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The optimal online control of the instantaneous power and the multiphase source's current

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Abstract. The paper presents the new optimal real-time control algorithm of the power source. The minimum of the square-instantaneous current was assumed as an optimal criterion, with an additional constraint on source instantaneous power. The mathematical model of a multiphase source was applied as a voltage-current convolution in the discrete time domain. The resulting control algorithm was the recursive digital filter with infinite recursion.

Key words: power source, real-time control, optimum, operators.

1. Introduction

The paper attempts to solve new problems of the electrical source optimal real-time control. In previous papers, optimal solutions were achieved i.e. for instantaneous power exceed criterion [1, 2], for minimum energy losses [3, 4], with usage of the similarity principle [5], etc.

This article is a significant generalization of previous studies for single-phase systems [6–8] and for three-phase systems [9, 10]. Those studies have been unsuccessful in the research for mathematical expressions for the so-called "inactive powers", while in this paper, a much more efficient approach based on optimization methods is applied.

In this paper, the two alternative cost functions were assumed as the optimal criterion: the achievement of the maximum instantaneous power or the achievement of the minimum instantaneous source's current ABS value with the given instantaneous source power. A mathematical model of multiphase power source in the discrete time domain (Fig. 1) was applied.

The source's voltage-current equation in this model is:

$$\mathbf{u}_n = \mathbf{e}_n - \sum_{m=1}^{\infty} \mathbf{z}_m \mathbf{i}_{n-m} \tag{1}$$

where: $\mathbf{u}_n = \mathrm{col}_\alpha[u_n^\alpha]$, $\mathbf{e}_n = \mathrm{col}_\alpha[e_n^\alpha]$, $\mathbf{i}_n = \mathrm{col}_\alpha[i_n^\alpha]$ – the column vectors of: the terminal phase voltage, source voltage and source output current; $\alpha = 0, 1, \ldots, M-1$ – the line (phase) number; $n = 0, 1, 2, \ldots$ – the voltage and current sample's number; $\mathbf{z}_m = \mathrm{mat}_{\alpha,\beta}[z_m^{\alpha,\beta}]$ – the square matrix of the impulse responses of source's internal impedance operator; $\alpha, \beta = 0, 1, \ldots, M-1$

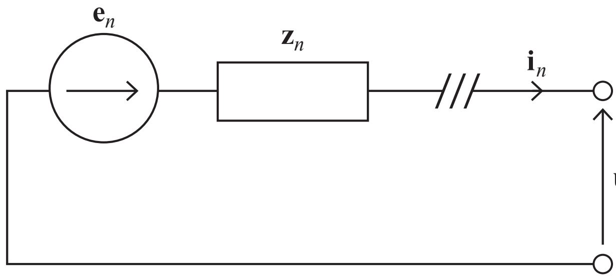

Fig. 1. A multiphase electrical power source

– the number of phase. This means that the source model assumed is a linear, time-invariant, i.e. the convolutional inner impedance operator. In particular, it is the discrete time convolution.

In the online optimal control system, the transition from the $n-1$ state to the n state takes place so as to meet the given quality criteria. Two alternate criteria are assumed:

– the maximum of instantaneous power criteria (called p_n^{MAX} algorithm), i.e.:

$$\mathbf{i}_n^T \mathbf{u}_n \to MAX$$

where T means function transposition,

– the minimum of instantaneous current ABS value at a given instantaneous source power p_n, called $|\mathbf{i}_n|^{MIN}$ algorithm:

$$\mathbf{i}_n^T \mathbf{i}_n \to MIN$$
$$\mathbf{i}_n^T \mathbf{u}_n - p_n = 0.$$

In order to solve these tasks, (1) is transformed in the form of recursive digital filter with infinite recursion:

$$\mathbf{u}_n = \mathbf{v}_n - \mathbf{z}_0 \mathbf{i}_n \tag{2}$$

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where:

$$\mathbf{v}_n = \mathbf{e}_n - \sum_{m=1}^{\infty} \mathbf{z}_m \mathbf{i}_{n-m}. \tag{3}$$

In expressions (2) and (3) the vector of sample's sequence $\{\mathbf{v}_n\}$ and the matrix $\mathbf{z}_0$ will be treated as given values, while the vector $\{\mathbf{i}_n\}$ of sample's sequence will be sought.

