

Fractional (or non-integer) derivatives are the generalizations of classical derivative to the case of an order being the real number. There are several definitions of derivatives of this type, the most popular and the most often discussed in literature are the derivatives of Riemann-Liouville, Caputo, and Riesz [1, 2]. The fractional calculus has found a wide range of applications in modelling the anomalous diffusion processes [3]. Examples of the fractional derivative applications can be found in [4-6].

Modelling the phenomenon of heat transfer in the porous materials is a complicated problem. For modelling this phenomenon one can apply the mathematical models using the fractional derivatives [7-9]. The usefulness of models with fractional order derivatives in modelling the thermal processes in porous materials is described in [8]. In literature one can also find the descriptions of experiments showing that in case of porous materials, the heat transfer equation with fractional derivative allows to reconstruct better the temperature distribution than the equation with classical derivative [10]. Paper [11] shows that the model described by the differential equation including the fractional derivative with respect to time, proposed for the process of heat conduction in a three-layer composite carrier, allows also to reconstruct very well the experimental data.

The literature also contains the papers dedicated to the inverse problems described by differential equations with fractional order derivatives. Murio's papers were the first works of this type [12-14]. More information about the inverse heat conduction problems for differential equations with fractional derivatives can be found in [15, 16].

This paper presents an algorithm dedicated for solving the inverse heat conduction problem based on the temperature measurements read in the selected points of region. The unknown parameter is the thermal conductivity coefficient. In the considered model we use the Riemann-Liouville fractional derivative. The direct problem is solved by using the finite difference method and the approximation of fractional derivative [17]. In order to find minimum of the function describing the error of approximate solution we apply the Ant Colony Optimization algorithm [18, 19].

2. Problem formulation

We investigate the unsteady one-dimensional problem defined in interval $[0, L]$. Let us consider the heat conduction equation with fractional derivative

$$c\rho \frac{\partial u(x,t)}{\partial t} = \hat{\lambda} \frac{\partial^\beta u(x,t)}{\partial x^\beta} \tag{1}$$

where $D \in \{(x,t): x \in [0,L], t \in [0,T], L, T \in \mathcal{R}_+\}$, $\beta \in (1,2)$ denotes the order of fractional derivative, $\hat{\lambda} = \widehat{\omega}\lambda$ is the scaled thermal conductivity coefficient $[W/(m^{3-\beta}K)]$, that is the thermal conductivity multiplied by scaling constant $\widehat{\omega}$ with numerical value of one and unit $[m^{\beta-2}]$, chosen so that the right and left units of the equation are the same, λ is the thermal conductivity coefficient $[W/(m\,K)]$, c is the specific heat $[J/(kg\,K)]$, ρ is the density $[kg/m^3]$, u $[K]$ denotes the temperature, x $[m]$ and t $[s]$ mean the spatial variable and the time, respectively. Equation (1) is completed by the following initial-boundary conditions

$$u(x,0) = f(x), \quad x \in [0,L],$$
$$-\lambda \frac{\partial u}{\partial x}(0,t) = q(t), \quad t \in (0,T],$$
$$-\lambda \frac{\partial u}{\partial x}(L,t) = h(t)(u(L,t) - u^\infty), \quad t \in (0,T],$$

where h is the heat transfer coefficient $[W/(m^2K)]$ and u^∞ denotes the ambient temperature $[K]$.

The fractional derivative occurring in equation (1) is the Riemann-Liouville derivative defined by formula [1,2]:

$$\frac{\partial^\beta u(x,t)}{\partial x^\beta} = \frac{1}{\Gamma(2-\beta)} \frac{\partial^2}{\partial x^2} \int_a^x u(s,t)(x-s)^{1-\beta}\, ds, \quad \beta \in (1,2).$$

The considered inverse problem consists in identification of the thermal conductivity coefficient λ which is unknown. The input data for inverse problem are the measurements of temperature taken from boundary of the region. We denote these data as follows

$$u(x_p, t) = \widehat{U}_j, \quad j = 1,2,\dots,N,$$

where x_p is the localization of measurement point and N denotes the number of measurements. Basing on this information (temperature measurements) and the values of temperature computed by using the described model for the fixed value of λ, we create the functional defining the error of approximate solution

$$J(\lambda) = \sqrt{\sum_{j=1}^N \left(U_j(\lambda) - \widehat{U}_j\right)^2}. \tag{2}$$

In order to find the unknown parameter λ we need to minimize the above functional. And to find the minimum of this functional we use the Ant Colony Optimization algorithm.

