ON THE DEFORMED BESOV–HANKEL SPACES

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Abstract. In this paper we introduce function spaces denoted by $BH_{\kappa,\beta}^{p,r}$ ($0 < \beta < 1$, $1 \le p, r \le +\infty$) as subspaces of L^p that we call deformed Besov–Hankel spaces. We provide characterizations of these spaces in terms of Bochner–Riesz means in the case $1 \le p \le +\infty$ and in terms of partial Hankel integrals in the case $1 < p < +\infty$ associated to the deformed Hankel operator by a parameter $\kappa > 0$. For $p = r = +\infty$, we obtain an approximation result involving partial Hankel integrals.

Keywords: deformed Hankel kernel, Besov spaces, Bochner–Riesz means, partial Hankel integrals.

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1. INTRODUCTION

In [2] the author introduced a new transform $\mathcal{F}_\kappa$ called deformed Hankel transform which is a deformation of the Hankel transform by a parameter $\kappa > 0$. Namely, for $\kappa > \frac{1}{4}$ and $f \in L^1(\mathbb{R}, d\mu_\kappa)$ $(d\mu_\kappa(x) = 2^{-1}\Gamma(2\kappa)^{-1}|x|^{2\kappa-1}dx)$, the integral transform $\mathcal{F}_\kappa$ is defined by

$$\mathcal{F}_\kappa(f)(\lambda) = \int_{\mathbb{R}} B_\kappa(\lambda, x)f(x)d\mu_\kappa(x), \quad \lambda \in \mathbb{R},$$

where the kernel $B_\kappa(\lambda, x)$ called deformed Hankel kernel is given by

$$B_\kappa(\lambda, x) = j_{2\kappa-1}(2\sqrt{|\lambda x|}) - \frac{\lambda x}{2\kappa(2\kappa+1)}j_{2\kappa+1}(2\sqrt{|\lambda x|}). \tag{1.1}$$

Here j_α is the modified Bessel function of the first kind and order α.

The same author established a product formula for the kernels $B_\kappa(\lambda, x)$, which induces a translation operator T_y^κ (see [2, §1] for more details).

In this paper we are concerned with the characterization of the deformed Besov––Hankel spaces. Our aim is to show that we can characterize the deformed Besov–Hankel spaces by means of Bochner–Riesz means and partial Hankel integrals.

Let $0 < \beta < 1$, $1 \leq p, r \leq +\infty$. We say that a measurable function f on $\mathbb{R}$ is in the deformed Besov–Hankel space $BH^{p,r}_{\kappa,\beta}$ if $f \in L^p(\mathbb{R}, d\mu_\kappa)$ and

$$\int\limits_0^{+\infty} \left(\frac{w_p(f,t)}{t^\beta} \right)^r \frac{dt}{t} < +\infty, \quad 1 \leq r < +\infty,$$

$$\operatorname*{ess\,sup}_{t>0} \left(\frac{w_p(f,t)}{t^\beta} \right) < +\infty, \quad r = +\infty,$$

where

$$w_p(f,t) = \|T^\kappa_t f + T^\kappa_{-t} f - 2f\|_{\kappa,p}, \quad t \in \mathbb{R},$$

denotes the modulus of continuity of second order of f and $\|\cdot\|_{\kappa,p}$ denotes the norm of the Lebesgue space $L^p(\mathbb{R}, d\mu_\kappa)$.

Note that the modulus of continuity has been used to characterize Sobolev spaces (see [6]).

The space $BH^{\infty,\infty}_{\kappa,\beta}$ is called the deformed Lipschitz–Hankel space denoted $\Lambda_{\kappa,\beta}$, defined by

$$\Lambda_{\kappa,\beta} = \left\{ f \in L^\infty(\mathbb{R}, d\mu_\kappa) : \operatorname*{ess\,sup}_{t>0} \left(\frac{w_\infty(f,t)}{t^\beta} \right) < +\infty \right\}.$$

We introduce the deformed Bochner–Riesz means σ^γ_T, where $T > 0$ and $\gamma \geq 0$, as operators on $L^1(\mathbb{R}, d\mu_\kappa)$ defined by

$$\sigma^\gamma_T f(x) = \int\limits_{-T}^{T} B_\kappa(\lambda, x) \left(1 - \frac{|\lambda|}{T} \right)^\gamma \mathcal{F}_\kappa f(\lambda) d\mu_\kappa(\lambda), \quad x \in \mathbb{R}.$$

$S_T = \sigma_{T,0}$ is called the deformed partial Hankel integral.

The contents of this paper are as follows.

In Section 2, we collect some results about harmonic analysis associated with the deformed Hankel transform.

In Section 3, by using the deformed Hankel transform $\mathcal{F}_\kappa$, we define the Bochner–Riesz mean σ^γ_T, where $T > 0$ and $\gamma \geq 0$ as an operator on $L^1(\mathbb{R}, d\mu_\kappa)$. Whenever $\gamma > 2\kappa - \frac{1}{2}$, σ^γ_T is a convolution operator on $L^1(\mathbb{R}, d\mu_\kappa)$. Next, we extend the definition of σ^γ_T on $L^p(\mathbb{R}, d\mu_\kappa)$, $1 \leq p < +\infty$ provided that $\gamma > 2\kappa - \frac{1}{2}$. Similarly, we define the partial deformed Hankel integral S_T, $T > 0$ on $L^p(\mathbb{R}, d\mu_\kappa)$, $1 \leq p < \frac{8\kappa}{4\kappa-1}$.

In Section 4, we introduce the deformed Besov–Hankel spaces $BH^{p,r}_{\kappa,\beta}$, $0 < \beta < 1$ and $1 \leq p, r \leq +\infty$. We provide their characterizations. Firstly, for $0 < \beta < 1$ and $1 \leq p, r \leq +\infty$, we give equivalent properties in terms of the deformed Bochner–Riesz mean ensuring that a function $f \in BH^{p,r}_{\kappa,\beta}$. Next, for $0 < \beta < 1$ and $\frac{8\kappa}{4\kappa+1} < p < \frac{8\kappa}{4\kappa-1}$, we derive equivalent assertions involving the deformed partial Hankel integrals. Finally,

for $p = r = +\infty$ we obtain equivalent results and also an approximation result involving partial Hankel integrals.

Analogous results were obtained by Giang and Móricz in [9] for the classical Fourier transform on $\mathbb{R}$, by Betancor and Rodríguez-Mesa in [3–5] for the Hankel transform on $(0, +\infty)$ and by Kamoun in [11] and Kamoun and Negzaoui in [12] for the Dunkl transform on the real line. Characterizations of other functions like Orlicz–Sobolev and Sobolev spaces has been studied by Rădulescu *et al.* (see [6, 13] and [15]).

Note that throughout the paper, C is a positive constant that can change from one line to another.

2. HARMONIC ANALYSIS ASSOCIATED WITH THE DEFORMED HANKEL TRANSFORM

2.1. PRELIMINARIES ON BESSEL FUNCTIONS

In this subsection we recall some properties of the Bessel functions and the normalized Bessel functions of the first kind and order α.

Let $\alpha \in \mathbb{R}$ such that $\alpha > -\frac{1}{2}$. The Bessel function of the first kind and order α is defined on $]0, +\infty[$ by

$$J_\alpha(x) = \left(\frac{x}{2}\right)^\alpha \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!\Gamma(n + \alpha + 1)} \left(\frac{x}{2}\right)^{2n}.$$

The function J_α possesses the following asymptotic behavior:

$$J_\alpha(x) \sim \frac{x^\alpha}{2^\alpha \Gamma(\alpha + 1)}, \quad \text{when } x \to 0^+ \tag{2.1}$$

and

$$J_\alpha(x) = \sqrt{\frac{2}{\pi x}} \cos\left(x - (2\alpha + 1)\frac{\pi}{4}\right) + \theta\left(\frac{1}{x^{3/2}}\right), \quad \text{when } x \to +\infty. \tag{2.2}$$

We also recall some useful integrals involving J_α. For $a \in \mathbb{R}_+^*$, we have (see [7, p. 49])

$$\int_0^{+\infty} J_\alpha(at) t^{\rho-1} dt = 2^{\rho-1} a^{-\rho} \frac{\Gamma\left(\frac{1}{2}\alpha + \frac{1}{2}\rho\right)}{\Gamma\left(1 + \frac{1}{2}\alpha - \frac{1}{2}\rho\right)}. \tag{2.3}$$

For $x, y \in \mathbb{R}$ and $T > 0$, we have (see [18, §5.1(8)])

$$\int_0^T J_\alpha(\lambda|x|) J_\alpha(\lambda|y|) \lambda d\lambda = \frac{T|x| J_{\alpha+1}(T|x|) J_\alpha(T|y|) - T|y| J_{\alpha+1}(T|y|) J_\alpha(T|x|)}{x^2 - y^2}. \tag{2.4}$$

For the normalized Bessel function j_α of the first kind and order α, we recall that

$$j_\alpha(u) = 2^\alpha \Gamma(\alpha+1) \frac{J_\alpha(u)}{u^\alpha} = \Gamma(\alpha+1) \sum_{m=0}^{+\infty} \frac{(-1)^m \left(\frac{u}{2}\right)^{2m}}{m! \Gamma(\alpha+m+1)}. \tag{2.5}$$

