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On the monotonicity of the relaxation spectrum of fractional Maxwell model of viscoelastic materials

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Abstract. This article focuses on the relaxation spectrum of fractional Maxwell model, which is a generalization of classic viscoelastic Maxwell model to non-integer order derivatives. The analytical formula for the spectrum of relaxation frequencies is derived. Theoretical analysis of the relaxation spectrum monotonicity is conducted by using simple analytical methods and illustrated by means of numerical examples. The necessary and sufficient conditions for the existence and uniqueness of the maximum of relaxation spectrum are stated and proved. The analytical formulas for minimum and maximum of the relaxation spectrum are derived. Also, a few useful properties concerning the relaxation spectrum monotonicity and concavity are given in the mathematical form of simple inequalities expressed directly in terms of the fractional Maxwell model parameters, which can be used to simplify the calculations and analysis.

Key words: fractional calculus, viscoelasticity, fractional Maxwell model, relaxation spectrum, maximum of relaxation spectrum.

INTRODUCTION

Rheology is concerned with time-dependent deformation of solids and fluids [4,13]. For over five decades classical exponential behavior models such as Maxwell, Kelvin-Voight and Zener models have been used for mathematical modelling stress relaxation and creep processes [4,13,23]. For these models the relationship between the stress and deformation of the material is approximated though an ordinary differential [4,11,13] or integral [4,13,20] equations.

By replacing the springs and dashpots of the classical viscoelastic models with the Scott-Blair fractional elements, several fractional models, including the fractional Maxwell, fractional Voigt and fractional Kelvin models, have been proposed [3,7,18]. The fractional Maxwell model is, perhaps, the most representative example of such models. To this end, fractional rheological models have proven to be a concise and elegant framework for predicting the response of complex viscoelastic materials using a small number of parameters [7,17,18,24]. In this paper fractional Maxwell model is considered, which relates the stress to the strain in the material by means of using differential fractional equation [7,18,23] and admit the closed form of analytical solution in terms of the known Mittag-Leffler function [5].

The mechanical properties of linear viscoelastic materials are characterized by relaxation spectrum [2,4,9,10,17,23]. From the relaxation spectrum other material functions such as the relaxation modulus or the creep compliance can be calculated without difficulty, and next both the constant and time-variable bulk or shear modulus or Poisson's ratio can be determined. Thus, the spectrum is vital not only for constitutive models but also for the insight into the properties of a viscoelastic material [9,10,23].

The spectrum is the density of distribution of relaxation modulus. The maximum of the spectrum corresponds to the concentration of relaxation processes [2,17]. Thus, the estimation of the maximum and, in general, the analysis of the spectrum monotonicity is basic for detailed knowledge of mechanical material [13].

The aim of the paper is to develop a concise analytical formula describing the relaxation spectrum. To examine the relaxation spectrum monotonicity and, in particular, the maximum of the spectrum is also a basic concern.

FRACTIONAL MAXWELL MODEL

The elementary fractional Scott-Blair model [7] is described by the fractional differential equation:

$$\sigma(t) = E\tau^{\alpha}\frac{d^{\alpha}\varepsilon(t)}{dt^{\alpha}}, \tag{1}$$

where: $\sigma(t)$ and $\varepsilon(t)$ denotes the stress and strain, respectively, E and τ are the elastic modulus and relaxation time, α is non-integer positive order of fractional derivative of the strain $\varepsilon(t)$. Here, $d^\alpha/dt^\alpha = D_t^\alpha$ means the fractional derivative operator in the sense of Caputo fractional derivative of a function $f(x)$ of non-integer order α with respect to variable t and with the starting point at $t = 0$, which is defined by [8, 14]:

$$D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-1)^{n-\alpha-1} \frac{d^n}{dt^n} f(t) dt,$$

where: $n - 1 < \alpha < n$ and $\Gamma(n)$ is Euler's gamma function [8, 14]. The fractional Scott Blair model is an intermediate model between ideal spring $\sigma(t) = E\varepsilon(t)$ and the Newton's model $\sigma(t) = \eta \frac{d\varepsilon(t)}{dt}$ of ideal fluids represented by means of an ideal dashpot of viscosity η. To illustrate the structure of a fractional model, a fractional element must be introduced [3] – see Fig. 1a. Assuming unit-step strain $\varepsilon(t)$, the uniaxial stress response of a fractional element (1), i.e. the time-dependent relaxation modulus $G(t)$ is given by [7, 18]:

$$G(t) = \frac{E}{\Gamma(1-\alpha)} \left(\frac{t}{\tau}\right)^{-\alpha}.$$

Thus, the elementary fractional element is uniquely described by three parameters (E, τ, α), as shown in Fig. 1a.