3. Direct problem

The direct problem is solved by using the finite difference method. In order to use this method we create the mesh

$$S = \{(x_i, t_k): x_i = i\Delta x, t_k = k\Delta t, i = 0,1,\dots,N, k = 0,1,\dots,M\},$$

of size N x M and with steps $\Delta x = \frac{L}{N}, \Delta t = \frac{T}{M}$. The approximate values of function u in points (x_i, t_k) are denoted by U_i^k. The Riemann-Liouville fractional derivative is approximated by formula

$$\frac{\partial^\beta u(x_i,t_k)}{\partial x^\beta} \approx \sum_{j=0}^{i+1} \omega(\beta,j) U_{i-j+1}^k, \tag{3}$$

where

$$\omega(\beta,j) = \frac{\Gamma(j-\beta)}{\Gamma(-\beta)\Gamma(j+1)}.$$

After discretizing equation (1) and using the approximation of fractional derivative (3), we get ($i = 1,2,\dots,M-1$) the equation

$$\frac{U_i^{k+1} - U_i^k}{\Delta t} = \frac{\lambda}{c\rho(\Delta x)^\beta} \sum_{j=0}^{i+1} \omega(\beta,j) U_{i-j+1}^{k+1}. \tag{4}$$

It is also necessary to approximate the boundary conditions. Boundary condition of the second kind is approximated as follows

$$-\lambda \frac{-U_2^{k+1} + 4U_1^{k+1} - 3U_0^{k+1}}{2\Delta x}\ q^{k+1}, \tag{5}$$

and boundary condition of the third kind is approximated as given below

$$-\lambda \frac{U_{N-2}^{k+1} - 4U_{N-1}^{k+1} + 3U_N^{k+1}}{2\Delta x} = h^{k+1}\left(U_N^{k+1} - u^\infty\right). \tag{6}$$

In result of the above approximation, for the fixed moment of time, we obtain the system of algebraic equations. By solving this system we get the approximate values of function u (temperatures) in the mesh points. More information about the numerical solution can be found in [17].

4. Minimization of the functional

In order to minimize the functional we use the Ant Colony Optimization (ACO) algorithm [18]. The inspiration for developing the ACO algorithm was the behavior of ant swarms in nature. The algorithm has been adapted for executing the parallel calculations. In order to describe the algorithm we introduce the following symbols:

$$J - minimized\ function,$$
$$n - number\ of\ unknown\ parameters,$$
$$nT - number\ of\ threads,$$
$$M = nT \cdot p - number\ of\ ants\ (p \in Z),$$
$$G - number\ of\ pheromone\ spots,$$
$$q = 0.9, \xi = 1.0 - parameters\ of\ the\ algorithm.$$

Let us present now the description of ACO algorithm.

Initialization of the algorithm

1. Setting parameters of the algorithm G, M, I, nT, q, ξ.
2. Random generating the G pheromone spots (solutions) and assigning them to set of solutions T_0.
3. Parallel computing the values of minimized function for every pheromone spot (solution). Sorting the elements in T_0 according to the quality of solution (descending).

Iterative process

4. Assigning the probabilities to the pheromone spots (solutions) according to formula

$$p_l = \frac{w_l}{\sum_{l=1}^{G} w_l} \qquad l = 1,2,\dots,G,$$

where weight w_l is associated to l-th solution and is given by formula

$$w_l = \frac{1}{qG\sqrt{2\pi}} \exp\left(\frac{-(l-1)^2}{2(qG)^2}\right).$$

5. Ant chooses the l-th pheromone spot (solution) with probability p_l.
6. Ant transforms the j-th coordinate ($j = 1,2,\dots,n$) of l-th pheromone spot (solution) s_j^l by sampling the neighborhood with the use of Gaussian probability density function

$$g(x, \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)$$

where $\mu = s_j^l$, $\sigma = \frac{\xi}{G-1}\sum_{p=1}^{G}\left|s_j^p - s_j^l\right|$.

7. Repeating steps 5-6 for each ant. M new pheromone spots (solutions) are obtained.
8. Dividing the population into nT sets. Calculating the value of minimized function for each new solution in population (parallel computation).
9. Adding new solutions to set T_i, sorting set T_i and removing M worst solutions.
10. Repeating steps 3-9 I times.

By knowing the values of parameters G, M and I we are able to determine the number of evaluations of the objective function during the algorithm execution, which is equal to $G + M \cdot I$.