Its representation integral formula is the following:

$$j_\alpha(\lambda x) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})} \int_{-1}^{1} (1-t^2)^{\alpha-\frac{1}{2}} e^{i\lambda x t} dt, \quad \lambda, x \in \mathbb{R}. \tag{2.6}$$

Now we recall Sonine's integral formula. Let $\alpha, \beta \in \mathbb{R}$ be such that $\alpha > \beta > -\frac{1}{2}$, then

$$j_\alpha(\lambda x) = \frac{2\Gamma(\alpha+1)}{\Gamma(\beta+1)\Gamma(\alpha-\beta+1)} \int_{0}^{1} (1-t^2)^{\alpha-\beta-1} j_\beta(\lambda x t) t^{2\beta+1} dt, \quad x \in \mathbb{R}, \ \lambda \in \mathbb{C}. \tag{2.7}$$

The product formula for j_α is given by

$$j_\alpha(\lambda x) j_\alpha(\lambda y) = \int_{0}^{\infty} j_\alpha(\lambda z) W_\alpha(x, y, z) z^{2\alpha+1} dz, \quad x, y \in \mathbb{R}_+^*, \tag{2.8}$$

where

$$W_\alpha(x, y, z) = \frac{\Gamma(\alpha+1)}{2^{2\alpha-1}\Gamma(\alpha+\frac{1}{2})\sqrt{\pi}} \frac{\left[((x+y)^2 - z^2)(z^2 - (x-y))^2\right]^{\alpha-\frac{1}{2}}}{(xyz)^{2\alpha}} \mathbb{1}_{]|x-y|, x+y[}(z). \tag{2.9}$$

Here $\mathbb{1}_A$ is the characteristic function of the set A.

We conclude this subsection by the derivative and the three-term recurrence formula

$$\frac{d}{du} j_\alpha(u) = -\frac{1}{2(\alpha+1)} u j_{\alpha+1}(u),$$

$$u^2 j_{\alpha+2}(u) = 4(\alpha+1)(\alpha+2)(j_{\alpha+1}(u) - j_\alpha(u)), \quad u \in \mathbb{R}_+. \tag{2.10}$$

2.2. THE DEFORMED HANKEL CONVOLUTION

In this subsection, we investigate a convolution product associated the deformed Hankel kernel.

For $x, y \in \mathbb{R}^*$ fixed, we denote by K_κ the function defined by

$$K_\kappa(x, y, z) = 2\Gamma(2\kappa) W_{2\kappa-1}(\sqrt{|x|}, \sqrt{|y|}, \sqrt{|z|}) \nabla_\kappa(x, y, z), \tag{2.11}$$

where $W_{2\kappa-1}$ is the positive kernel given by (2.9) with $\alpha = 2\kappa - 1$,

$$\nabla_\kappa(x, y, z) = \frac{1}{4}\left\{ 1 + \frac{\operatorname{sgn}(xy)}{4\kappa - 1}\left[4\kappa\Delta(|x|, |y|, |z|)^2 - 1 \right] \right.$$
$$+ \frac{\operatorname{sgn}(xz)}{4\kappa - 1}\left[4\kappa\Delta(|z|, |x|, |y|)^2 - 1 \right]$$
$$\left. + \frac{\operatorname{sgn}(yz)}{4\kappa - 1}\left[4\kappa\Delta(|z|, |y|, |x|)^2 - 1 \right] \right\}.$$

and

$$\Delta(u, v, w) = \frac{1}{2\sqrt{uv}}(u + v - w), \quad u, v, w \in \mathbb{R}_+^*.$$

Remark 2.1.

(i) The function $z \to K_\kappa(x, y, z)$ is supported on

$$(\sqrt{|x|} - \sqrt{|y|})^2 < |z| < (\sqrt{|x|} + \sqrt{|y|})^2.$$

(ii) The kernel $K_\kappa(x, y, z)$ is not well-defined as a function if x or y is zero; in this case $K_\kappa(0, y, z)|z|^{2\kappa-1}dz$ (resp. $K_\kappa(x, 0, z)|z|^{2\kappa-1}dz$) can be considered as $d\delta_y(z)$ (resp. $d\delta_x(z)$), δ_x being the Dirac measure.

Theorem 2.2 (see [2]). *Let $\kappa > \frac{1}{4}$. Then the following assertions hold.*

(i) *For any $\lambda \in \mathbb{R}$ and $x, y \in \mathbb{R}$, the product formula for the kernels B_κ is of the form*

$$B_\kappa(\lambda, x)B_\kappa(\lambda, y) = \int_{\mathbb{R}} B_\kappa(\lambda, z)K_\kappa(x, y, z)d\mu_\kappa(z),$$

where the function K_κ is defined in (2.11).

(ii) *The function $K_\kappa(x, y, z)$ is unchanged by permutation of the three variables, it is non-positive and*

$$\int_{\mathbb{R}} K_\kappa(x, y, z)d\mu_\kappa(z) = 1.$$

(iii) *There exists a constant A_κ independent of x and y such that*

$$\int_{\mathbb{R}} |K_\kappa(x, y, z)|d\mu_\kappa(z) \le A_\kappa.$$

Moreover $A_\kappa \overset{<}{\to} 2$ as $\kappa \to +\infty$, whenever $xy < 0$, and $A_\kappa \overset{<}{\to} 3$ as $\kappa \to +\infty$, elsewhere.

Definition 2.3. Let f be a suitable function on $\mathbb{R}$. The translation operator T_y^κ is defined by

$$T_y^\kappa(f)(x) = \int_{\mathbb{R}} f(z)K_\kappa(x, y, z)d\mu_\kappa(z).$$

Remark 2.4. For $x, y \in \mathbb{R}^*$, $T_y^\kappa(f)(0) = f(y)$ for $y \in \mathbb{R}^*$ and $T_0^\kappa(f)(x) = f(x)$ for all $x \in \mathbb{R}$. It satisfies $T_y^\kappa(B_\kappa(\lambda, \cdot))(x) = B_\kappa(\lambda, x)B_\kappa(\lambda, y)$ and $T_y^\kappa(f)(x) = T_x^\kappa(f)(y)$.

Lemma 2.5. *Let* $\varphi \in \mathcal{S}(\mathbb{R})$. *We put*

$$\varphi_{1,z,\phi}(x) = \varphi_e(\langle x, z\rangle_\phi), \quad \varphi_{2,z,\phi}(x) = \varphi_o(\langle x, z\rangle_\phi) \quad and \quad \varphi_{3,z,\phi}(z_t) = -\frac{\varphi_{2,z,\phi}^{'}(z_t)}{\langle z_t, z\rangle_\phi},$$

with $z_t = x + t(y - x)$. *Then for all* $x, y \in \mathbb{R}$, *we have*

$$\|T_x^\kappa \varphi - T_y^\kappa \varphi\|_{\kappa,1} \leq C|x - y| \left(\|\varphi_{1,z,\phi}^{'}\|_{\kappa,1} + \|\varphi_{2,z,\phi}^{'}\|_{\kappa,1} + \|\varphi_{3,z,\phi}\|_{\kappa,1} \right).$$

Proof. By [2, Lemma 3.4], one can write

$T_x^\kappa \varphi(y)$

$$= \int_0^\pi \varphi_e(\langle x, y\rangle_\phi) \left\{ 1 + \frac{\text{sgn}(xy)}{4\kappa - 1}(4\kappa \cos^2 \phi - 1) \right\} (\sin \phi)^{4\kappa - 2} d\phi$$

$$+ \int_0^\pi \varphi_o(\langle x, y\rangle_\phi) \left\{ (\text{sgn}(x) + \text{sgn}(y)) - \frac{4\kappa}{4\kappa - 1}\text{sgn}(xy)(x + y)\frac{\sin^2 \phi}{\langle x, y\rangle_\phi} \right\} (\sin \phi)^{4\kappa - 2} d\phi.$$