The classic Maxwell model is a viscoelastic body that stores energy like a linearized elastic spring and dissipates energy like a classical fluid dashpot. Precisely, the classic viscoelastic Maxwell model is the arrangement of ideal spring in series with a dashpot (see Fig. 1b), described by the first order differential equation [4, 23]:

$$\frac{d\sigma(t)}{dt} + \frac{E}{\eta}\sigma(t) = E\frac{d\varepsilon(t)}{dt},$$

which for unit-step strain $\varepsilon(t)$ has exponential type response $G(t) = Ee^{-t/\tau}$, with the relaxation time $\tau = \eta/E$.

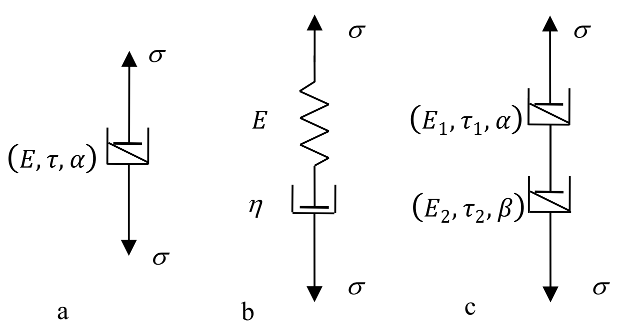

Fig. 1. Elementary fractional element (a) followed by the Maxwell model (b) and the fractional Maxwell model (c)

Connecting in series, by analogy to classic Maxwell model, two elementary fractional Scott-Blair elements (E_1, τ_1, α) and (E_2, τ_2, β) – see Fig. 1c – we obtain fractional Maxwell model described by the fractional differential equation [7, 18, 24]:

$$\tau^{\alpha-\beta} \frac{d^{\alpha-\beta}\sigma(t)}{dt^{\alpha-\beta}} + \sigma(t) = E\tau^\alpha \frac{d^\alpha \varepsilon(t)}{dt^\alpha}, \tag{2}$$

where the parameters of the FMM (Fractional Maxwell Model) are functions of the model components parameters given by [24]:

$$\tau = \left[\frac{E_1(\tau_1)^\alpha}{E_2(\tau_2)^\beta}\right]^{\frac{1}{\alpha-\beta}},$$

$$E = \left[\frac{(E_1\tau_1)^{-\beta}(\tau_1)^{\alpha(1-\alpha)}}{[E_2(\tau_2)^\beta]^{-\alpha}}\right]^{\frac{1}{\alpha-\beta}}.$$

The assumption $\alpha \geq \beta$ is taken, usually [7, 18]. For details of the fractional Maxwell model (2) construction see, e.g. [7]. For the unit-step strain the solution $\sigma(t) = G(t)$ of FMM (2) is known for an arbitrary $1 \geq \alpha \geq \beta \geq 0$ and given by the formula [7, 18]:

$$G(t) = E\left(\frac{t}{\tau}\right)^{-\beta} E_{\alpha-\beta,1-\beta}\left(-\left(\frac{t}{\tau}\right)^{\alpha-\beta}\right), \tag{3}$$

where $E_{a,b}(x)$ is the generalized Mittag-Leffler function defined by series representation, convergent in the whole z-complex plane [5, 14]:

$$E_{a,b}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(an+b)}. \tag{4}$$

RELAXATION SPECTRUM OF FMM

In the rheological literature it is commonly assumed that the modulus $G(t)$ has the following integral representation [4,13,23]:

$$G(t) = \int_0^\infty H(v)e^{-vt} dv, \tag{5}$$

where a non-negative relaxation spectrum $H(v)$ characterizes the distribution of relaxation frequencies $v \geq 0$. Equation (5) yields a formal definition of a relaxation spectrum [4, 21].

The spectrum representation of (5) guarantees that the modulus $G(t)$ is a completely monotone function, i.e. that $G(t)$ has derivatives of all orders and satisfies:

$$(-1)^n \frac{d^n}{dt^n} G(t) \geq 0 \quad \text{for all } t > 0 \text{ and } n = 0,1,2,\ldots.$$

The above means, in particular, that $G(t)$ is a monotonically decreasing function. The completely monotonic character of $G(t)$ is a necessary condition for the relaxation spectrum existence, however, it is not the sufficient one. The necessary and sufficient conditions of the existence of relaxation spectrum can be found in [21, 22].

In [19] Schneider proved that the generalized Mittag-Leffler function $E_{a,b}(-x)$, $x \geq 0$ was completely monotonic for $a > 0$, $b > 0$ if and only if $0 < a \leq 1$ and $b > a$. His proof is based on the application of the corresponding probability measures and the Hankel contour integration [19]. The same result was obtained by Miller and Samko [12] as an immediate corollary of the known Pollard's result [15] concerning the complete monotonicity of one parameter Mittag-Leffler function $E_a(x) = E_{a,1}(x)$.

