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1D and 2D finite-difference operators for periodic functions on arbitrary mesh

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Abstract: This paper presents novel discrete differential operators for periodic functions of one- and two-variables, which relate the values of the derivatives to the values of the function itself for a set of arbitrarily chosen points over the function's area. It is very characteristic, that the values of the derivatives at each point depend on the function values at all points in that area. Such operators allow one to easily create finite-difference equations for boundary-value problems. The operators are addressed especially to nonlinear differential equations.

Key words: arbitrary meshes, finite-difference operators, partial finite difference operators, periodic functions, two-variable periodic functions

1. Introduction

The finite-difference method is an important approach for solving boundary-value problems of nonlinear ordinary and partial differential equations. In this approach, derivatives and partial derivatives are substituted by finite-difference operators and finite-difference algebraic equations are developed. A large class of discrete operators exists and has been presented in many books on numerical methods. A few exemplary are [1–6].

The finite-difference methods are still subject to investigations for quite different subjects. Some exemplary works from previous years are mentioned. The finite-difference method has been used in [7] to solve the 3D magnetic field problem. In [8], it has been proved that the finite-difference method with hexahedral elements and the edge element method, applied to magnetic field 3D problems, have common features. The new structures of finite-difference schemes are proposed in [9–11]. In [12], high-order finite-difference schemes for Navier-Stokes 2D equations are proposed. Trends combining the finite-difference method with other approaches also are observed. In [13], finite-difference and finite-element methods are mixed.

However, in all these books and papers the finite-difference operators are defined based on the values of the adjacent points with respect to the point, in which the derivative is determined. Existing finite-difference operators are created for the meshes rectangular for 2D or cuboidal for 3D problems, even if, the meshes are irregular with respect to an individual axis. This is an important disadvantage of the Finite-Difference Method (FDM), which does not occur in the case of the Finite-Element Method (FEM). This paper presents novel discrete Finite-Difference Operators (FDOs) dedicated to solving boundary value problems for 1D and 2D cases for arbitrary located points of the mesh. The first- and second-order FDOs are presented, applicable to a class of second-order Partial Differential Equations (PDEs), the most important for engineering applications.

The FDOs of periodic and two-periodic time functions are presented in [14] and [15] and have been successfully tested in [16, 17] for steady-state analysis of electromagnetic circuits. The same methodology has been used in [18] to develop FDOs to solve boundary-value problems for Ordinary Differential Equations (ODEs). The FDOs for 2D problems are shown in [19] and successfully tested in [20]. All those FDOs are determined for point sets regularly distributed over the rectangular function's area. This paper presents FDOs for an arbitrary chosen point set. An important advantage is that using new FDOs finite-difference equations can be created directly for a given PDE and any functional related to the PDE is not required.

2. Discrete differential operators for one-variable periodic function for an arbitrary point set

2.1. First-order discrete operator

If one assumes that the function $y(x)$ is approximated by the Fourier series with a limited number of terms

$$y(x) = y(x + 2\pi) = \sum_{r=-R}^{R} Y_r \cdot e^{\mathrm{j}rx}, \tag{1}$$

unique relations can be found between values and Fourier coefficients when selecting an arbitrary set of $(2R + 1)$ points: $\{x_1, x_2, \ldots, x_n, \ldots, x_{2R+1}\}$ for $0 < x_n < 2\pi$ (Fig. 1).

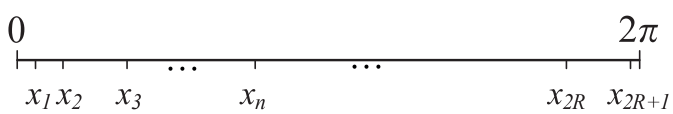

Fig. 1. An arbitrary point set $\{x_n\}$

This relation, after proper ordering, can be written as:

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{2R} \\ y_{2R+1} \end{bmatrix} = \begin{bmatrix} e^{\mathrm{j}\cdot R\cdot x_1} & \cdots & 1 & \cdots & e^{-\mathrm{j}\cdot R\cdot x_1} \\ e^{\mathrm{j}\cdot R\cdot x_2} & \cdots & 1 & \cdots & e^{-\mathrm{j}\cdot R\cdot x_2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ e^{\mathrm{j}\cdot R\cdot x_{2R}} & \cdots & 1 & \cdots & e^{-\mathrm{j}\cdot R\cdot x_{2R}} \\ e^{\mathrm{j}\cdot R\cdot x_{2R+1}} & \cdots & 1 & \cdots & e^{-\mathrm{j}\cdot R\cdot x_{2R+1}} \end{bmatrix} \begin{bmatrix} Y_R \\ \vdots \\ Y_0 \\ \vdots \\ Y_{-R} \end{bmatrix}, \quad y = F \cdot Y. \tag{2}$$

The $\boldsymbol{F}$ matrix is the square and non-singular.

The relationship between the Fourier coefficients of function (1) and its first derivative (3) can be written as:

$$\begin{gathered} \boldsymbol{Y}' = \mathrm{j}\boldsymbol{R}^{(1)} \cdot \boldsymbol{Y}, \\ \text{where} \quad \boldsymbol{Y}' = \begin{bmatrix} Y'_R & \cdots & Y'_1 & Y'_0 & Y'_{-1} & \cdots & Y'_{-R} \end{bmatrix}^T. \end{gathered} \tag{3}$$

The $\boldsymbol{R}^{(1)}$ matrix is the differential operator in the frequency domain and it is the diagonal

$$\boldsymbol{R}^{(1)} = \mathrm{diag}\begin{bmatrix} R & \cdots & 1 & 0 & -1 & \cdots & -R \end{bmatrix}^T.$$

To obtain the first-order deterministic design optimization (DDO), relating the values of function (1) and its first derivative, Eq. (3) can be rewritten in the form:

$$\boldsymbol{F} \cdot \boldsymbol{Y}' = \mathrm{j}\left(\boldsymbol{F} \cdot \boldsymbol{R}^{(1)} \cdot \boldsymbol{F}^{-1}\right) \cdot (\boldsymbol{F} \cdot \boldsymbol{Y}),$$

or shortly as:

$$\begin{gathered} \boldsymbol{y}' = \boldsymbol{D}^{(1)} \cdot \boldsymbol{y}, \\ \text{where} \quad \boldsymbol{y}' = \begin{bmatrix} y'_1 & y'_2 & \cdots & y'_{2R} & y'_{2R+1} \end{bmatrix}^T. \end{gathered} \tag{4}$$

The first-order DDO $\boldsymbol{D}^{(1)}$ is defined as:

$$\boldsymbol{D}^{(1)} = \boldsymbol{F} \cdot \left(\mathrm{j}\boldsymbol{R}^{(1)}\right) \cdot \boldsymbol{F}^{-1}. \tag{5}$$

The $\boldsymbol{D}^{(1)}$ matrix is the first-order discrete differential operator of the periodic function for an arbitrary set of discretization points. It is a singular skew-Hermitian matrix because its eigenvalues are purely imaginary and one eigenvalue equals zero.

$$\boldsymbol{D}^{(1)} = -\overset{*}{\left(\boldsymbol{D}^{(1)}\right)^T}. \tag{6}$$

It should be noticed that the $\boldsymbol{F}$ matrix becomes Hermitian when choosing the uniform set of points, for $\Delta x = 2\pi/(2R+1)$ [15]. Its invers matrix has the form:

$$\boldsymbol{F}^{-1} = \frac{1}{(2R+1)}\left(\overset{*}{\boldsymbol{F}^T}\right). \tag{7}$$

The operator $\boldsymbol{D}^{(1)}$ for arbitrary located points over period (6) requires finding the invers matrix $\boldsymbol{F}^{-1}$. To omit that operation the operator $\boldsymbol{D}^{(1)}$ can be developed based on the expressions defining a set of the Fourier coefficients $\{Y_R \quad \cdots \quad Y_1 \quad Y_0 \quad Y_{-1} \quad \cdots \quad Y_{-R}\}$.

$$Y_r = \frac{1}{2\pi} \int_0^{2\pi} \left(y(x) \cdot e^{-\mathrm{j}rx} \right) \mathrm{d}x \,. \tag{8}$$