So we get

$T_x^\kappa \varphi(z) - T_y^\kappa \varphi(z)$

$$= \int_0^\pi \varphi_e(\langle x, z\rangle_\phi) \left\{ 1 + \frac{\text{sgn}(xz)}{4\kappa - 1}(4\kappa \cos^2 \phi - 1) \right\} (\sin \phi)^{4\kappa - 2} d\phi$$

$$+ \int_0^\pi \varphi_o(\langle x, z\rangle_\phi) \left\{ (\text{sgn}(x) + \text{sgn}(z)) - \frac{4\kappa}{4\kappa - 1}\text{sgn}(xz)(x + z)\frac{\sin^2 \phi}{\langle x, z\rangle_\phi} \right\} (\sin \phi)^{4\kappa - 2} d\phi$$

$$- \int_0^\pi \varphi_e(\langle y, z\rangle_\phi) \left\{ 1 + \frac{\text{sgn}(yz)}{4\kappa - 1}(4\kappa \cos^2 \phi - 1) \right\} (\sin \phi)^{4\kappa - 2} d\phi$$

$$- \int_0^\pi \varphi_o(\langle y, z\rangle_\phi) \left\{ (\text{sgn}(y) + \text{sgn}(z)) - \frac{4\kappa}{4\kappa - 1}\text{sgn}(yz)(y + z)\frac{\sin^2 \phi}{\langle y, z\rangle_\phi} \right\} (\sin \phi)^{4\kappa - 2} d\phi.$$

$$= \int_0^\pi \left[\varphi_e(\langle x, z\rangle_\phi) - \varphi_e(\langle y, z\rangle_\phi) \right.$$

$$+ \frac{4\kappa \cos^2 \phi - 1}{4\kappa - 1} \left(\text{sgn}(xz)\varphi_e(\langle x, z\rangle_\phi) \right.$$

$$\left. \left. - \text{sgn}(yz)\varphi_e(\langle y, z\rangle_\phi) \right) \right] (\sin \phi)^{4\kappa - 2} d\phi$$

$$+ \int_0^\pi \Bigg[\varphi_o(\langle x, z\rangle_\phi)\,(\mathrm{sgn}(x) + \mathrm{sgn}(z)) - \varphi_o(\langle y, z\rangle_\phi)\,(\mathrm{sgn}(y) + \mathrm{sgn}(z))$$

$$+ \frac{4\kappa}{4\kappa - 1}\sin^2\phi\Bigg(\mathrm{sgn}(yz)\frac{(y+z)}{\langle y, z\rangle_\phi}\varphi_o(\langle y, z\rangle_\phi)$$

$$- \mathrm{sgn}(xz)\frac{(x+z)}{\langle x, z\rangle_\phi}\varphi_o(\langle x, z\rangle)_\phi)\Bigg) \Bigg](\sin\phi)^{4\kappa-2}d\phi.$$

Since

$$\varphi'_{1,z,\phi}(x) = \varphi'_e(\langle x, z\rangle_\phi)(\langle x, z\rangle_\phi)'$$

and

$$\varphi'_{2,z,\phi}(x) = \varphi'_o(\langle x, z\rangle_\phi)(\langle x, z\rangle_\phi)',$$

we get by the mean value theorem that

$$\frac{\varphi_{1,z,\phi}(x) - \varphi_{1,z,\phi}(y)}{x - y}$$

$$= \int_0^1 \varphi'_{1,z,\phi}(x + t(x-y))dt$$

$$= \int_0^1 \varphi'_{1,z,\phi}(x + t(x-y))(\langle x, z\rangle_\phi)'(x + t(x-y))dt.$$

If x and y are of the same sign, then we use the change $z_t = x + t(y-x)$. We obtain

$$T_x^\kappa\varphi(z) - T_y^\kappa\varphi(z)$$

$$= (x - y)\int_0^1 \int_0^\pi \Bigg[\varphi'_{1,z,\phi}(z_t)$$

$$+ \frac{4\kappa\cos^2\phi - 1}{4\kappa - 1}\mathrm{sgn}(z_t z)\varphi'_{1,z,\phi}(z_t)$$

$$+ (\mathrm{sgn}(z_t) + \mathrm{sgn}(z))\varphi'_{2,z,\phi}(z_t)$$

$$- \frac{4\kappa}{4\kappa - 1}\sin^2\phi\Bigg[\mathrm{sgn}(z_t z)(z_t + z)\frac{\varphi'_{2,z,\phi}(z_t)}{\langle z_t, z\rangle_\phi}\Bigg]\Bigg](\sin\phi)^{4\kappa-2}d\phi dt.$$

Therefore,

$$
T_x^\kappa \varphi(z) - T_y^\kappa \varphi(z)
$$

$$
= (x - y) \int_0^1 \int_0^\pi \left[\varphi_e'(\langle z_t, z \rangle_\phi) \left(1 + |z| - \frac{|z|}{\sqrt{|z_t z|}} \cos \phi \right) \right.
$$

$$
+ \frac{4 \cos^2 \phi - 1}{4\kappa - 1} \operatorname{sgn}(z_t z) \varphi_e'(\langle z_t, z \rangle_\phi) \left(1 + |z| - \frac{|z|}{\sqrt{|z_t z|}} \cos \phi \right)
$$

$$
+ (\operatorname{sgn}(z_t) + \operatorname{sgn}(z)) \varphi_o'(\langle z_t, z \rangle_\phi) \left(1 + |z| - \frac{|z|}{\sqrt{|z_t z|}} \cos \phi \right)
$$

$$
\left. + \frac{4\kappa \sin^2 \phi}{4\kappa - 1} \operatorname{sgn}(z_t z)(z_t + z) \varphi_{3,z,\phi}(z_t) \right] (\sin \phi)^{4\kappa - 2} d\phi dt
$$

$$
= (x - y) \int_0^1 \int_0^\pi \left[\varphi_{1,z,\phi}'(z_t) \left(1 + |z| - \frac{|z|}{\sqrt{|z_t z|}} \cos \phi \right) \right.
$$

$$
+ \frac{4 \cos^2 \phi - 1}{4\kappa - 1} \operatorname{sgn}(z_t z) \varphi_{1,z,\phi}'(z_t) \left(1 + |z| - \frac{|z|}{\sqrt{|z_t z|}} \cos \phi \right)
$$

$$
+ (\operatorname{sgn}(z_t) + \operatorname{sgn}(z)) \varphi_{2,z,\phi}'(z_t) \left(1 + |z| - \frac{|z|}{\sqrt{|z_t z|}} \cos \phi \right)
$$

$$
\left. + \frac{4\kappa \sin^2 \phi}{4\kappa - 1} \operatorname{sgn}(z_t z)(z_t + z) \varphi_{3,z,\phi}(z_t) \right] (\sin \phi)^{4\kappa - 2} d\phi dt.
$$

$$
= (x - y) \int_0^1 \int_0^\pi \left[\varphi_{1,z,\phi}'(z_t) \left(\frac{4 \cos^2 \phi - 1}{4\kappa - 1} \operatorname{sgn}(z_t z) + 1 \right) \left(1 + |z| - \frac{|z|}{\sqrt{|z_t z|}} \cos \phi \right) \right.
$$

$$
+ \varphi_{2,z,\phi}'(z_t)(\operatorname{sgn}(z_t) + \operatorname{sgn}(z)) \left(1 + |z| - \frac{|z|}{\sqrt{|z_t z|}} \cos \phi \right)
$$

$$
\left. + \varphi_{3,z,\phi}(z_t) \operatorname{sgn}(z_t z)(z_t + z) \frac{4\kappa \sin^2 \phi}{4\kappa - 1} \right] (\sin \phi)^{4\kappa - 2} d\phi dt.
$$

Since there exists a positive constant C such that

$$
\left| \frac{4\kappa \cos^2 \phi - 1}{4\kappa - 1} \operatorname{sgn}(z_t z) + 1 \right| \left| 1 + |z| - \frac{|z|}{\sqrt{|z_t z|}} \cos \phi \right| \leq C,
$$

$$
|\operatorname{sgn}(z_t) + \operatorname{sgn}(z)| \left| 1 + |z| - \frac{|z|}{\sqrt{|z_t z|}} \cos \phi \right| \leq C,
$$

and

$$\left| \text{sgn}(z_t z)|z_t + z| \frac{4\kappa \sin^2 \phi}{4\kappa - 1} \right| \leq C,$$

we deduce that

$$|T_x^\kappa \varphi(z) - T_y^\kappa \varphi(z)|$$

$$\leq C|x - y| \int_0^1 \int_0^\pi \left[|\varphi'_{1,z,\phi}(z_t)| + |\varphi'_{2,z,\phi}(z_t)| + |\varphi_{3,z,\phi}(z_t)| \right] (\sin \phi)^{4\kappa - 2} d\phi dt.$$

By making a change of variable $u \to \langle z_t, z \rangle_\phi$,

$$|T_x^\kappa \varphi(z) - T_y^\kappa \varphi(z)|$$

$$\leq C|x - y| \int_0^1 \int_{\mathbb{R}} \left[|\varphi'_{1,z,\phi}(u)| + |\varphi'_{2,z,\phi}(u)| + |\varphi_{3,z,\phi}(u)| \right] K_\kappa(z_t, z, u) d\mu_\kappa(u) dt.$$

By Fubini's theorem and the fact that $\int_{\mathbb{R}} K_\kappa(z_t, z, u) d\mu_\kappa(u) = 1$, we have

$$|T_x^\kappa \varphi(z) - T_y^\kappa \varphi(z)|$$

$$\leq C|x - y| \int_0^1 \left[|\varphi'_{1,z,\phi}(u)| + |\varphi'_{2,z,\phi}(u)| + |\varphi_{3,z,\phi}(u)| \right] \left(\int_{\mathbb{R}} K_\kappa(z_t, z, u) d\mu_\kappa(u) \right) dt$$

$$\leq C|x - y| \int_0^1 \left[|\varphi'_{1,z,\phi}(u)| + |\varphi'_{2,z,\phi}(u)| + |\varphi_{3,z,\phi}(u)| \right] dt$$

$$\leq C|x - y| \left(\|\varphi'_{1,z,\phi}\|_{\kappa,1} + \|\varphi'_{2,z,\phi}\|_{\kappa,1} + \|\varphi_{3,z,\phi}\|_{\kappa,1} \right)$$

which finishes the proof of the lemma. $\qquad \square$

Lemma 2.6 (Bernstein's lemma). *Let $f \in L^1(\mathbb{R}, d\mu_\kappa)$ and $T > 0$ be such that* $\text{supp}(\mathcal{F}_\kappa f) \subset [-T, T]$, *then for all $x, y \in \mathbb{R}$ we have*

$$\|T_x^\kappa f - T_y^\kappa f\|_{\kappa,1} \leq CT|x - y| \|f\|_{\kappa,1}.$$

Proof. Take the function

$$\phi_T(x) = \frac{T^{2\kappa}}{\Gamma(2\kappa + 1)} j_{2\kappa}(2\sqrt{|x|T}) \in \mathcal{S}(\mathbb{R}).$$