Thus, the FMM relaxation modulus (3) is a completely monotone function, since the product of two completely monotone functions is completely monotone [1; Lemma 6]. The necessary condition of the existence of nonnegative relaxation spectrum is satisfied, see [21; Remark 5].

Let us define:

$$e_{a,b}(t;\lambda) = t^{b-1}E_{a,b}(-\lambda t^a). \tag{6}$$

In [6] for the case $0 < a \leq b < 1$, $\lambda > 0$, the following integral representation is obtained using the complex Bromwich formula to invert the Laplace transform of (6) and bending the Bromwich path into the Hankel path:

$$e_{a,b}(t;\lambda) = \int_0^\infty e^{-rt}K_{a,b}(r;\lambda)dr, \tag{7}$$

with the non-negative spectral function:

$$K_{a,b}(r;\lambda) = \frac{1}{\pi}\frac{\lambda\, sin[(b-a)\pi] + r^a sin(b\pi)}{r^{2a} + 2\lambda r^a cos(a\pi) + \lambda^2} r^{a-b}. \tag{8}$$

Note, that for $a = \alpha - \beta$, $b = 1 - \beta$ the above inequalities $0 < a \leq b < 1$ hold. Indeed:

$$0 < \alpha - \beta \leq 1 - \beta < 1,$$

iff (throughout, iff = if and only if) $0 < \beta < \alpha \leq 1$.

Putting in (6) $a = \alpha - \beta$, $b = 1 - \beta$ and $\lambda = 1$ the modulus (3) can be rewritten as:

$$G(t) = E e_{\alpha-\beta,1-\beta}\left(\frac{t}{\tau};1\right),$$

whence, in view of the spectral representation (7) we have:

$$G(t) = E\int_0^\infty e^{-r\frac{t}{\tau}}K_{\alpha-\beta,1-\beta}(r;1)dr,$$

and now, using the juxtposition $v = \frac{r}{\tau}$ we obtain:

$$G(t) = E\tau\int_0^\infty e^{-vt}K_{\alpha-\beta,1-\beta}(v\tau;1)dv.$$

Taking into account definition (5) the spectrum of FMM for the relaxation frequencies $v > 0$ is equal to:

$$H(v;\alpha,\beta) = E\tau K_{\alpha-\beta,1-\beta}(v\tau;1),$$

and in view of (8) turns out to be:

$$H(v;\alpha,\beta) = E\tau\frac{1}{\pi}\frac{sin[(1-\alpha)\pi] + (v\tau)^{\alpha-\beta}sin[(1-\beta)\pi]}{(v\tau)^{2(\alpha-\beta)} + 2(v\tau)^{\alpha-\beta}cos[(\alpha-\beta)\pi] + 1}(v\tau)^{\alpha-1}.$$

In view of the above and taking into account that $sin[(1-\alpha)\pi] = sin(\alpha\pi)$, the following result can be formulated.

Corollary 1. If $0 < \beta < \alpha \leq 1$, then the non-negative integrable relaxation spectrum $H(v;\alpha,\beta)$ of the fractional Maxwell model there exists and for $v > 0$ is given by the formula:

$$H(v;\alpha,\beta) = E\tau\frac{1}{\pi}\frac{sin(\alpha\pi) + (v\tau)^{\alpha-\beta}sin(\beta\pi)}{(v\tau)^{2(\alpha-\beta)} + 2(v\tau)^{\alpha-\beta}cos[(\alpha-\beta)\pi] + 1}(v\tau)^{\alpha-1}. \tag{9}$$

Denote for simplicity:

$$x = v\tau, \tag{10}$$

$$A = sin(\alpha\pi), \tag{11}$$

$$B = sin(\beta\pi), \tag{12}$$

$$C = cos[(\alpha-\beta)\pi]. \tag{13}$$

Under taken assumptions concerning α and β the two first parameters are such that $0 < A \leq 1$ and $0 < B \leq 1$, while the sign of C depends on specific values of α and β.

Let us define for real x the function:

$$\phi(x) = \frac{Ax^{\alpha-1} + Bx^{2\alpha-\beta-1}}{x^{2(\alpha-\beta)} + 2Cx^{\alpha-\beta} + 1}. \tag{14}$$

From (9), (10) and (14) we have:

$$H(v;\alpha,\beta) = E\tau\frac{1}{\pi}\phi(x)\Big|_{x=v\tau} = E\tau\frac{1}{\pi}\phi(v\tau). \tag{15}$$

Thus, in order to study the properties of the relaxation spectrum $H(v;\alpha,\beta)$ it is enough to analyze the properties of $\phi(x)$ (14). Moreover, according to the value of α, the analytical form of $H(v;\alpha,\beta)$ (9) have different expressions and properties. Thus, it is important to distinguish two different cases when α is equal to one or not.