These integrals can be approximated by a linear combination of function values

$$\{y_1,\ y_2,\ \ldots,\ y_r,\ \ldots,\ y_{2R+1}\} \,.$$

and written in the matrix form as:

$$\boldsymbol{Y} = \boldsymbol{F}_{\mathrm{int}} \cdot \boldsymbol{y} \,, \tag{9}$$

where

$$\boldsymbol{Y} = \begin{bmatrix} Y_R & \cdots & Y_1 & Y_0 & Y_{-1} & \cdots & Y_{-R} \end{bmatrix}^T$$

and

$$\boldsymbol{y} = [y_1,\ y_2,\ y_r,\ y_{2R+1}]^T \,.$$

In the simplest case, the integrals in (8) can be approximated by the sums

$$\begin{aligned} Y_r &= \frac{1}{2\pi} \sum_{l=1}^{2R+1} \int_{x_l}^{x_{l+1}} \left(y(x) \cdot e^{-\mathrm{j}rx} \right) \mathrm{d}x = \\ &= \frac{1}{2\pi} \sum_{l=1}^{2R+1} \left(\frac{1}{2} \left(y_{l+1} \cdot e^{-\mathrm{j}rx_{l+1}} + y_l \cdot e^{-\mathrm{j}rx_l} \right) \cdot (x_{l+1} - x_l) \right). \end{aligned} \tag{10}$$

This leads to the relation of (9) with a nonsingular $\boldsymbol{F}_{\mathrm{int}}$ matrix. Now, the discrete operator $\boldsymbol{D}^{(1)}$ is determined by the formula:

$$\boldsymbol{D}^{(1)} = \mathrm{j}\boldsymbol{F}_{\mathrm{int}}^{-1} \cdot \boldsymbol{R}^{(1)} \cdot \boldsymbol{F}_{\mathrm{int}} \,. \tag{11}$$

Combining those two definitions, the first-order DDO can be approximated by the formula:

$$\boldsymbol{D}^{(1)} \approx \mathrm{j}\boldsymbol{F} \cdot \boldsymbol{R}^{(1)} \cdot \boldsymbol{F}_{\mathrm{int}} \,, \tag{12}$$

eliminating operation of inversing matrices $\boldsymbol{F}$ or $\boldsymbol{F}_{\mathrm{int}}$.

2.2. Second-order discrete operator

The relationship between the Fourier coefficients of function (1) and its second derivative using (8) can be written as:

$$\boldsymbol{Y}'' = \mathrm{j}\boldsymbol{R}^{(1)} \cdot \boldsymbol{Y}' = -\boldsymbol{R}^{(2)} \cdot \boldsymbol{Y} \,, \tag{13}$$

where

$$\boldsymbol{R}^{(2)} = \boldsymbol{R}^{(1)} \cdot \boldsymbol{R}^{(1)} = \mathrm{diag} \begin{bmatrix} R^2 & \cdots & 4 & 1 & 0 & 1 & 4 & \cdots & R^2 \end{bmatrix}^T .$$

The second-order discrete operator $\boldsymbol{D}^{(2)}$ fulfilling the relation

$$\boldsymbol{y}'' = \boldsymbol{D}^{(2)} \cdot \boldsymbol{y} \tag{14}$$

can be obtained by the formulas:

$$\boldsymbol{D}^{(2)} = -\boldsymbol{F} \cdot (\boldsymbol{R}^{(2)}) \cdot \boldsymbol{F}^{-1},$$

or

$$\boldsymbol{D}^{(2)} = -F_{\text{int}}^{-1} \cdot (\boldsymbol{R}^{(2)}) \cdot \boldsymbol{F}_{\text{int}}\,,$$

or

$$\boldsymbol{D}^{(2)} \approx -\boldsymbol{F} \cdot (\boldsymbol{R}^{(2)}) \cdot \boldsymbol{F}_{\text{int}}\,. \tag{15}$$

All such operators can be extended for an N-dimensional vector-function $\boldsymbol{y}(x)$.