Since $\mathcal{S}(\mathbb{R}) = \overline{L^p(\mathbb{R}, d\mu_\kappa)}$, $p \in [1, +\infty]$, we have

$$\mathcal{F}(\phi_T)(x) = \chi_{[-T,T]}(x),$$

where $\chi_{[-T,T]}$ is the characteristic function of the interval $[-T, T]$. Thus

$$\mathcal{F}_\kappa(\phi)(x) = \chi_{[-T,T]}(x),$$

the function ϕ is homogeneous of degree -4κ, i.e.

$$\phi_T(x) = T^{4\kappa}\phi(Tx),$$

and then

$$\mathcal{F}_\kappa(\phi_T)(x) = \mathcal{F}_\kappa(\phi)\left(\frac{x}{T}\right) = 1$$

in the interval $[-T, T]$ and we can write

$$\begin{aligned}
T_x^\kappa(f)(z) - T_y^\kappa(f)(z) &= \phi_T *_\kappa (T_x^\kappa f - T_y^\kappa f)(z) \\
&= f *_\kappa (T_x^\kappa \phi_T - T_y^\kappa \phi_T)(z).
\end{aligned}$$

By Lemma 2.5, we obtain

$$\begin{aligned}
\|T_x^\kappa f - T_y^\kappa f\|_{\kappa,1} &\leq C\|f\|_{\kappa,1}\|T_x^\kappa \phi_T - T_y^\kappa \phi_T\|_{\kappa,1} \\
&\leq C\|f\|_{\kappa,1}\|T_x^\kappa T(\phi) - T_y^\kappa T(\phi)\|_{\kappa,1} \\
&\leq CT|x-y|\|f\|_{\kappa,1}
\end{aligned}$$

which completes the proof of the lemma. $\qquad\square$

By means of translation operator we can define the deformed Hankel convolution product.

Definition 2.7. The deformed Hankel convolution product of two suitable functions f and g on $\mathbb{R}$ is defined by

$$(f *_\kappa g)(x) = \int_\mathbb{R} f(y)T_x^\kappa(g)(y)d\mu_\kappa(y) = \int_\mathbb{R}\int_\mathbb{R} f(y)g(z)K_\kappa(x,y,z)d\mu_\kappa(z)d\mu_\kappa(y).$$

Remark 2.8. For $f, g \in L^1(\mathbb{R}, d\mu_\kappa)$ and $y \in \mathbb{R}$, we have

$$T_y^\kappa(f *_\kappa g) = T_y^\kappa(f) *_\kappa g = f *_\kappa T_y^\kappa(g). \tag{2.12}$$

2.3. THE DEFORMED HANKEL TRANSFORM

First, we shall prove the boundedness of the deformed Hankel kernel.

Lemma 2.9. *Let $\kappa \in \mathbb{R}$, $\kappa > \frac{1}{4}$. Then*

$$|B_\kappa(\lambda, x)| \leq 1, \quad \lambda, x \in \mathbb{R}.$$

Proof. The relation (2.10) implies that the deformed Hankel kernel $B_\kappa(\lambda, x)$ can be expressed in terms of the normalized Bessel function

$$B_\kappa(\lambda, x) = j_{2\kappa-1}\left(2\sqrt{|\lambda x|}\right) - \operatorname{sgn}(\lambda x)\left[j_{2\kappa}\left(2\sqrt{|\lambda x|}\right) - j_{2\kappa-1}\left(2\sqrt{|\lambda x|}\right)\right].$$

If λ and x are of different signs, we have

$$B_\kappa(\lambda, x) = j_{2\kappa}\left(2\sqrt{|\lambda x|}\right),$$

so the assertion is clear.

Now, if λ and x are of the same sign, then from (2.6) we get

$$B_\kappa(\lambda, x) = 2j_{2\kappa-1}\left(2\sqrt{\lambda x}\right) - j_{2\kappa}\left(2\sqrt{\lambda x}\right).$$

$$= \frac{\Gamma(2\kappa)}{\sqrt{\pi}\Gamma(2\kappa - \frac{1}{2})}\frac{2\kappa - 1}{2\kappa - \frac{1}{2}}\int_{-1}^{1} \exp\left(2it(\sqrt{\lambda x})\right)\left(1 + \frac{2\kappa t^2}{2\kappa - 1}\right)\left(1 - t^2\right)^{2\kappa - \frac{3}{2}} dt.$$

Using the fact that

$$\int_{-1}^{1}\left(1 - t^2\right)^{\alpha - \frac{1}{2}} dt = \frac{\sqrt{\pi}\Gamma(\alpha - \frac{1}{2})}{\Gamma(\alpha + 1)},$$

we obtain

$$|B_\kappa(\lambda, x)| \leq \frac{2\kappa - 1}{2\kappa - \frac{1}{2}} + \frac{\Gamma(2\kappa + 1)}{\sqrt{\pi}\Gamma(2\kappa + \frac{1}{2})}\int_{-1}^{1}(1 - t^2)^{2\kappa - \frac{3}{2}} dt$$

$$- \frac{\Gamma(2\kappa + 1)}{\sqrt{\pi}\Gamma(2\kappa + \frac{1}{2})}\int_{-1}^{1}(1 - t^2)^{2\kappa - \frac{1}{2}} dt = 1$$

which gives the assertion. $\qquad\square$

Lemma 2.9 permits to define the deformed Hankel transform.

Definition 2.10. For $f \in L^1(\mathbb{R}, d\mu_\kappa)$, the deformed Hankel transform $\mathcal{F}_\kappa(f)$ is defined by

$$\mathcal{F}_\kappa(f)(\lambda) = \int_{\mathbb{R}} f(x)B_\kappa(\lambda, x)d\mu_\kappa(x), \quad \lambda \in \mathbb{R}.$$

Remark 2.11. If $f \in L^1(\mathbb{R}, d\mu_\kappa)$, then $\mathcal{F}_\kappa(f) \in \mathcal{C}_0(\mathbb{R})$ and $\|\mathcal{F}_\kappa f\|_{\kappa,\infty} \leq \|f\|_{\kappa,1}$.

For $\alpha \in \mathbb{R}$, $\alpha > -\frac{1}{2}$, we denote by $\mathcal{H}_\alpha$ the Hankel transform defined for a suitable function f by

$$\mathcal{H}_\alpha(f)(\nu) = \frac{1}{2^{\alpha-1}\Gamma(\alpha + 1)}\int_0^{+\infty} f(t)j_\alpha(t\nu)t^{2\alpha+1}dt, \quad \nu \in]0, +\infty[.$$

Using the superposition (1.1) for the function B_κ, we obtain the following lemma.

Lemma 2.12. *For all $f \in L^1(\mathbb{R}, d\mu_\kappa)$, we have*

$$\mathcal{F}_\kappa(f)(\lambda) = \frac{1}{2^{2\kappa}} \mathcal{H}_{2\kappa-1}(g)(\sqrt{|\lambda|}) - \frac{\lambda}{2^{2\kappa+2}} \mathcal{H}_{2\kappa+1}(h)(\sqrt{|\lambda|}), \quad \lambda \in \mathbb{R},$$

where g, h are the functions defined on $\mathbb{R}_+$ by

$$g(t) = \frac{f_e\left(\frac{t^2}{4}\right)}{2}, \quad h(t) = \frac{f_o\left(\frac{t^2}{4}\right)}{\frac{t^2}{4}},$$

and f_e and f_o are the even and odd part of the function f.