Assumed control algorithms take the following forms:

– p_n^{MAX} algorithm:

$$\mathbf{i}_n^T \mathbf{u}_n = \mathbf{i}_n^T (\mathbf{v}_n - \mathbf{z}_0 \mathbf{i}_n) = \mathbf{i}_n^T \mathbf{v}_n - \mathbf{i}_n^T \mathbf{z}_0 \mathbf{i}_n = \\ = \mathbf{i}_n^T \mathbf{v}_n - \mathbf{i}_n^T \mathbf{r}_0 \mathbf{i}_n \rightarrow MAX$$

or equivalently:

$$\mathbf{i}_n^T \mathbf{r}_0 \mathbf{i}_n - \mathbf{i}_n^T \mathbf{v}_n \rightarrow MIN \tag{4}$$

where:

$$\mathbf{r}_0 = \frac{1}{2}\left(\mathbf{z}_0 + \mathbf{z}_0^T\right)$$

is positive defined, time-invariant matrix;

– $|\mathbf{i}_n|^{MIN}$ algorithm:

$$\mathbf{i}_n^T \mathbf{i}_n \longrightarrow MIN \\ \mathbf{i}_n^T (\mathbf{v}_n - \mathbf{r}_0 \mathbf{i}_n) - p_n = 0,$$

or using Lagrange's factor:

$$\mathbf{i}_n^T (\mathbf{1} + \lambda \mathbf{r}_0) \mathbf{i}_n - \lambda \mathbf{v}_n^T \mathbf{i}_n \longrightarrow MIN \tag{5}$$

where $\mathbf{1}$ is the identity matrix.

The solution of both optimization tasks, at a given $\mathbf{e}_n$ and $\mathbf{z}_n$ for $n = 0, 1, 2, \ldots$ is the $\mathbf{i}_n$ sequence as a function of $\mathbf{v}_n$, this in turn is function of $\mathbf{i}_{n-1}, \mathbf{i}_{n-2}, \ldots, \mathbf{i}_1, \mathbf{i}_0$. Thus, the solutions of those tasks should have form of a theoretically infinite recursive digital filter.

2. The solution of the optimal tasks for the single phase source

For the single phase source its voltage-current equation takes the simpler (scalar) form:

$$u_n = e_n - \sum_{m=0}^{\infty} z_m i_{n-m} = e_n - \sum_{m=1}^{\infty} z_m i_{n-m} - z_0 i_n = \\ = v_n - z_0 i_n \tag{6}$$

where:

$$v_n = e_n - \sum_{m=1}^{\infty} z_m i_{n-m}.$$

The p_n^{MAX} task has the form:

$$i_n (v_n - z_0 i_n) \longrightarrow MAX \tag{7}$$

and the $\left[(i_n)^2\right]^{MIN}$ task has the form:

$$(i_n)^2 \longrightarrow MIN \\ i_n (v_n - z_0 i_n) = p_n \tag{8}$$

or yet another form using Lagrange's factor:

$$(1 + \lambda z_0)(i_n)^2 - v_n i_n \longrightarrow MIN. \tag{9}$$

In order to solve these tasks the discrete time index is skipped in (7) and (8), therefore the p_n^{MAX} takes form:

$$vi - r_0 i^2 \longrightarrow MAX \tag{10}$$

where $r_0 = z_0$, and the $(i^2)^{MIN}$ takes form:

$$(1 + \lambda r_0) i^2 - vi \longrightarrow MIN. \tag{11}$$

The solving equations for (10) and (11) tasks take form:

$$v - 2r_0 i = 0$$

thus:

$$i \equiv i^d = \frac{v}{2r_0}$$

and:

$$(1 + \lambda r_0) i = \frac{1}{2} \lambda v$$

thus:

$$i \equiv i^{opt} = \Lambda v$$

where Λ is an undetermined real scalar. The obtained currents i^d and i^{opt} will be called adjustment current and optimal current respectively. Using these currents in the source's balance condition:

$$r_0 i^2 - vi + p = 0$$

the maximum power is obtained:

$$p^{MAX} = \frac{v^2}{4r_0},$$

and the equation which allows to calculate the Λ factor takes form:

$$r_0 v^2 \Lambda^2 - v^2 \Lambda + p = 0.$$

This quadratic equation can be formed as follows:

$$r_0 \Lambda^2 - \Lambda + \frac{1}{4r_0} x = 0,$$

where $x = \frac{p}{p^{MAX}}$ is the so–called fraction of the source's load. The one and only allowed solution for the quadratic equation is:

$$i = i^{opt} = \frac{v}{2r_0}\left(1 - \sqrt{1 - x}\right) = i^d \left(1 - \sqrt{1 - x}\right).$$

The same result can be obtained using the source's balance condition:

$$r_0 i^2 - vi + p = 0$$

which takes form:

$$\frac{r_0}{v} i^2 - i + \frac{v}{4r_0} x = 0,$$

and solving it with respect to i yields:

$$i = \frac{v}{2r_0}\left(1 - \sqrt{1-x}\right) = i^d\left(1 - \sqrt{1-x}\right).$$

Thus, the following theorem of the single phase source has been proved: the current that delivers the given instantaneous power from the source has the minimum or the maximum instantaneous-square value.

The $\left[(i_n)^2\right]^{MIN}$ algorithm works recursively, i.e. $(i_0, i_1, i_2, \ldots, i_{n-1}) \rightarrow i_n$ according to the scheme:

$$(i_n)^2 \longrightarrow MIN$$
$$i_n u_n = p_n$$

and is solved by the following sequences of operations:

$$u_n = e_n - \sum_{m=1}^{\infty} z_m i_{n-m},$$

$$i_n^d = \frac{v_n}{2r_0}$$

where $r_0 = z_0$,

$$p_n^{MAX} = \frac{(v_n)^2}{4r_0},$$

$$x_n = \frac{p_n}{p_n^{MAX}},$$

$$i_n = \begin{cases} i_n^d \sqrt{1-x_n} & \text{for} \quad x_n < 1 \\ i_n^d & \text{for} \quad x_n > 1 \end{cases} \tag{12}$$

where $n = 0, 1, 2, \ldots$ is the discrete time index.

The sequence of the given instantaneous power p_n is limited by the (12) condition. It means that the source cannot provide current that transfers greater power than the source's maximum power p_n^{MAX}. In such case the source is delivering the adjustment current i_n^d, which transfers the maximum power p_n^{MAX} at the moment. It will be shown in an example in Section 4.

3. The solution of the optimal tasks for the multiphase source

Skipping the discrete time index n in (4), the p^{MAX} task takes the form:

$$\mathbf{i}^T \mathbf{r}_0 \mathbf{i} - \mathbf{i}^T \mathbf{v} \longrightarrow MIN. \tag{13}$$

Doing the same with (5), the $|\mathbf{i}|^{MIN}$ task takes the form:

$$\mathbf{i}^T\left(\mathbf{1} + \lambda \mathbf{r}_0\right)\mathbf{i} - \lambda \mathbf{v}^T \mathbf{i} \longrightarrow MIN. \tag{14}$$