3. Discrete differential operators of two-variable periodic function for an arbitrary point set

3.1. First-order discrete partial differential operators

Discrete partial differential operators for the two-variable periodic function $z(x, y)$ can be specified using the Fourier series approach

$$z(x, y) = z(x + 2\pi, y) = z(x, y + 2\pi) = \sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} Z_{r,s} \cdot e^{\mathrm{j}rx} \cdot e^{\mathrm{j}sy}. \tag{16}$$

If one assumes that the sought function has first partial derivatives, it follows that

$$\begin{aligned} \frac{\partial z(x, y)}{\partial y} &= \sum_{s=-\infty}^{\infty} \left(\sum_{r=-\infty}^{\infty} \left(\mathrm{j} \cdot r \cdot Z_{r,s} \cdot e^{\mathrm{j} \cdot r \cdot x} \right) \right) \cdot e^{\mathrm{j} \cdot s \cdot y}, \\ \frac{\partial z(x, y)}{\partial y} &= \sum_{r=-\infty}^{\infty} \left(\sum_{s=-\infty}^{\infty} \left(\mathrm{j} \cdot s \cdot Z_{r,s} \cdot e^{\mathrm{j} \cdot s \cdot y} \right) \right) \cdot e^{\mathrm{j} \cdot r \cdot x}. \end{aligned} \tag{17}$$

To obtain the discrete differential operators relating the values of that function and its first partial derivatives, the relations between the values of the periodic function and its Fourier coefficients should be determined for the series in (17), when limited to a finite number of terms $-R < r < R$ and $-S < s < S$. For such a function, unique relations can be found between values of the function $z(x, y)$ and its Fourier coefficients $Z_{r,s}$, when selecting an arbitrary set of $(2R + 1) \cdot (2S + 1)$ points $\{x_n, y_n\}$, where: $0 < x_n < 2\pi$ for $n \in \{1, 2, \ldots, 2R + 1\}$, and $0 < y_n < 2\pi$ for $m \in \{1, 2, \ldots, 2S + 1\}$. It is illustrated in Fig. 2.

This relation can be written in the matrix form:

$$\boldsymbol{z} = \boldsymbol{F} \cdot \boldsymbol{Z}\,. \tag{18}$$

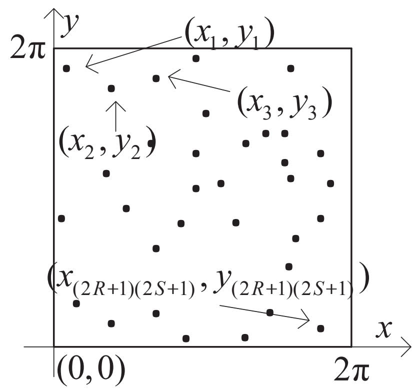

Fig. 2. An arbitrary set of points over the rectangular area

The vector z contains the function values z_n in the selected set of $(2R+1)\cdot(2S+1)$ points $(x_n y_n)$, ordered arbitrary from the first, numbered by $n = 1$, to the last, with the number $n = (2R+1)\cdot(2S+1)$.

$$z = \begin{bmatrix} z_1 & z_2 & \cdots & z_n & \cdots & z_{(2R+1)\cdot(2S+1)} \end{bmatrix}^T . \tag{19}$$

The vector $\boldsymbol{Z}$ contains the Fourier coefficients $Z_{r,s}$, limited to $|r| \le R$ and $|s| \le S$, and is arranged in the hyper-vectors form as follows:

$$\begin{aligned} \boldsymbol{Z} &= \begin{bmatrix} \boldsymbol{Z}_R & \cdots & \boldsymbol{Z}_1 & \boldsymbol{Z}_0 & \boldsymbol{Z}_{-1} & \cdots & \boldsymbol{Z}_{-R} \end{bmatrix}^T , \\ \boldsymbol{Z}_r &= \begin{bmatrix} Z_{r,S} & \cdots & Z_{r,1} & Z_{r,0} & Z_{r,-1} & \cdots & Z_{r,-S} \end{bmatrix}^T . \end{aligned} \tag{20}$$