Proof. Writing $f = f_e + f_o$, we obtain

$$\mathcal{F}_\kappa(f) = \mathcal{F}_\kappa(f_e) + \mathcal{F}_\kappa(f_o). \tag{2.13}$$

By a change of variable, we get

$$\begin{aligned}
\mathcal{F}_\kappa(f_e)(\lambda) &= \frac{1}{2\Gamma(2\kappa)} \int_{\mathbb{R}} f_e(x) j_{2\kappa-1}(2\sqrt{|\lambda x|})|x|^{2\kappa-1} dx \\
&= \frac{1}{\Gamma(2\kappa)} \int_0^\infty f_e(x) j_{2\kappa-1}(2\sqrt{|\lambda|x}) x^{2\kappa-1} dx \\
&= \frac{1}{2^{4\kappa-1}\Gamma(2\kappa)} \int_0^\infty f_e\left(\frac{t^2}{4}\right) j_{2\kappa-1}(t\sqrt{|\lambda|}) t^{4\kappa-1} dt \\
&= \frac{1}{2^{2\kappa}} \mathcal{H}_{2\kappa-1}(g)(\sqrt{|\lambda|})
\end{aligned} \tag{2.14}$$

and

$$\begin{aligned}
\mathcal{F}_\kappa(f_o)(\lambda) &= \frac{-\lambda}{\Gamma(2\kappa+2)} \int_{\mathbb{R}} f_o(x) j_{2\kappa+1}(2\sqrt{|\lambda x|}) x^{2\kappa} dx \\
&= -\frac{\left(\frac{1}{2}\right)^{4\kappa+1} \lambda}{\Gamma(2\kappa+2)} \int_{\mathbb{R}} f_o\left(\frac{t^2}{4}\right) j_{2\kappa+1}(t\sqrt{|\lambda|}) t^{4\kappa+1} dt \\
&= -\frac{\lambda}{2^{2\kappa+2}} \mathcal{H}_{2\kappa+1}(h)(\sqrt{|\lambda|}).
\end{aligned} \tag{2.15}$$

The result follows from (2.13), (2.14) and (2.15).   □

Theorem 2.13 (see [2]). *Let $\kappa > \frac{1}{4}$ and A_κ be as in Theorem 2.2 (iii).*

(i) *For all $f \in L^1_{loc}(\mathbb{R}, d\mu_\kappa)$ and for all $x, y \in \mathbb{R}$, we have*

$$T^\kappa_x(f, y) = T^\kappa_y(f, x) \quad and \quad T^\kappa_x \circ T^\kappa_y = T^\kappa_y \circ T^\kappa_x.$$

(ii) *For all $1 \leq p \leq \infty$ and $f \in L^p(\mathbb{R}, d\mu_\kappa)$, there exists a constant A_κ such that*

$$\|T^\kappa_x(f, \cdot)\|_{\kappa, p} \leq A_\kappa \|f\|_{\kappa, p}, \quad x \in \mathbb{R}.$$

(iii) *If $f \in L^p(\mathbb{R}, d\mu_\kappa)$, $1 \leq p \leq 2$ and $x \in \mathbb{R}$, then*

$$\mathcal{F}_\kappa(T^\kappa_x(f, \cdot))(\lambda) = B_\kappa(\lambda, x)\mathcal{F}_\kappa(f)(\lambda)$$

for almost every $\lambda \in \mathbb{R}$.

(iv) (*Young's inequality*) *For p, q, r such that $1 \leq p, q, r \leq \infty$ and $\frac{1}{p} + \frac{1}{q} - 1 = \frac{1}{r}$, and for $f \in L^p(\mathbb{R}, d\mu_\kappa)$ and $g \in L^q(\mathbb{R}, d\mu_\kappa)$, the convolution product $f *_\kappa g$ is a well-defined element in $L^r(\mathbb{R}, d\mu_\kappa)$ and*

$$\|f *_\kappa g\|_{\kappa, r} \leq A_\kappa \|f\|_{\kappa, p} \|g\|_{\kappa, q}.$$

(v) *For p, q, r such that $1 \leq p, q, r \leq 2$ and $\frac{1}{p} + \frac{1}{q} - 1 = \frac{1}{r}$, and for $f \in L^p(\mathbb{R}, d\mu_\kappa)$ and $g \in L^q(\mathbb{R}, d\mu_\kappa)$, we have*

$$\mathcal{F}_\kappa(f *_\kappa g) = \mathcal{F}_\kappa(f)\mathcal{F}_\kappa(g).$$

In particular $*_\kappa$ is associative in $L^1(\mathbb{R}, d\mu_\kappa)$.

(vi) *For f and g in $L^1_{loc}(\mathbb{R}, d\mu_\kappa)$ such that*

$$\text{supp}(f) \subset \{x : |x| \leq t\} \quad and \quad \text{supp}(g) \subset \{x : r \leq |x| \leq s\},$$

with $s > r > 0$ and $t > 0$, we have

$$\text{supp}(f *_\kappa g) \subset \{x : (\sqrt{r} - \sqrt{t})^2 \leq |x| \leq (\sqrt{s} + \sqrt{t})^2\}.$$

Theorem 2.14. *Let $\kappa > \frac{1}{4}$. Then the following assertions hold.*

(i) *$\mathcal{F}_\kappa$ is an involutional unitary operator on $L^1(\mathbb{R}, d\mu_\kappa)$.*
(ii) *If $f \in L^1(\mathbb{R}, d\mu_\kappa) \cap L^2(\mathbb{R}, d\mu_\kappa)$ then $\mathcal{F}_\kappa(f) \in L^2(\mathbb{R}, d\mu_\kappa)$ and $\|\mathcal{F}_\kappa f\|_{\kappa, 2} = \|f\|_{\kappa, 2}$.*
(iii) *There exists a unique isometry on $L^2(\mathbb{R}, d\mu_\kappa)$ that coincides with $\mathcal{F}_\kappa$ on $L^1(\mathbb{R}, d\mu_\kappa) \cap L^2(\mathbb{R}, d\mu_\kappa)$.*

Proof. (i) Let $f = f_e + f_o$ for $f \in L^1(\mathbb{R}, d\mu_\kappa)$. We put

$$g(t) = \frac{f_e\left(\frac{t^2}{4}\right)}{2} \quad and \quad h(t) = \frac{f_o\left(\frac{t^2}{4}\right)}{\frac{t^2}{4}}.$$

Then using the inversion formula for the Hankel transform and by a change of variable, we obtain

$$
f_e(x) = 2g(2\sqrt{|x|}) = \frac{1}{2^{2\kappa-1}\Gamma(2\kappa)} \int_0^\infty \mathcal{H}_{2\kappa-1}(g)(\nu) j_{2\kappa-1}(2\sqrt{|x|}\nu)\nu^{4\kappa-1} d\nu
$$

$$
= \frac{2}{2^{2\kappa}} \int_{\mathbb{R}} \mathcal{H}_{2\kappa-1}(g)(\sqrt{\lambda}) j_{2\kappa-1}(2\sqrt{\lambda|x|}) d\mu_\kappa(\lambda)
$$

$$
= \frac{1}{2^{2\kappa}} \int_{\mathbb{R}} \mathcal{H}_{2\kappa-1}(g)(\sqrt{\lambda}) B_\kappa(\lambda, x) d\mu_\kappa(\lambda)
$$

$$
= \int_{\mathbb{R}} \mathcal{F}_\kappa(f_e)(\lambda) B_\kappa(\lambda, x) d\mu_\kappa(\lambda).
$$

On the other hand,

$$
h(2\sqrt{|x|}) = \frac{1}{2^{2\kappa+1}\Gamma(2\kappa+2)} \int_0^\infty \mathcal{H}_{2\kappa+1}(h)(\nu) j_{2\kappa+1}(2\sqrt{|x|}\nu)\nu^{4\kappa+3} d\nu
$$

$$
= \frac{1}{2^{2\kappa+2}\Gamma(2\kappa+2)} \int_0^\infty \mathcal{H}_{2\kappa+1}(h)(\sqrt{\lambda}) j_{2\kappa+1}(2\sqrt{\lambda|x|})\lambda^{2\kappa+1} d\lambda.
$$

Thus from (2.15) we get

$$
f_o(x) = xh(2\sqrt{|x|})
$$

$$
= -\frac{1}{2^{2\kappa+2}\Gamma(2\kappa)2\lambda} \int_0^\infty \mathcal{H}_{2\kappa+1}(h)(\sqrt{\lambda}) \left(B_\kappa(\lambda, x) - B_\kappa(-\lambda, x) \right) \lambda^{2\kappa+1} d\lambda
$$

$$
= -\frac{1}{2^{2\kappa+2}\Gamma(2\kappa)2\lambda} \int_{\mathbb{R}} \mathcal{H}_{2\kappa+1}(h)(\sqrt{\lambda}) B_\kappa(\lambda, x)\lambda^{2\kappa+1} d\lambda
$$

$$
= \frac{1}{2\Gamma(2\kappa)} \int_{\mathbb{R}} \mathcal{F}_\kappa(f_o)(\lambda) B_\kappa(\lambda, x)\lambda^{2\kappa-1} d\lambda
$$

$$
= \int_{\mathbb{R}} \mathcal{F}_\kappa(f_o)(\lambda) B_\kappa(\lambda, x) d\mu_\kappa(\lambda)
$$

which gives (i).