These minimum tasks were considered in paper [11] and its solving equations are:

$$\mathbf{r}_0 \mathbf{i}^d = \frac{1}{2}\mathbf{v} \tag{15}$$

for the p^{MAX} task, and:

$$\left(\mathbf{1} + \lambda \mathbf{r}_0\right)\mathbf{i}^\lambda = \lambda \mathbf{r}_0 \mathbf{i}^d \tag{16}$$

for the $|\mathbf{i}|^{MIN}$ task. Solving (15) and (16) the multiphase vector of adjustment current is obtained:

$$\mathbf{i}^d = \frac{1}{2}\mathbf{r}_0^{-1}\mathbf{v}$$

and the λ – based current vectors:

$$\mathbf{i}^\lambda = \frac{1}{2}\left(\lambda^{-1}\mathbf{1} + \mathbf{r}_0\right)^{-1}\mathbf{v}. \tag{17}$$

Simultaneously the source's power balance condition is met:

$$\mathbf{i}^T\left(\mathbf{v} - \mathbf{r}_0 \mathbf{i}\right) - p = 0$$

which allows to calculate the maximum power:

$$p^{MAX} = \frac{1}{4}\mathbf{v}^T \mathbf{r}_0^{-1}\mathbf{v}.$$

The Lagrange's factor λ in (17) can be calculated from the so–called power equation:

$$F(\lambda) = P.$$

where

$$F(\lambda) = \left(\mathbf{i}^\lambda\right)^T\left(\mathbf{v} - \mathbf{r}_0 \mathbf{i}^\lambda\right)$$

is the so–called source's energy function. That function can be approximated by expression [11]:

$$F(\lambda) = \frac{2 + \lambda r}{(1 + \lambda r)^2}\lambda r p^{MAX},$$

where

$$r = \frac{\mathbf{v}^T \mathbf{v}}{\mathbf{v}^T \mathbf{r}_0^{-1}\mathbf{v}}$$

is the so-called source normative resistance. Solving the energy equation by using the approximation function, the Lagrange's factor is obtained:

$$\lambda_*^{-1} = r\frac{\sqrt{1-x}}{1 - \sqrt{1-x}},$$

where $x = \frac{p}{p^{MAX}}$ is fraction of the source's load. An instance of the optimal current is defined by (17) which now takes the form:

$$\mathbf{i}^{opt} = \frac{1}{2}\left(\lambda_*^{-1}\mathbf{1} + \mathbf{r}_0\right)^{-1}\mathbf{v}.$$

Introducing again the discrete time index causes appropriate values to take the sense of the instantaneous values. In this way, the sequential algorithm for determining the current's instantaneous-value vector is obtained:

$$\left(\mathbf{i}_{n-1}, \mathbf{i}_{n-2}, \mathbf{i}_{n-3}, \ldots\right) \longrightarrow \mathbf{i}_n,$$

which operates as follows:

– the instantaneous value of $\mathbf{v}_n$ is calculated:

$$\mathbf{v}_n = \mathbf{e}_n - \sum_{m=1}^{\infty}\mathbf{z}_m\mathbf{i}_{n-m}, \tag{18}$$

– then the maximum instantaneous power:

$$p_n^{MAX} = \frac{1}{4}\mathbf{v}_n^T\mathbf{r}_0^{-1}\mathbf{v}_n,$$

– next the instantaneous fraction of the source's load:

$$x_n = \frac{p_n}{p_n^{MAX}},$$

– next the source's normative-instantaneous resistance:

$$r_n = \frac{\mathbf{v}_n^T\mathbf{v}_n}{\mathbf{v}_n^T\mathbf{r}_0^{-1}\mathbf{v}_n}$$

– next the instantaneous Lagrange's factor:

$$\lambda_n^{-1} = \begin{cases} r_n\dfrac{\sqrt{1-x_n}}{1-\sqrt{1-x_n}} & \text{for } x_n < 1 \\ 0 & \text{for } x_n > 1 \end{cases},$$

– and finally the instantaneous optimal current:

$$\mathbf{i}^{opt} = \frac{1}{2}\left(\lambda_n^{-1}\mathbf{1} + \mathbf{r}_0\right)^{-1}\mathbf{v}_n.$$