The square, nonsingular matrix $\boldsymbol{F}$ combines those two sets of values and has the form:

$$\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_{(2R+1)\cdot(2R+1)} \end{bmatrix} = \begin{bmatrix} F_{R,1} & \cdots & F_{0,1} & \cdots & F_{R,1} \\ F_{R,2} & \cdots & F_{0,1} & \cdots & F_{R,2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ F_{R,(2R+1)\cdot(2R+1)} & \cdots & F_{0,(2R+1)\cdot(2R+1)} & \cdots & F_{R,(2R+1)\cdot(2R+1)} \end{bmatrix} \begin{bmatrix} z_R \\ \vdots \\ z_0 \\ \vdots \\ z_R \end{bmatrix} , \tag{21}$$

where

$$\boldsymbol{F}_{r,k} = \begin{bmatrix} e^{\mathrm{j}\cdot r\cdot x_n}\cdot e^{\mathrm{j}\cdot S\cdot y_n} & \cdots & e^{\mathrm{j}\cdot r\cdot x_n}\cdot e^{\mathrm{j}\cdot 0\cdot y_n} & \cdots & e^{\mathrm{j}\cdot r\cdot x_n}\cdot e^{-\mathrm{j}\cdot S\cdot y_n} \end{bmatrix} .$$

So, the invers relation also exists.

$$\boldsymbol{Z} = \boldsymbol{F}^{-1}\cdot z . \tag{22}$$

The relation in (18) is valid also for the first partial derivatives:

$$\begin{aligned} z'_x &= \boldsymbol{F}\cdot \boldsymbol{Z}'_x , \\ z'_y &= \boldsymbol{F}\cdot \boldsymbol{Z}'_y . \end{aligned} \tag{23}$$

The vectors $\boldsymbol{Z}'_x$ and $\boldsymbol{Z}'_y$ contain the Fourier coefficients of the respective partial derivatives from (17), and are ordered analogously as the $\boldsymbol{Z}$ vector.

$$\boldsymbol{Z}'_x = \begin{bmatrix}\boldsymbol{Z}'_{x,R} & \cdots & \boldsymbol{Z}'_{x,1} & \boldsymbol{Z}'_{x,0} & \boldsymbol{Z}'_{x,-1} & \cdots & \boldsymbol{Z}'_{x,-R}\end{bmatrix}^T,$$
$$\boldsymbol{Z}'_{x,r} = \begin{bmatrix}Z'_{x,r,S} & \cdots & Z'_{x,r,1} & Z'_{x,r,0} & Z'_{x,r,-1} & \cdots & Z'_{x,r,-R}\end{bmatrix}^T$$

and

$$\boldsymbol{Z}'_y = \begin{bmatrix}\boldsymbol{Z}'_{y,R} & \cdots & \boldsymbol{Z}'_{y,1} & \boldsymbol{Z}'_{y,0} & \boldsymbol{Z}'_{y,-1} & \cdots & \boldsymbol{Z}'_{y,-R}\end{bmatrix}^T,$$
$$\boldsymbol{Z}'_{y,r} = \begin{bmatrix}Z'_{y,r,S} & \cdots & Z'_{y,r,1} & Z'_{y,r,0} & Z'_{y,r,-1} & \cdots & Z'_{y,r,-R}\end{bmatrix}^T.$$

In addition, the relationships between the Fourier coefficients of the series in (17) can be easy find and written in the matrix forms:

$$\begin{aligned}\boldsymbol{Z}'_x &= \mathrm{j}\cdot\boldsymbol{R}_x^{(1)}\cdot\boldsymbol{Z},\\ \boldsymbol{Z}'_y &= \mathrm{j}\cdot\boldsymbol{R}_y^{(1)}\cdot\boldsymbol{Z}.\end{aligned} \tag{24}$$

The matrices $\boldsymbol{R}_x^{(1)}$ and $\boldsymbol{R}_y^{(1)}$ are the differential operators of a two-variable periodic function in frequency domain with respect to each of variables. They have hyper-diagonal forms. The matrix $\boldsymbol{R}_x^{(1)}$ is constituted by the $(2R+1)$ diagonal matrices $\boldsymbol{R}_r^{(1)}$.