MONOTONICITY. CASE $\alpha < 1$

Here we assume that $0 < \beta < \alpha < 1$. The analysis of the spectrum asymptotic properties as $v \to 0^+$ and $v \to \infty$, can be reduced to the study of the asymptotic properties of $\phi(x)$ as $x \to 0^+$ and $x \to \infty$, respectively. Since $\alpha - 1 < 0$ and $\alpha - \beta > 0$ we have:

$$lim_{x\to 0^+} \phi(x) = \lim_{x\to 0^+} \frac{Ax^{\alpha-1} + Bx^{2\alpha-\beta-1}}{x^{2(\alpha-\beta)} + 2Cx^{\alpha-\beta} + 1} = \infty,$$

regardless of the sign of the power $(2\alpha - \beta - 1)$, and:

$$lim_{x\to\infty} \phi(x) = \lim_{x\to\infty} \frac{Ax^{\beta-\alpha} + B}{x^{1-\beta} + 2Cx^{1-\alpha} + x^{1-2\alpha+\beta}} = 0.$$

Property 1. Let $0 < \beta < \alpha < 1$. For the relaxation spectrum $H(v;\alpha,\beta)$ (9) we have:

$$lim_{v\to 0^+} H(v;\alpha,\beta) = \infty, \tag{16}$$

and

$$lim_{v\to\infty} H(v;\alpha,\beta) = 0. \tag{17}$$

Thus, the spectrum of FMM is unbounded. In view of [21; Theorem 1] the last is not a surprise, since it can be proved that for the relaxation modulus (3) if $\alpha < 1$, then $G(t) \to \infty$ as $t \to 0^+$, i.e., the condition that $lim_{t\to 0^+} G(t) < \infty$ required to ensure the boundedness of the relaxation spectrum is not satisfied here, see [21; Theorem 1].

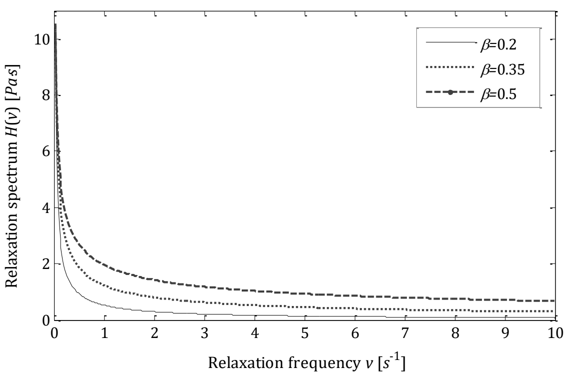

Fig. 2. Relaxation spectra of FMM for $\alpha = 0.6$, $\tau = 100\ [s]$, $E = 1\ [Pa]$

Two typical curves of relaxation spectrum $H(v;\alpha,\beta)$, shown in Fig. 2 and 3, plots the spectrum $H(v;\alpha,\beta)$ versus frequency and represent the two characteristic shapes – monotonically decreasing in Fig. 2 and having extrema in Fig. 3. The course of $H(v;\alpha,\beta)$ depends on the values of α and β parameters. To study the influence of the order parameters α and β on the spectrum, a more detailed analysis of the component function $\phi(x)$ (14), which allows for a deeper insight into the spectrum properties will be made. From (14) after straightforward manipulations the derivative $\phi'(x)$ can be obtained and expressed as:

$$\phi'(x) = x^{\alpha-2} \frac{\varphi(x)}{g^2(x)}, \tag{18}$$

the notation $\frac{d}{dx}\phi(x) = \phi'(x)$ is used for brevity, the numerator and denominator are given by:

$$\varphi(x) = c_3 x^{3(\alpha-\beta)} + c_2 x^{2(\alpha-\beta)} + c_1 x^{\alpha-\beta} + c_0, \tag{19}$$

and

$$g(x) = \left[x^{\alpha-\beta} + C\right]^2 + 1 - C^2, \tag{20}$$

with the coefficients defined as follows:

$$c_0 = (\alpha - 1)A, \tag{21}$$

$$c_1 = 2(\beta - 1)AC + (2\alpha - \beta - 1)B, \tag{22}$$

$$c_2 = 2(\alpha - 1)BC - (\alpha - 2\beta + 1)A, \tag{23}$$

$$c_3 = (\beta - 1)B. \tag{24}$$

It can be easily verified that for $0 < \beta < \alpha < 1$ both c_0 and c_3 are negative, while the signs of c_2 and c_1 depend on the values of α and β. Note that since in view of the definition (13) for $0 < \beta < \alpha < 1$ we have $-1 < C < 1$, the function $g(x)$ (20) is positive definite. Notice, that also the multiplicand $x^{\alpha-2}$ in (18) is positive for all $x > 0$, thus both the sign of the derivative $\phi'(x)$ as well as their real roots are identical to those of $\varphi(x)$.