5. Results

In the numerical example we consider the following data:

$$L = 3.825\,[mm], \quad T = 71.82\,[s], \quad c = 900[J/kgK],$$
$$\rho = 2106\,[kg/m^3], \quad u^\infty = 298\,[K], \quad f(x) = 573.15\,[K]$$
$$q(t) = 0, \quad h(t) = 2.42t^2 - 5t + 78.07, \quad \beta = 1.08$$

In the presented example the investigated inverse problem consists in identification of the thermal conductivity coefficient $\lambda = a, (a \in \mathbb{R})$ in the form of a constant. The experiment assumes that the sample was characterized by 22.5% degree of porosity, hence the exact value of thermal conduction coefficient is equal to $\lambda = 0.225\lambda_p + 0.775\lambda_a \approx 184$. The restored parameter was searched in interval $[120, 250]$. In order to generate the input data we used the mesh 200×3990 (space × time), while for solving the inverse problem we used the mesh 100×1995 ($\Delta x = 0.03825, \Delta t = 0.036$). To run the ACO algorithm we used the following parameters: $a \in [120, 250]$, $G = 12$, $M = 16$, $I = 55$, $nT = 4$.

Table 1 presents the results of parameter λ reconstruction for various input data (disturbed by various pseudorandom error). In each considered case we received similar results. The errors of thermal conductivity reconstruction are minimal and they do not exceed 0.5%.

Table 1.
Results of calculations (λ – reconstructed value of thermal conductivity coefficient)

noise	λ	error [%]	functional value
0%	184.53	0.29	0.101
2%	184.48	0.26	22103.097
5%	184.80	0.43	143170.148

Table 2 compiles the errors of temperature reconstruction in the measurement point for the identified values of thermal conductivity coefficient. These errors are minimal and of similar level for various input data. This is due to the fact that the reconstructed λ coefficients are close to each other and oscillate around 184.

Table 2.
Errors of temperature reconstruction in measurement point (Δ_{mean} – mean absolute error, Δ_{max} – maximal absolute error, δ_{mean}- mean relative error, δ_{max} – maximal relative error)

	0%	**2%**	**5%**
$\boldsymbol{\Delta_{mean}}$ **[K]**	$6.2 \cdot 10^{-3}$	$1.7 \cdot 10^{-2}$	$7.4 \cdot 10^{-2}$
$\boldsymbol{\Delta_{max}}$ **[K]**	$1.2 \cdot 10^{-2}$	$2.2 \cdot 10^{-2}$	$1.3 \cdot 10^{-1}$
$\boldsymbol{\delta_{mean}}$ **[%]**	$1.2 \cdot 10^{-3}$	$3.4 \cdot 10^{-3}$	$1.6 \cdot 10^{-2}$
$\boldsymbol{\delta_{max}}$ **[%]**	$2.2 \cdot 10^{-3}$	$4.6 \cdot 10^{-3}$	$3.1 \cdot 10^{-2}$

Finally, Figure 1 presents the reconstruction of temperature in the measurement point in case of 5% error input data. Both lines (reconstruction and measurements) are very similar, so the temperature calculated from the model fits very well to the measurements.

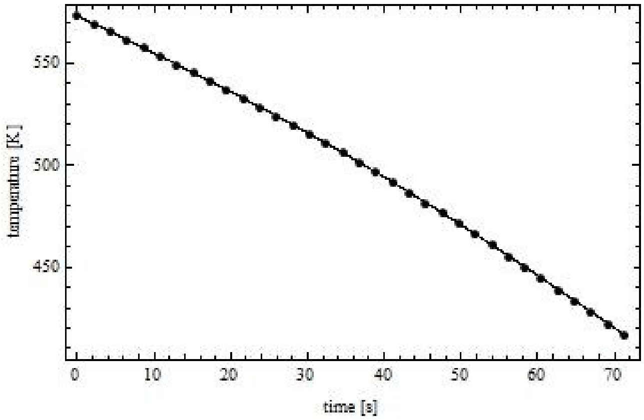

Fig. 1. Reconstructed temperatures (dotted line) and measurements (solid line) in case of 5% error input data

6. Conclusions

This article presented the one-dimensional heat conduction model with fractional Riemann-Liouville derivative. Model of this kind can be used for describing the phenomenon of heat conduction in the porous materials. The algorithm for solving the discussed inverse heat conduction problems, based on the Ant Colony Optimization algorithm, was also presented. The executed numerical example showed that the considered algorithm reconstructs very well the value of thermal conductivity coefficient, despite the disturbances of input data.

Acknowledgments

Publication supported within the framework of grants in the area of scientific research and developmental works founded by Rector of the Silesian University of Technology, 09/010/RGJ19/0041.

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