$$\boldsymbol{R}_x^{(1)} = \mathrm{diag}\begin{bmatrix}\boldsymbol{R}_R^{(1)} & \cdots & \boldsymbol{R}_1^{(1)} & \boldsymbol{R}_0^{(1)} & \boldsymbol{R}_{-1}^{(1)} & \cdots & \boldsymbol{R}_{-R}^{(1)}\end{bmatrix},$$
$$\boldsymbol{R}_r^{(1)} = r\cdot\boldsymbol{E}_s,$$

where $\boldsymbol{E}_s$ is the unit matrix with dimensions $(2S+1)$. The matrix $\boldsymbol{R}_y^{(1)}$ is also constituted by the $(2R+1)$ diagonal matrices $\boldsymbol{R}_s^{(1)}$.

$$\boldsymbol{R}_y^{(1)} = \mathrm{diag}\begin{bmatrix}\boldsymbol{R}_s^{(1)} & \cdots & \boldsymbol{R}_s^{(1)} & \boldsymbol{R}_s^{(1)} & \boldsymbol{R}_s^{(1)} & \cdots & \boldsymbol{R}_s^{(1)}\end{bmatrix}.$$

The matrix $\boldsymbol{R}_s^{(1)}$ is diagonal, has dimensions $(2S+1)$ with respective harmonic numbers on the main diagonal

$$\boldsymbol{R}_s^{(1)} = \mathrm{diag}\begin{bmatrix}S & \cdots & 1 & 0 & -1 & \cdots & -S\end{bmatrix}.$$

The compounds relations of (18), (23) and (24) allow for writing:

$$\left(\boldsymbol{F}\cdot\boldsymbol{Z}'_x\right) = \mathrm{j}\cdot\left(\boldsymbol{F}\cdot\boldsymbol{R}_x^{(1)}\cdot\boldsymbol{F}^{-1}\right)\cdot\left(\boldsymbol{F}\cdot\boldsymbol{Z}\right),$$
$$\left(\boldsymbol{F}\cdot\boldsymbol{Z}'_y\right) = \mathrm{j}\cdot\left(\boldsymbol{F}\cdot\boldsymbol{R}_y^{(1)}\cdot\boldsymbol{F}^{-1}\right)\cdot\left(\boldsymbol{F}\cdot\boldsymbol{Z}\right).$$

These lead to relationships between the values of the first partial derivatives and of the function itself at the point set $\{x_n,\ y_n\}$.

$$\begin{aligned}\boldsymbol{z}'_x &= \boldsymbol{D}_x^{(1)}\cdot\boldsymbol{z},\\ \boldsymbol{z}'_y &= \boldsymbol{D}_y^{(1)}\cdot\boldsymbol{z}.\end{aligned} \tag{25}$$

The matrices $\boldsymbol{D}_x^{(1)}$ and $\boldsymbol{D}_y^{(1)}$ are the sought discrete partial differential operators of a two-variable function, periodic with respect to each of variables

$$\boldsymbol{D}_x^{(1)} = \mathrm{j} \cdot \boldsymbol{F} \cdot \boldsymbol{R}_x^{(1)} \cdot \boldsymbol{F}^{-1}, \qquad \boldsymbol{D}_y^{(1)} = \mathrm{j} \cdot \boldsymbol{F} \cdot \boldsymbol{R}_y^{(1)} \cdot \boldsymbol{F}^{-1}. \tag{26}$$

It should be notice that the $\boldsymbol{F}$ matrix becomes Hermitian when choosing the points regularly distributed [15].