(ii) From (i) we deduce that

$$
f(x)\overline{g(x)} = \int_{\mathbb{R}} \mathcal{F}_\kappa(f)(\lambda) B_\kappa(\lambda, x)\overline{g(x)} d\mu_\kappa(\lambda).
$$

The application of Fubini's theorem yields

$$\int_{\mathbb{R}} f(x)\overline{g(x)}d\mu_\kappa(x) = \int_{\mathbb{R}} \mathcal{F}_\kappa(f)(\lambda)\left(\int_{\mathbb{R}} \overline{g(x)}B_\kappa(\lambda,x)d\mu_\kappa(x)\right)d\mu_\kappa(\lambda)$$
$$= \int_{\mathbb{R}} \mathcal{F}_\kappa(f)(\lambda)\overline{\mathcal{F}_\kappa(g)}(\lambda)d\mu_\kappa(\lambda).$$

(iii) Let $f \in L^2(\mathbb{R}, d\mu_\kappa)$, then $f_n = f\mathbb{1}_{[-n,n]}$ is in $L^1(\mathbb{R}, d\mu_\kappa)$ and

$$\|f - f_n\|_{\kappa,2}^2 = \int_{|x|>n} |f(x)|^2 d\mu_\kappa(x) \underset{n\to+\infty}{\to} 0 .$$

From (ii) we deduce that the operator

$$\mathcal{F}_\kappa : L^1(\mathbb{R}, d\mu_\kappa) \cap L^2(\mathbb{R}, d\mu_\kappa) \to L^2(\mathbb{R}, d\mu_\kappa)$$

is continuous for the norm $\| \cdot \|_{\kappa,2}$. As the space $L^1(\mathbb{R}, d\mu_\kappa) \cap L^2(\mathbb{R}, d\mu_\kappa)$ is a dense part of $L^2(\mathbb{R}, d\mu_\kappa)$ and $L^2(\mathbb{R}, d\mu_\kappa)$ is complete, so by the theorem of extension of uniformly continuous applications (a linear application is continuous if it is uniformly continuous) there exists a unique extension of $\mathcal{F}_\kappa$ in $L^2(\mathbb{R}, d\mu_\kappa)$. The extension is still an isometry of the norm $\| \cdot \|_{\kappa,2}$ by passing through the limit in the equality of (ii). If we also know that the image is dense, then the operator is surjective and the inverse is continuous because $\mathcal{F}_\kappa$ is an isometry. □

3. THE DEFORMED BOCHNER–RIESZ MEANS AND THE DEFORMED PARTIAL HANKEL INTEGRALS

3.1. THE DEFORMED BOCHNER–RIESZ MEANS

In this subsection, we will define and study the deformed Bochner–Riesz mean operator.
For $\kappa > \frac{1}{4}$, $\gamma \geq 0$ and $T > 0$, we put

$$\Phi_{T,\gamma}(x) = \frac{T^{2\kappa}\Gamma(\gamma+1)}{\Gamma(2\kappa+\gamma+1)}j_{2\kappa+\gamma}(2\sqrt{|x|T}), \quad x \in \mathbb{R}. \tag{3.1}$$

Remark 3.1. We note that

$$|\Phi_{T,\gamma}(x)| \leq \begin{cases} C\,T^{2\kappa}, & \text{near the origin,} \\ C\,T^{\kappa-\frac{\gamma}{2}-\frac{1}{4}}\,|x|^{-\kappa-\frac{\gamma}{2}-\frac{1}{4}}, & \text{as } x \to +\infty. \end{cases} \tag{3.2}$$

Proposition 3.2. *Let $f \in L^1(\mathbb{R}, d\mu_\kappa)$, $\kappa > \frac{1}{4}$, $\gamma \geq 0$ and $T > 0$, then the deformed Bochner–Riesz mean verifies the convolution relation*

$$\sigma_T^\gamma f = \Phi_{T,\gamma} *_\kappa f. \tag{3.3}$$

Proof. Let $x \in \mathbb{R}$. Using Fubini's theorem, we obtain

$$\sigma_T^\gamma f(x) = \int_{-T}^{T} B_\kappa(\lambda, x) \left(1 - \frac{|\lambda|}{T}\right)^\gamma \mathcal{F}_\kappa(f)(\lambda) d\mu_\kappa(\lambda)$$

$$= \int_{\mathbb{R}} \left[\int_{-T}^{T} B_\kappa(\lambda, x) B_\kappa(\lambda, y) \left(1 - \frac{|\lambda|}{T}\right)^\gamma d\mu_\kappa(\lambda)\right] f(y) d\mu_\kappa(y).$$

By (2.8), a change of variable and Fubini's theorem applied again we have

$$\int_{-T}^{T} B_\kappa(\lambda, x) B_\kappa(\lambda, y) \left(1 - \frac{|\lambda|}{T}\right)^\gamma d\mu_\kappa(\lambda)$$

$$= T^{2\kappa} \int_{-1}^{1} B_\kappa(\lambda T, x) B_\kappa(\lambda T, y) \left(1 - |\lambda|\right)^\gamma d\mu_\kappa(z)$$

$$= T^{2\kappa} \int_{-1}^{1} \left[\int_{\mathbb{R}} B_\kappa(\lambda T, z) K_\kappa(x, y, z) d\mu_\kappa(z)\right] \left(1 - |\lambda|\right)^\gamma d\mu_\kappa(\lambda) \tag{3.4}$$

$$= T^{2\kappa} \int_{\mathbb{R}} \left[\int_{-1}^{1} B_\kappa(\lambda T, z)(1 - |\lambda|)^\gamma d\mu_\kappa(\lambda)\right] K_\kappa(x, y, z) d\mu_\kappa(z).$$

Using the obvious relation

$$B_\kappa^{even}(\lambda T, z) = j_{2\kappa-1}(2\sqrt{|\lambda T z|}),$$

a change of variable and (2.7), we get

$$\int_{-1}^{1} B_\kappa(\lambda T, z)(1 - |\lambda|)^\gamma d\mu_\kappa(\lambda) = \int_{-1}^{1} j_{2\kappa-1}(2\sqrt{|\lambda z T|})(1 - |\lambda|)^\gamma d\mu_\kappa(\lambda)$$

$$= 2 \int_{0}^{1} j_{2\kappa-1}(2\sqrt{|z|T}\mu)(1 - \mu^2)^\gamma \mu^{4\kappa-1} d\mu$$

$$= \frac{\Gamma(2\kappa)\Gamma(\gamma + 1)}{\Gamma(2\kappa + \gamma + 1)} j_{2\kappa+\gamma}(2\sqrt{|z|T}).$$

Thus,

$$\int\limits_{-T}^{T} B_\kappa(\lambda, x) B_\kappa(\lambda, y) \left(1 - \frac{|\lambda|}{T}\right)^\gamma d\mu_\kappa(\lambda) = T_y^\kappa(\Phi_{T,\gamma})(x). \tag{3.5}$$

Combining (3.4) and (3.5), we arrive to the desired assertion for σ_T^γ. $\square$

Lemma 3.3. *Let $\kappa > \frac{1}{4}$, $\gamma > 2\kappa - \frac{1}{2}$, and $T > 0$. Then*

(i) $\int_\mathbb{R} \Phi_{T,\gamma}(x) d\mu_\kappa(x) = 1$,

(ii) *for $q \in [1, +\infty[$, $\Phi_{T,\gamma} \in L^q(\mathbb{R}, d\mu_\kappa)$,*

(iii) $\lim\limits_{T \to \infty} \int_{|x| \geq \epsilon} |\Phi_{T,\gamma}(x)| \, d\mu_\kappa(x) = 0$ *for all $\epsilon > 0$.*

Proof. (i) By a change of variable, (2.5) and (2.3), we obtain

$$\int\limits_\mathbb{R} \Phi_{T,\gamma}(x) d\mu_\kappa(x) = \frac{T^{2\kappa}\Gamma(\gamma+1)}{\Gamma(2\kappa+\gamma+1)} \int\limits_0^{+\infty} j_{2\kappa+\gamma}(2\sqrt{xT}) x^{2\kappa-1} dx$$

$$= \frac{\Gamma(\gamma+1)}{2^{4\kappa-1}\Gamma(2\kappa)\Gamma(2\kappa+\gamma+1)} \int\limits_0^{+\infty} j_{2\kappa+\gamma}(t) t^{4\kappa-1} dt$$

$$= \frac{2^{\gamma+1-2\kappa}\Gamma(\gamma+1)}{\Gamma(2\kappa)} \int\limits_0^{+\infty} J_{2\kappa+\gamma}(t) t^{2\kappa-\gamma-1} dt = 1$$

which gives (i).

(ii) Let $q \in [1, +\infty[$. Then using (3.2), we get

$$\int\limits_\mathbb{R} |\Phi_{T,\gamma}(x)|^q \, d\mu_\kappa(x) \leq C + C \int\limits_1^{+\infty} |x|^{2\kappa-1-q(\kappa+\frac{\gamma}{2}+\frac{1}{4})} dx.$$

Since

$$\left(\kappa + \frac{\gamma}{2} + \frac{1}{4}\right) q \geq \kappa + \frac{\gamma}{2} + \frac{1}{4} > 2\kappa,$$

the assertion (ii) is an immediate consequence of the above inequality.