4. Examples for the single phase source

Two similar examples are used for the single phase source in the discrete time domain. In both of them the source's internal impedance was assumed as follows:

$$Z(s) = R + sL \xrightarrow[4f = 1/\tau]{s = f(1-z)} R + X_L - X_L z,$$

where: $R = 4$, $X_L = fL = 2$ (so-called digital reactance); from which the internal impedance sequence is obtained:

$$\{z_n\} = \{R + X_L; -X_L; 0; 0; \ldots\} = \{6, -2, 0, 0, \ldots\},$$

thus: $z_0 = r = 6$, $z_1 = -2$.

Also, in both cases, the constant sequence of the instantaneous power is assumed:

$$\{p_n\} = \{200, 200, 200, \ldots\}.$$

According to the internal impedance operator, voltage v_n will take the form:

$$v_n = e_n - z_1 i_{n-1}.$$

In Example 1 the source's non-sinusoidal voltage sequence is assumed as follows (Fig. 2):

$$\{e_n\} = \begin{Bmatrix} 80, 90, 100, 90, 80, -80, -90, -100, \\ -90, -80, 80, 90, 100, 90, 80, -80, \ldots \end{Bmatrix}.$$

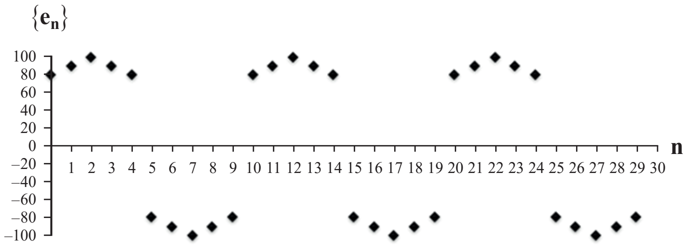

Fig. 2. The sequence of the source's voltage (Example 1)

The optimal online control algorithm begins with sample $n = 0$:

$$v_0 = e_0 - z_1 i_{-1} = e_0 = 80,$$

$$i_0^d = \frac{v_0}{2r} = 6{,}666667,$$

$$p_0^{MAX} = \frac{(v_0)^2}{4r} = 266{,}666667,$$

$$x_0 = \frac{p_0}{p_0^{MAX}} = 0{,}75,$$

$$i_0 = i_0^d\left(1 - \sqrt{1-x_0}\right) = 3{,}333333,$$

then sample $n = 1$:

$$v_1 = e_1 - z_1 i_0 = 96{,}666667,$$

$$i_1^d = \frac{v_1}{2r} = 8{,}055556,$$

$$p_1^{MAX} = \frac{(v_1)^2}{4r} = 389{,}351852,$$

$$x_1 = \frac{p_1}{p_1^{MAX}} = 0{,}513674,$$

$$i_1 = i_1^d\left(1 - \sqrt{1-x_1}\right) = 2{,}437848,$$

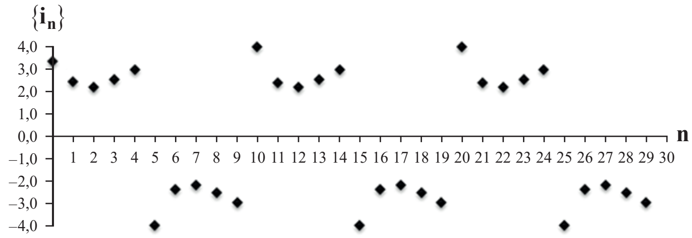

Fig. 3. The sequence of the optimal current (Example 1)

and so on. The optimal current $\left(i^2\right)^{MIN}$ is shown in Fig. 3.
In this example, all samples of current are optimal, which means that in each sample the required power p_n is smaller than the maximum power p_n^{MAX}.