The operators $\boldsymbol{D}_x^{(1)}$ and $\boldsymbol{D}_y^{(1)}$ can be developed based on the expressions defining a set of the Fourier coefficients of the series in (17).

$$Z_{r,s} = \left(\frac{1}{2\pi}\right)^2 \cdot \int\limits_0^{2\pi} \int\limits_0^{2\pi} \left(z(x,y) \cdot e^{-\mathrm{j}rx} \cdot e^{-\mathrm{j}sy} \right) \mathrm{d}x\, \mathrm{d}y \tag{27}$$

and finding the matrix $\boldsymbol{F}_{\mathrm{int}}$ satisfying the relation

$$\boldsymbol{Z} = \boldsymbol{F}_{\mathrm{int}} \cdot \boldsymbol{z} \tag{28}$$

analogously to Formula (9) for one-variable functions. The integrals in (27) can be approximated by the sums

$$Z_{r,s} \approx \left(\frac{1}{2\pi}\right)^2 \cdot \sum_k \frac{1}{3} S_k \cdot \Big(z_{k,1} \cdot e^{-\mathrm{j}r x_{k,1}} \cdot e^{-\mathrm{j}s y_{k,1}} + z_{k,2} \cdot e^{-\mathrm{j}r x_{k,2}} \cdot e^{-\mathrm{j}s y_{k,2}} + \\ + z_{k,3} \cdot e^{-\mathrm{j}r x_{k,3}} \cdot e^{-\mathrm{j}s y_{k,3}} \Big), \tag{29}$$

where S_k is the surface of an elementary triangle as it is shown in Fig. 3.

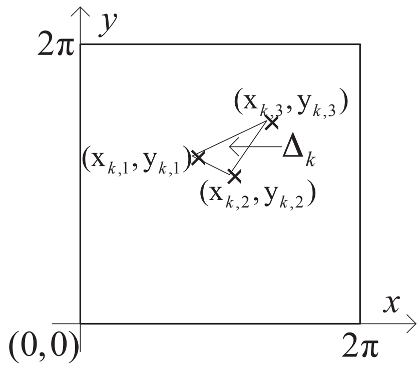

Fig. 3. An elementary triangle in the rectangular area

This surface can be calculated by the elementary formulas:

$$S_k = \sqrt{p_k \cdot (p_k - d_{k,1}) \cdot (p_k - d_{k,2}) \cdot (p_k - d_{k,3})},$$
$$p_k = \frac{1}{2} \left(d_{k,1} + d_{k,2} + d_{k,3} \right),$$

$$
\begin{aligned}
d_{k,1} &= \sqrt{\left(x_{k,2} - x_{k,1}\right)^2 + \left(y_{k,2} - y_{k,1}\right)^2}\,,\\
d_{k,2} &= \sqrt{\left(x_{k,3} - x_{k,2}\right)^2 + \left(y_{k,3} - y_{k,2}\right)^2}\,,\\
d_{k,3} &= \sqrt{\left(x_{k,1} - x_{k,3}\right)^2 + \left(y_{k,1} - y_{k,3}\right)^2}\,.
\end{aligned}
$$

In consequence, the operators $\boldsymbol{D}_x^{(1)}$ and $\boldsymbol{D}_y^{(1)}$ can take the form:

$$
\begin{aligned}
\boldsymbol{D}_x^{(1)} &= \mathrm{j}\,\boldsymbol{F}_{\mathrm{int}}^{-1} \cdot \boldsymbol{R}_x^{(1)} \cdot \boldsymbol{F}_{\mathrm{int}}\,,\\
\boldsymbol{D}_y^{(1)} &= \mathrm{j}\,\boldsymbol{F}_{\mathrm{int}}^{-1} \cdot \boldsymbol{R}_y^{(1)} \cdot \boldsymbol{F}_{\mathrm{int}}\,.
\end{aligned}
\tag{30}
$$

Omitting calculations of the inverse matrices $\boldsymbol{F}^{-1}$ in (26) or $\boldsymbol{F}_{\mathrm{int}}^{-1}$ in (30), the operators can be approximated by the formulas:

$$
\begin{aligned}
\boldsymbol{D}_x^{(1)} &\approx \mathrm{j}\,\boldsymbol{F} \cdot \boldsymbol{R}_x^{(1)} \cdot \boldsymbol{F}_{\mathrm{int}}\,,\\
\boldsymbol{D}_y^{(1)} &\approx \mathrm{j}\,\boldsymbol{F} \cdot \boldsymbol{R}_y^{(1)} \cdot \boldsymbol{F}_{\mathrm{int}}\,.
\end{aligned}
\tag{31}
$$