(iii) We have

$$\lim_{T\to\infty} \int\limits_{|x|\geq\epsilon} |\Phi_{T,\gamma}(x)|d\mu_\kappa(x)$$

$$\leq \lim_{T\to\infty} \int\limits_{|x|\geq\epsilon} CT^{\kappa-\frac{\gamma}{2}-\frac{1}{4}}|x|^{-\kappa-\frac{\gamma}{2}-\frac{1}{4}}d\mu_\kappa(x)$$

$$= \lim_{T\to\infty} \left[\int\limits_{-\infty}^{-\epsilon} CT^{\kappa-\frac{\gamma}{2}-\frac{1}{4}}|x|^{-\kappa-\frac{\gamma}{2}-\frac{1}{4}}d\mu_\kappa(x) + \int\limits_{\epsilon}^{+\infty} CT^{\kappa-\frac{\gamma}{2}-\frac{1}{4}}|x|^{-\kappa-\frac{\gamma}{2}-\frac{1}{4}}d\mu_\kappa(x) \right]$$

$$= \lim_{T\to\infty} CT^{\kappa-\frac{\gamma}{2}-\frac{1}{4}} \left[\int\limits_{-\infty}^{-\epsilon} |x|^{-\kappa-\frac{\gamma}{2}-\frac{1}{4}}d\mu_\kappa(x) + \int\limits_{\epsilon}^{+\infty} |x|^{-\kappa-\frac{\gamma}{2}-\frac{1}{4}}d\mu_\kappa(x) \right]$$

$$= \lim_{T\to\infty} CT^{\kappa-\frac{\gamma}{2}-\frac{1}{4}} \left[\int\limits_{-\infty}^{-\epsilon} |x|^{\kappa-\frac{\gamma}{2}-\frac{5}{4}}dx + \int\limits_{\epsilon}^{+\infty} x^{\kappa-\frac{\gamma}{2}-\frac{5}{4}}dx \right]$$

$$= \lim_{T\to\infty} CT^{\kappa-\frac{\gamma}{2}-\frac{1}{4}} \left[-\epsilon^{\kappa-\frac{\gamma}{2}-\frac{1}{4}} - \epsilon^{\kappa-\frac{\gamma}{2}-\frac{1}{4}} \right]$$

$$= \lim_{T\to\infty} CT^{\kappa-\frac{\gamma}{2}-\frac{1}{4}}\epsilon^{\kappa-\frac{\gamma}{2}-\frac{1}{4}} = 0 \text{ if } \gamma > 2\kappa - \frac{1}{2}$$

which furnishes the assertion. □

Remark 3.4. Let $f \in L^p(\mathbb{R}, d\mu_\kappa)$, $1 \leq p \leq +\infty$, and assume that $\gamma > 2\kappa - \frac{1}{2}$. Since $\Phi_{T,\gamma} \in L^1(\mathbb{R}, d\mu_\kappa)$ we have by virtue of Theorem 2.13 (iv)

$$\|\Phi_{T,\gamma} *_\kappa f\|_{\kappa,p} \leq A_\kappa \|\Phi_{T,\gamma}\|_{\kappa,1}\|f\|_{\kappa,p}.$$

That suggests us to define the operator σ_T^γ on $L^p(\mathbb{R}, d\mu_\kappa)$ by

$$\sigma_T^\gamma f = \Phi_{T,\gamma} *_\kappa f.$$

Lemma 3.5. Let $\gamma > 2\kappa - \frac{1}{2}$ and $1 \leq p < +\infty$. For every function $f \in L^p(\mathbb{R}, d\mu_\kappa)$, we have

$$f(x)\log(2) = \int\limits_{0}^{+\infty} [\sigma_{2T}^\gamma f(x) - \sigma_T^\gamma f(x)] \frac{dT}{T} \quad a.e \quad x \in \mathbb{R}.$$

Proof. Let $f \in L^p(\mathbb{R}, d\mu_\kappa)$. Then for every $T > 0$ we can write

$$\sigma_{2T}^\gamma f(x) - \sigma_T^\gamma f(x) = \int\limits_{T}^{2T} \frac{d}{dt}\{\sigma_t^\gamma f(x)\}dt \quad a.e \quad x \in \mathbb{R}.$$

By integrating both sides and using Fubini's theorem, we obtain

$$
\int\limits_0^{+\infty} [\sigma_{2T}^\gamma f(x) - \sigma_T^\gamma f(x)] \frac{dT}{T} = \int\limits_0^{+\infty} \left(\int\limits_T^{2T} \frac{d}{dt}\{\sigma_t^\gamma f(x)\} dt \right) \frac{dT}{T}
$$

$$
= \log(2) \int\limits_0^{+\infty} \frac{d}{dt}\{\sigma_T^\gamma f(x)\} dt
$$

$$
= \log(2) \left(\lim_{T \to +\infty} \sigma_T^\gamma f(x) - \lim_{T \to 0} \sigma_T^\gamma f(x) \right).
$$

Since $\Phi_{T,\gamma}$ is an approximation to the unity, then we can prove as in [8] that $\sigma_T^\gamma f(x) \to f(x)$ as $T \to +\infty$, almost everywhere $x \in \mathbb{R}$.

Moreover, according to Proposition 3.2 and Theorem 2.13 (iv), we get

$$
|\sigma_T^\gamma f(x)| \le A_\kappa \|\Phi_{T,\gamma}\|_{\kappa,p'} \|f\|_{\kappa,p},
$$

here $p \in [1, +\infty[$ and p' its conjugate exponent. By a change of variable, it follows that

$$
\|\Phi_{T,\gamma}\|_{\kappa,p'} = \frac{2\Gamma(\gamma+1)T^{2\kappa}}{\Gamma(2\kappa+\gamma+1)} \left(\int\limits_0^\infty |j_{2\kappa+\gamma}(2\sqrt{xt})|^{p'} x^{2\kappa-1} dx \right)^{\frac{1}{p'}}
$$

$$
= CT^{\frac{2\kappa}{p}},
$$

where

$$
C = \frac{\Gamma(\gamma+1)}{2^{\frac{4\kappa-1}{p'}}\Gamma(2\kappa+\gamma+1)} \left(\int\limits_0^\infty |j_{2\kappa+\gamma}(y)|^{p'} y^{4\kappa-1} dy \right)^{\frac{1}{p'}}.
$$

Hence, $\sigma_T^\gamma f(x) \to 0$ as $T \to 0$, uniformly in $x \in \mathbb{R}$ which proves the assertion. $\qquad\square$

3.2. THE DEFORMED PARTIAL HANKEL INTEGRALS

For $T > 0$, we define the deformed partial Hankel integral $S_T f$ of a function $f \in L^1(\mathbb{R}, d\mu_\kappa)$ as follows:

$$
S_T f(x) = \int\limits_{-T}^T B_\kappa(\lambda, x) \mathcal{F}_\kappa(f)(\lambda) d\mu_\kappa(\lambda), \quad x \in \mathbb{R}, f \in L^1(\mathbb{R}, d\mu_\kappa).
$$

Proposition 3.6. *Let $f \in L^1(\mathbb{R}, d\mu_\kappa)$. Then we have*

$$S_T f(x) = \int_{\mathbb{R}} \Phi_T(z) T_z^\kappa(f, x) d\mu_\kappa(z), \quad x \in \mathbb{R}. \tag{3.6}$$

Furthermore, if $1 < p < \frac{8\kappa}{4\kappa-1}$, then the partial deformed Hankel integral given by the above identity is well-defined on $L^p(\mathbb{R}, d\mu_\kappa)$.

Proof. The first assertion is a consequence of (3.3).

Let $1 < p < \frac{8\kappa}{4\kappa-1}$, and q its conjugate exponent. Since $\left(\kappa + \frac{1}{4}\right) q > 2\kappa$ one has $\Phi_T \in L^q(\mathbb{R}, d\mu_\kappa)$. Hölder's inequality and Theorem 2.13 (ii) ensure that (3.6) has a sense for $f \in L^p(\mathbb{R}, d\mu_\kappa)$. $\qquad\square$

Proposition 3.7. *Suppose that $\frac{8\kappa}{4\kappa+1} < p < \frac{8\kappa}{4\kappa-1}$. Then $\{S_T\}_{T>0}$ is a uniformly bounded family of operators from $L^p(\mathbb{R}, d\mu_\kappa)$ into itself.*

Proof. Let $f \in \mathcal{C}_c(\mathbb{R})$ and $T > 0$. Fubini's theorem gives us

$$S_T f(x)$$

$$= \int_{\mathbb{R}} f(y) \left(\int_{-T}^{T} B_\kappa(\lambda, x) B_\kappa(\lambda, y) d\mu_\kappa(\lambda) \right) d\mu_\kappa(y)$$

$$= 2 \int_{\mathbb{R}} f(y) \left(\int_{0}^{T} \left(j_{2\kappa-1}(2\sqrt{\lambda|x|}) j_{2\kappa-1}(2\sqrt{\lambda|y|}) \right. \right.$$

$$\left. \left. + \frac{\lambda^2 xy}{(2\kappa(2\kappa+1))^2} j_{2\kappa+1}(2\sqrt{\lambda|x|}) j_{2\kappa+1}(2\sqrt{\lambda|y|}) \right) d\mu_\kappa(\lambda) \right) d\mu_\kappa(y)$$

$$= 4 \int_{0}^{\infty} f_e(y) \left(\int_{0}^{T} j_{2\kappa-1}(2\sqrt{\lambda|x|}) j_{2\kappa-1}(2\sqrt{\lambda y}) d\mu_\kappa(\lambda) \right) d\mu_\kappa(y)$$

$$+ 4 \int_{0}^{\infty} f_o(y) \left(\int_{0}^{T} \frac{\lambda^2 xy}{(2\kappa(2\kappa+1))^2} j_{2\kappa+1}(2\sqrt{\lambda|x|}) j_{2\kappa+1}(2\sqrt{\lambda|y|}) d\mu_\kappa(\lambda) \right) d\mu_\kappa(y)$$

$$= 2|x|^{-\kappa+\frac{1}{2}} \int_{0}^{\infty} f_e(y) \left(\int_{0}^{T} J_{2\kappa-1}(2\sqrt{|x|}s) J_{2\kappa-1}(2\sqrt{y}s) s ds \right) y^{\kappa-\frac{1}{2}} dy$$

$$+ 2|x|^{-\kappa-\frac{1}{2}} \mathrm{sgn}(x) \int_{0}^{\infty} f_o(y) \left(\int_{0}^{T} J_{2\kappa+1}(2\sqrt{|x|}s) J_{2\kappa+1}(2\sqrt{y}s) s ds \right) y^{\kappa-\frac{3}{2}} dy.$$