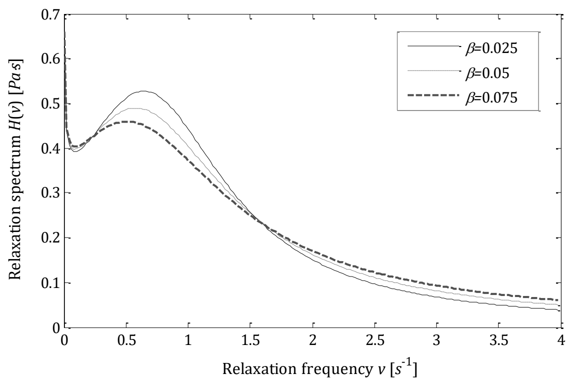

Fig. 3. Relaxation spectra of FMM for $\alpha = 0.8$, $\tau = 1\ [s]$, $E = 1\ [Pa]$

Let:

$$2\alpha-\beta-1 \le 0. \tag{25}$$

Since (25) means that $\alpha < \frac{\beta}{2}+\frac{1}{2}$, the next inequality results:

$$\alpha < \beta+\frac{1}{2}, \tag{26}$$

whence $(\alpha-\beta)\pi < \frac{\pi}{2}$, which implies that $C = cos[(\alpha-\beta)\pi] > 0$ and $c_1 < 0$. Since, under the assumption $0<\beta<\alpha<1$ we have $\alpha-2\beta+1>0$, also the coefficient $c_2<0$. Thus in (25) case, all the coefficients of the function $\varphi(x)$ (19) are negative, and in view of (18), the derivative $\phi'(x)<0$ for all $x>0$, whence, the relaxation spectrum $H(v;\alpha,\beta)$ (15) is monotonically decreasing function. From (18) in straightforward way we have:

$$\phi''(x) = \frac{(\alpha-2)x^{\alpha-3}\varphi(x)+x^{\alpha-2}\varphi'(x)}{[g(x)]^2} - \frac{2x^{\alpha-2}\varphi(x)g'(x)}{[g(x)]^3}, \tag{27}$$

where (20) gives:

$$g'(x) = 2(\alpha-\beta)x^{\alpha-\beta-1}\left[x^{(\alpha-\beta)}+C\right] > 0,$$

for all $x>0$ when (25) holds, whence the second summand of the right hand side of (27) is positive if $x>0$. From (19), taking into account (22)-(24), we have:

$$\varphi'(x) = (\alpha-\beta)x^{\alpha-\beta-1}\left(3c_3x^{2(\alpha-\beta)}+2c_2x^{\alpha-\beta}+c_1\right),$$

thus the numerator:

$$\psi(x) = (\alpha-2)x^{\alpha-3}\varphi(x)+x^{\alpha-2}\varphi'(x),$$

of the first summand of $\phi''(x)$ (27) in view of (19) and (22)-(24) is given by:

$$\psi(x) = (\alpha-2)x^{\alpha-3}\left[c_3x^{3(\alpha-\beta)}+c_2x^{2(\alpha-\beta)}+c_1x^{\alpha-\beta}+c_0\right] \\ +(\alpha-\beta)x^{\alpha-2}x^{\alpha-\beta-1}\left(3c_3x^{2(\alpha-\beta)}+2c_2x^{\alpha-\beta}+c_1\right),$$

and can be expressed as:

$$\psi(x) = x^{\alpha-3}\{(4\alpha-3\beta-2)c_3x^{3(\alpha-\beta)} \\ +(3\alpha-2\beta-2)c_2x^{2(\alpha-\beta)} \\ +(2\alpha-\beta-2)c_1x^{\alpha-\beta}+(\alpha-2)c_0\}. \tag{28}$$

The inequality (25) implies, in particular, that:

$$4\alpha-2\beta-2 \le 0,$$

what, in turn, implies $4\alpha-3\beta-2<0$, thus the coefficient $(4\alpha-3\beta-2)c_3>0$, since $c_3<0$. Next, from (25) and (26) we have:

$$3\alpha-2\beta-\frac{3}{2}<0,$$

what together with $c_2<0$ implies that $(3\alpha-2\beta-2)c_2>0$. The positivity of two next coefficients $(2\alpha-\beta-2)c_1$ and $(\alpha-2)c_0$ are obvious and the positive definiteness of $\psi(x)$ (28), and in consequence of $\phi''(x)$ (27) follows. Thus, the relaxation spectrum is convex in (25) case and the following sufficient condition is stated.

Corollary 2. Let $0<\beta<\alpha<1$. If additionally $2\alpha-\beta-1\le 0$, then the relaxation spectrum $H(v;\alpha,\beta)$ (9) is monotonically decreasing convex function.

The next property results immediately from Corollary 2 and the above analysis by contradiction.

Corollary 3. Let $0<\beta<\alpha<1$. If the relaxation spectrum $H(v;\alpha,\beta)$ (9) has local minimum and local maximum, then:

$$2\alpha-\beta-1>0. \tag{29}$$

Note that from (29) it follows, in particular, that $\alpha>\frac{1}{2}$.