Such $\boldsymbol{D}_x^{(1)}$ and $\boldsymbol{D}_y^{(1)}$ matrices should have purely imaginary eigenvalues, i.e. should be close to skew-Hermitian and fulfil conditions $\boldsymbol{D}_x^{(1)} \approx -\left(\overset{*}{\boldsymbol{D}}{}_x^{(1)}\right)^T$ and $\boldsymbol{D}_y^{(1)} \approx -\left(\overset{*}{\boldsymbol{D}}{}_y^{(1)}\right)^T$. These conditions are equivalent to the requirement that a product of matrices $\boldsymbol{F}$ and $\boldsymbol{F}_{\mathrm{int}}$ should be close to the unit matrix $\boldsymbol{F} \cdot \boldsymbol{F}_{\mathrm{int}} \approx \boldsymbol{E}$.

3.2. Second-order discrete partial differential operators

Second-order discrete operators for the function $z(x, y)$ in (17), if the second-order partial derivatives exist, can be found based on the discrete operators developed in the previous chapters. Three discrete operators could be determined to three second-order partial derivatives when represented by their Fourier series:

$$
\begin{aligned}
\frac{\partial^2}{\partial x^2} z(x, y) &= -\sum_{s=-\infty}^{\infty}\left(\sum_{r=-\infty}^{\infty}\left(r^2 \cdot Z_{r,s} \cdot e^{\mathrm{j}rx}\right)\right) \cdot e^{\mathrm{j}sy},\\
\frac{\partial^2}{\partial y^2} &= -\sum_{r=-\infty}^{\infty}\left(\sum_{s=-\infty}^{\infty}\left(s^2 \cdot Z_{r,s} \cdot e^{\mathrm{j}sy}\right)\right) \cdot e^{\mathrm{j}rx},\\
\frac{\partial^2}{\partial x \partial y} &= -\sum_{r=-\infty}^{\infty} r \cdot \left(\sum_{s=-\infty}^{\infty}\left(s \cdot Z_{r,s} \cdot e^{\mathrm{j}sy}\right)\right) \cdot e^{\mathrm{j}rx}.
\end{aligned}
$$

Those operators should relate the values of respective derivatives to the values of the function in the point set $x_n y_m$, choosing as before. Those relations can be denoted as:

$$
\begin{aligned}
\boldsymbol{z}''^{xx} &= \boldsymbol{D}_{xx}^{(2)} \cdot \boldsymbol{z}\,,\\
\boldsymbol{z}''^{yy} &= \boldsymbol{D}_{yy}^{(2)} \cdot \boldsymbol{z}\,,\\
\boldsymbol{z}''^{xy} &= \boldsymbol{D}_{xy}^{(2)} \cdot \boldsymbol{z} = \boldsymbol{D}_{yx}^{(2)} \cdot \boldsymbol{z}\,.
\end{aligned}
\tag{32}
$$

The operators $\boldsymbol{D}_{xx}^{(2)}$, $\boldsymbol{D}_{yy}^{(2)}$ and $\boldsymbol{D}_{xy}^{(2)}$ can be found by multiplying the respective first-order operators

$$\begin{aligned} \boldsymbol{D}_{xx}^{(2)} &= \boldsymbol{D}_{x}^{(1)} \cdot \boldsymbol{D}_{x}^{(1)}, \\ \boldsymbol{D}_{yy}^{(2)} &= \boldsymbol{D}_{y}^{(1)} \cdot \boldsymbol{D}_{y}^{(1)}, \\ \boldsymbol{D}_{xy}^{(2)} &= \boldsymbol{D}_{x}^{(1)} \cdot \boldsymbol{D}_{y}^{(1)}. \end{aligned} \tag{33}$$

4. Conclusions

The new class of the discrete difference operators for periodic functions is presented in this paper. The main advantages of those operators are: they operate on an arbitrary discretization mesh and allow creating discrete-difference equations directly from the partial differential equation. It can make the finite-difference approach more flexible. It is also important that any functional is not necessary to create finite difference equations, which is required for the finite element method.

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