In Example 2, the source's voltage sequence is assumed as follows (Fig. 4):

$$\{e_n\} = \begin{Bmatrix} 40,\ 80,\ 100,\ 50,\ 10,\ -40,\ -80,\ -100, \\ -50,\ -10,\ 40,\ 80,\ 100,\ 50,\ 10,\ \ldots \end{Bmatrix}.$$

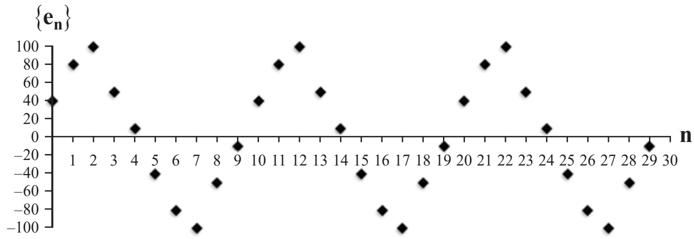

Fig. 4. The sequence of the source's voltage (Example 2)

The optimal online control algorithm for $n = 0$:

$$v_0 = e_0 - z_1 i_{-1} = e_0 = 40,$$

$$i_0^d = \frac{v_0}{2r} = 3{,}333333,$$

$$p_0^{MAX} = \frac{(v_0)^2}{4r} = 66{,}666667,$$

$$x_0 = \frac{p_0}{p_0^{MAX}} = 3,$$

$$i_0 = i_0^d = 3{,}333333,$$

and for $n = 1$:

$$v_1 = e_1 - z_1 i_0 = 86{,}666667,$$

$$i_1^d = \frac{v_1}{2r} = 7{,}222222,$$

$$p_1^{MAX} = \frac{(v_1)^2}{4r} = 312{,}962963,$$

$$x_1 = \frac{p_1}{p_1^{MAX}} = 0{,}639053,$$

$$i_1 = i_1^d\left(1 - \sqrt{1 - x_1}\right) = 2{,}883195,$$

and so on. The optimal current $\left(i^2\right)^{MIN}$ is shown in Fig. 5.

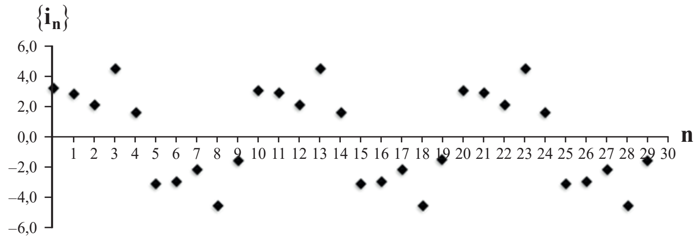

Fig. 5. The sequence of the optimal current (Example 2)

In sample $n = 0$ it can be seen that the source's maximum power is lower than required p_0, therefore the optimal current is limited to the adjustment current (delivering the maximum power from the source) (12).

5. Conclusion

A new current control algorithm for the real-voltage source was formulated in the paper. The discrete time convolution voltage-current model with the internal impedance operator was used. First, the single-phase model was considered, then the multiphase one. The optimal source's current control brings to searching such a current signal as to provide both its minimum square value all the time and the given instantaneous power value. The control algorithm is recursive, i.e. the actual samples are obtained from the preceding ones. For the single-phase source, the problem of seeking current signal that provides the given instantaneous power has the unequivocal solution and thus, the obtained current signal has minimum square value all the time. The multiphase source works in a different way. The vector space of currents which deliver the given instantaneous power have infinite dimension and only one of them has the minimum RMS value.

It may be concluded that the obtained optimal current–control algorithm is a recursive digital filter with infinite recursion. However, in practice the infinite recursion can be approximately replaced by a finite one. Due to the stability of inner impedance operator, it can be assumed that

$$\mathbf{z}_m = 0$$

(the zero matrix) for $m > M > 0$. Then the recursion (18) takes the finite form:

$$\mathbf{v}_n = \mathbf{e}_n - \sum_{m=1}^{\infty} \mathbf{z}_m \mathbf{i}_{n-m}.$$

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