Let us introduce a new variable:

$$y = x^{\alpha-\beta} > 0, \tag{30}$$

and define a new function:

$$\bar{\varphi}(y) = \bar{\varphi}\left(x^{\alpha-\beta}\right) = \varphi(x). \tag{31}$$

From (19) we have:

$$\bar{\varphi}(y) = c_3y^3+c_2y^2+c_1y+c_0, \tag{32}$$

where the coefficients are defined by (21)-(24).

In order to state the necessary and sufficient conditions for the existence of the relaxation spectrum $H(v;\alpha,\beta)$ extrema, note that from the stationary point condition follows that the spectrum has a maximum for the relaxation frequency $v=v_{max}>0$ iff the respective $y_{max}=(v_{max}\tau)^{\alpha-\beta}>0$ (compare (30) and (10)) is the root of the cubic function $\bar{\varphi}(y)$ (32). Similar property holds for the minimum frequency $v=v_{min}>0$ and corresponding $y_{min}=(v_{min}\tau)^{\alpha-\beta}>0$, also being the root of $\bar{\varphi}(y)$. Thus, the analysis of the relaxation spectrum extremal properties has been reduced to the analysis of the properties and roots of the cubic function (third order polynomial) $\bar{\varphi}(y)$ (32) for $y>0$.

In view of Property 1 (16), if the spectrum $H(v;\alpha,\beta)$ has maximum, a minimum exists too. Since $\bar{\varphi}(0)=c_0<0$ and $lim_{y\to-\infty}\bar{\varphi}(y)=\infty$, the cubic function $\bar{\varphi}(y)$ has at least one real root on the negative real axis. Thus, from the point of view of the course of the relaxation spectrum, and in particular from the point of view of the existence of its extrema, the existence or not of positive real roots of the function $\bar{\varphi}(y)$ is basic. The necessary and sufficient conditions of the existence of three real roots of third order polynomials are known, as well as the analytical methods for their computation. The algebraic solution of the cubic equation can be derived in a number of different ways. The Cardano's method dated 1545 and Vieta's method published in 1615 are the most known. The two methods are combined here and applied to the cubic equation $\bar{\varphi}(y) = 0$, which takes the standard form:

$$c_3 y^3 + c_2 y^2 + c_1 y + c_0 = 0. \tag{33}$$

Applying the standard substitution:

$$y = z - \frac{c_2}{3c_3}, \tag{34}$$

and dividing equation (33) by c_3 we get so-called depressed cubic equation with the zero quadratic term coefficient:

$$z^3 + 3pz + 2q = 0, \tag{35}$$

where:

$$3p = \frac{3c_3c_1 - [c_2]^2}{3[c_3]^2}, \tag{36}$$

$$2q = \frac{2[c_2]^3}{27[c_3]^3} - \frac{c_2c_1}{3[c_3]^2} + \frac{c_0}{c_3}. \tag{37}$$

The number and types of roots are uniquely determined by the determinant of the cubic equation defined as follows:

$$D = q^2 + p^3. \tag{38}$$

The depressed cubic equation (35) has three real roots iff $D \le 0$. If the determinant $D = 0$, then the equation (35), and whence (33), has a multiple real root, all of its roots are real. If $p = q = 0$, i.e., $3c_3c_1 - [c_2]^2 = 0$ and $2[c_2]^3 - 9c_3c_2c_1 + 27[c_3]^2c_0 = 0$, what implies $27[c_3]^2c_0 = [c_2]^3$, the triple root is such that $y_{1\_3} = -\frac{c_2}{3c_3} = -\frac{9c_3c_0}{[c_2]^2} < 0$ (for derivation the equation (39) below may be used). Thus $\bar{\varphi}(y) < 0$ for all $y > 0$. If $p^3 = -q^2 \neq 0$, then the equation (35) has two real roots, one of them is double. It may be proved that simple (single) root is negative. Even if the double root is positive, the function $\bar{\varphi}(y)$, and hence the derivative $\phi'(x)$ (18) is negative on both sides of the root. The relaxation spectrum decreases in the neighbourhood of respective relaxation frequency, what due to the asymptotic properties (16) and (17) results in the next property.

Corollary 4. Let $0 < \beta < \alpha < 1$. If α and β are such that the determinant $D = 0$, i.e.:

$$[2[c_2]^3 - 9c_3c_2c_1 + 27[c_3]^2c_0]^2 = -4[3c_3c_1 - [c_2]^2]^3,$$

where the coefficients c_i, $i = 0,1,2,3$ are defined by (21)-(24), then the relaxation spectrum $H(v;\alpha,\beta)$ (9) is monotonically decreasing function for $v > 0$.

If $D < 0$, then the cubic equation (35) roots are obtained by Viète's formulas [16] in terms of trigonometric functions (except when $p = 0$, but it is not the case for $D < 0$), which in view of (34) for the original third order equation (35) results in the three different real roots:

$$y_1 = -2r\,cos\left(\frac{\theta}{3}\right) - \frac{c_2}{3c_3}, \tag{39}$$

$$y_2 = 2r\,cos\left(\frac{\pi-\theta}{3}\right) - \frac{c_2}{3c_3}, \tag{40}$$

$$y_3 = 2r\,cos\left(\frac{\pi+\theta}{3}\right) - \frac{c_2}{3c_3}, \tag{41}$$

where:

$$r = sgn(q)\sqrt{|p|},$$

and the angle θ is such that:

$$cos(\theta) = \frac{q}{r^3}. \tag{42}$$

Note, that $D < 0$, which in view of (38) implies $p < 0$, and next:

$$0 \le |q| < |p|\sqrt{|p|},$$

guarantee that $cos(\theta)$ given by (42) is such that:

$$0 \le cos(\theta) = \frac{q}{sgn(q)|p|\sqrt{|p|}} = \frac{|q|}{|p|\sqrt{|p|}} < 1,$$

whence:

$$0 < \theta = arc\,cos\left(\frac{q}{r^3}\right) \le \frac{\pi}{2}.$$

If $q > 0$, then $r > 0$ and the inequalities occur:

$$-2r\,cos\left(\frac{\theta}{3}\right) < 0 < 2r\,cos\left(\frac{\pi+\theta}{3}\right) < 2r\,cos\left(\frac{\pi-\theta}{3}\right),$$

whence we have:

$$y_1 < y_3 < y_2.$$

If $q < 0$, then $r < 0$ and we have the inequalities:

$$-2r\,cos\left(\frac{\theta}{3}\right) > 0 > 2r\,cos\left(\frac{\pi+\theta}{3}\right) > 2r\,cos\left(\frac{\pi-\theta}{3}\right),$$

whence the relation follows:

$$y_1 > y_3 > y_2.$$

If $q = 0$, then $r > 0$ and $cos(\theta) = 0$, whence $\theta = \frac{\pi}{2}$ and the three real roots are such that:

$$y_1 < y_3 < y_2.$$

The basic property has been proved.

Theorem 1. Let $0 < \beta < \alpha < 1$. If α and β are such that the determinant $D < 0$, i.e.:

$$[2[c_2]^3 - 9c_3c_2c_1 + 27[c_3]^2c_0]^2 < -4[3c_3c_1 - [c_2]^2]^3, \quad (43)$$

where the coefficients c_i, $i = 0,1,2,3$ are defined by (21)-(24), then the relaxation spectrum $H(v;\alpha,\beta)$ (9) has local minimum $v_{min} = \frac{1}{\tau}(y_3)^{\frac{1}{\alpha-\beta}}$ and local maximum v_{max} given by:

$$v_{max} = \frac{1}{\tau}(y_2)^{\frac{1}{\alpha-\beta}} \quad \text{if} \quad q \geq 0,$$

$$v_{max} = \frac{1}{\tau}(y_1)^{\frac{1}{\alpha-\beta}} \quad \text{if} \quad q < 0.$$

Note that the necessary and sufficient condition for the existence of local extrema is the inequality $D < 0$, which itself implies that $p < 0$. Whence, by contradiction the next property holds.

Corollary 5. Let $0 < \beta < \alpha < 1$. If α and β are such that $p \geq 0$, which in view of (36) takes the form $3c_3c_1 - [c_2]^2 \geq 0$, then the relaxation spectrum $H(v;\alpha,\beta)$ (9) does not have local minimum and local maximum, i.e. is monotonically decreasing function.

The inequality $p \geq 0$ obviously implies $D \geq 0$, thus the condition from Corollary 4 is a special case of that stated above.

The useful necessary, but not sufficient, simple condition for the existence of relaxation spectrum local maximum and minimum also results:

$$3c_3c_1 - [c_2]^2 < 0, \quad (44)$$

which is satisfied, in particular, if $c_1 > 0$. Thus, checking if (44) holds, avoids calculation of the determinant D in the case, when it is not satisfied.

MONOTONICITY. CASE $\alpha = 1$

Up to now, the case where $\alpha \neq 1$ has been considered. In assuming that $\alpha = 1$, the relaxation spectrum $H(v;\alpha,\beta)$ (9) is described by new analytical formula:

$$H_1(v;\beta) = H(v;\alpha = 1,\beta) = E\tau\frac{1}{\pi}\frac{(v\tau)^{1-\beta}sin[\beta\pi]}{(v\tau)^{2(1-\beta)} - 2(v\tau)^{1-\beta}cos[\beta\pi] + 1}. \quad (45)$$

Now, the asymptotic properties of the spectrum $H_1(v;\beta)$ are summarized in next property.

Property 2. Let $0 < \beta < \alpha = 1$. The relaxation spectrum $H_1(v;\beta)$ (45) is such that:

$$lim_{v\to 0^+} H_1(v;\beta) = 0,$$

and

$$lim_{v\to\infty} H_1(v;\beta) = 0.$$

By analogy to general model (15) the further analysis of spectrum $H_1(v;\beta)$ (45) will use the multiplicative form:

$$H_1(v;\beta) = E\tau\frac{1}{\pi}\phi_1(x)\Big|_{x=v\tau} = E\tau\frac{1}{\pi}\phi_1(v\tau),$$

here the function $\phi_1(x)$, being an analogue of $\phi(x)$ (14), is given for $x > 0$ by:

$$\phi_1(x) = \frac{Bx^{1-\beta}}{x^{2(1-\beta)} + 2Cx^{1-\beta} + 1}, \quad (46)$$

where now $C = cos[\beta\pi]$. From (46) in straightforward way we have:

$$\phi_1'(x) = (1-\beta)B\frac{x^{-\beta}[-x^{2(1-\beta)} + 1]}{[x^{2(1-\beta)} + 2x^{1-\beta}C + 1]^2}.$$

The stationary point equation $\phi_1'(x) = 0$ has for $x > 0$ unique real solution $x = 1$, and since $\phi_1'(x) > 0$ for $0 < x < 1$ and $\phi_1'(x) < 0$ for $x > 1$, the component function $\phi_1(x)$ has maximum at $x = 1$. Thus, due to (10), the relaxation spectrum has maximum at:

$$v_{max} = \frac{1}{\tau},$$

and the maximal value of the spectrum:

$$H_{1,max} = H_1(v_{max};\beta) = E\tau\frac{1}{2\pi}\frac{sin(\beta\pi)}{1-cos(\beta\pi)}. \qquad (47)$$

While v_{max} depends only on the relaxation time of the FMM, the maximum $H_{1,max}$ is expressed in terms of all model parameters, and in particular depend on the order parameter β. Based on the last expression in the right hand side of (47) we have:

$$\frac{d}{d\beta}H_{1,max} = E\tau\frac{1}{2\pi}\frac{\beta}{cos(\beta\pi)-1}.$$

Hence, $\frac{d}{d\beta}H_{1,max} < 0$ for any $0 < \beta < 1$. Thus the bigger is β, the lower is $H_{1,max}$ (47) – see Fig. 4. The greater is β, the more restricted is the spectrum and the higher its maximum is. Thus, the order parameter β characterizes the 'height' of the spectrum and its 'width'.

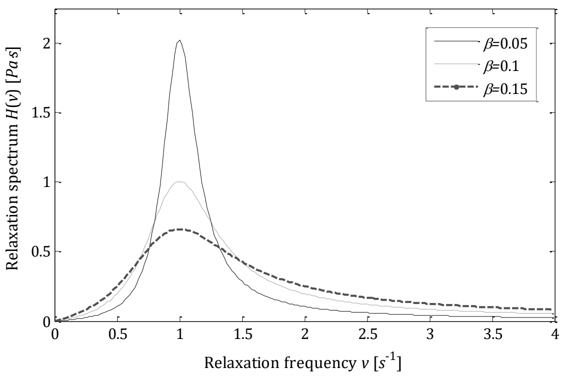

Fig. 4. Relaxation spectra of fractional Maxwell model for $\alpha = 1, E = 1\,[Pa], \tau = 1\,[s]$

Theorem 2. Let $0 < \beta < \alpha = 1$. The relaxation spectrum $H_1(v;\beta)$ (45) has unique maximum for the frequency $v_{max} = 1/\tau$ equal to $H_{1,max}$ (47).

FINAL REMARKS

Two different cases, when the derivative order α is equal to one and when it is not, have been considered separately because of the different mathematical formula and properties of relaxation spectrum, especially its boundedness and monotonicity. It has been proved that the relaxation spectrum monotonicity character is uniquely determined by the sign of the determinant D of the cubic stationary point equation, which also uniquely determines the number and types of its roots. The analytical formulas for the local minimum and local maximum of the spectrum have been given. Also, some necessary conditions for the local extrema existence and sufficient conditions for the monotonic convex decreasing character of the spectrum have been derived in the form of useful simple inequalities expressed directly in terms of the FMM parameters, which do not require the determinant computation, and thus could be used to simplify the analysis.

In the special case of order $\alpha = 1$ it has been proved that the greater the maximum density of spectrum is, the lower the β parameter is. In this case the maximum density of spectrum is independent on other FMM parameters. The analysis of the influence of FMM parameters on the spectrum maximum will be the subject of future research. In many issues of the dynamical properties the analysis of relaxation spectrum properties plays an important role, thus the investigations are useful wherever the determination of mechanical properties of rheological materials [11,17,20,25] is important from a cognitive and engineering point of view.

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