By relation (2.4) and a change of variable, we get

$$S_T f(x) = -2T|x|^{-\kappa+1} J_{2\kappa}(2T\sqrt{|x|}) \int_0^\infty \frac{f_e(z^2)}{z^2 - |x|} J_{2\kappa-1}(2Tz) z^{2\kappa} dz$$

$$+ 2T|x|^{-\kappa+\frac{1}{2}} J_{2\kappa-1}(2T\sqrt{|x|}) \int_0^\infty \frac{f_e(z^2)}{z^2 - |x|} J_{2\kappa}(2Tz) z^{2\kappa+1} dz$$

$$- 2T\mathrm{sgn}(x)|x|^{-\kappa+1} J_{2\kappa+2}(2T\sqrt{|x|}) \int_0^\infty \frac{f_o(z^2)}{z^2 - |x|} J_{2\kappa+1}(2Tz) z^{2\kappa-2} dz$$

$$+ 2T\mathrm{sgn}(x)|x|^{-\kappa+\frac{1}{2}} J_{2\kappa+1}(2T\sqrt{|x|}) \int_0^\infty \frac{f_o(z^2)}{z^2 - |x|} J_{2\kappa+2}(2Tz) z^{2\kappa-1} dz.$$

Therefore,

$$
\begin{aligned}
S_T f(x) \\
= -\pi T|x|^{-\kappa+\frac{1}{2}} J_{2\kappa}(2T\sqrt{|x|}) H_-(z^{2k} J_{2k-1}(2Tz) f_e(z^2))(\sqrt{|x|}) \\
+ \pi T|x|^{-\kappa+\frac{1}{2}} \mathrm{sgn}(x) J_{2\kappa+2}(2T\sqrt{|x|}) H_-(z^{2k-2} J_{2k+1}(2Tz) f_o(z^2))(\sqrt{|x|}) \\
- \pi T|x|^{-\kappa+\frac{1}{2}} J_{2\kappa-1}(2T\sqrt{|x|}) |x|^{-\frac{1}{2}} H_+(z^{2k+1} J_{2k}(2Tz) f_e(z^2))(\sqrt{|x|}) \\
- \pi T|x|^{-\kappa+\frac{1}{2}} \mathrm{sgn}(x)|x|^{-\frac{1}{2}} J_{2\kappa+1}(2T\sqrt{|x|}) H_+(z^{2k-1} J_{2k+2}(2Tz) f_o(z^2))(\sqrt{|x|}),
\end{aligned}
$$

where H_- and H_+ are odd and even Hilbert transforms (see [16, p. 1028]). By taking into account the behavior of the Hilbert transform on weighted L^p-spaces it easy to see that the right term of the above inequality defines a bounded linear operator from $L^p(\mathbb{R}, d\mu_\kappa)$ into itself when $\frac{8\kappa}{4\kappa+1} < p < \frac{8\kappa}{4\kappa-1}$, i.e. since S_T, $T > 0$ is bounded on $L^p(\mathbb{R}, d\mu_\kappa)$, then

$$\|S_T\|_{\kappa,p} = \|\Phi_T *_\kappa f\|_{\kappa,p} \le \|f\|_{\kappa,1} \|\Phi_T\|_{\kappa,p},$$

and

$$\|\Phi_T\|_{\kappa,p} \le C + C \int_1^{+\infty} |x|^{2\kappa-1-p(\kappa+\frac{1}{4})} dx = C + CI.$$

Therefore, I is convergent provided that $p > \frac{8\kappa}{4\kappa+1}$.

Furthermore, the constant in $L^p(\mathbb{R}, d\mu_\kappa)$ boundedness does not depend on T. The result arises from the density of $\mathcal{C}_c(\mathbb{R})$ in $L^p(\mathbb{R}, d\mu_\kappa)$. □

4. CHARACTERIZATIONS OF THE DEFORMED BESOV–HANKEL SPACES

4.1. CHARACTERIZATION IN TERMS OF THE DEFORMED RIESZ MEAN

In the following, we characterize the deformed Besov–Hankel spaces through the deformed Riesz mean.

Theorem 4.1. *Let* $0 < \beta < 1$, $\gamma > 2\kappa + \beta - \frac{1}{2}$, $1 \le p < +\infty$, $1 \le r \le +\infty$ *and* $f \in L^p(\mathbb{R}, d\mu_\kappa)$. *Then following three properties are equivalent:*

(i) $f \in BH_{\kappa,\beta}^{p,r}$,

(ii) $T^\beta \|\sigma_T^\gamma f - f\|_{\kappa,p} \in L^r\left((0, +\infty), \frac{dT}{T}\right)$,

(iii) $T^\beta \|\sigma_{2T}^\gamma f - \sigma_T^\gamma f\|_{\kappa,p} \in L^r\left((0, +\infty), \frac{dT}{T}\right)$.

Proof. (i)$\Rightarrow$(ii) Let $T > 0$. Since $\Phi_{T,\gamma}$ is even, then using the properties of the generalized translation operator T_y^κ, $y \in \mathbb{R}$, we can write

$$\sigma_T^\gamma f(x) - f(x) = \int\limits_0^{+\infty} \Phi_{T,\gamma}(y) \left(T_y^\kappa f(x) + T_{-y}^\kappa f(x) - 2f(x)\right) d\mu_\kappa(y), \quad x \in \mathbb{R}. \quad (4.1)$$

By the generalized Minkowski inequality ([14, p. 21]) and (3.2), we can assert that

$$\|\sigma_T^\gamma f - f\|_{\kappa,p} \le \int\limits_0^{+\infty} |\Phi_{T\gamma}(y)| w_p(f, y) d\mu_\kappa(y)$$

$$\le C \left(T^{2\kappa} \int\limits_0^{\frac{1}{T}} w_p(f, y) y^{2\kappa - 1} dy + T^{\kappa - \frac{\gamma}{2} - \frac{1}{4}} \int\limits_{\frac{1}{T}}^{+\infty} w_p(f, y) y^{\kappa - \frac{\gamma}{2} - \frac{5}{4}} dy \right).$$

So

$$\left(\int\limits_0^{+\infty} (T^\beta \|\sigma_T^\gamma f - f\|_{\kappa,p})^r \frac{dT}{T} \right)^{\frac{1}{r}} \le C \left(\|G\|_{L^r\left(]0,\infty[, \frac{dT}{T}\right)} + \|H\|_{L^r\left(]0,\infty[, \frac{dT}{T}\right)} \right),$$

where

$$G(T) = T^{2\kappa + \beta} \int\limits_0^{\frac{1}{T}} w_p(f, y) y^{2\kappa - 1} dy$$

and

$$H(T) = T^{\beta + \kappa - \frac{\gamma}{2} - \frac{1}{4}} \int\limits_{\frac{1}{T}}^{\infty} w_p(f, y) y^{\kappa - \frac{\gamma}{2} - \frac{5}{4}} dy.$$

By a change of variable, we get

$$G(T) = T^\beta \int\limits_0^1 w_p\left(f, \frac{y}{T}\right) y^{2\kappa - 1} dy$$

and

$$H(T) = T^\beta \int\limits_1^{+\infty} w_p\left(f, \frac{y}{T}\right) y^{\kappa - \frac{\gamma}{2} - \frac{5}{4}} dy.$$

Case $r = 1$. From the Fubini–Tonelli theorem and a change of variable we obtain

$$
\begin{aligned}
\int\limits_0^{+\infty} G(T) \frac{dT}{T} &= \int\limits_0^{+\infty} T^\beta \left(\int\limits_0^1 w_p\left(f, \frac{u}{T}\right) u^{2\kappa-1} du \right) \frac{dT}{T} \\
&= \int\limits_0^1 u^{2\kappa-1} \left(\int\limits_0^{+\infty} T^\beta w_p\left(f, \frac{u}{T}\right) \frac{dT}{T} \right) du \\
&= \frac{1}{2\kappa + \beta} \int\limits_0^{+\infty} \frac{w_p(f, t)}{t^\beta} \frac{dt}{t}.
\end{aligned}
\tag{4.2}
$$

In the same manner, we deduce that

$$\int\limits_0^{+\infty} H(T) \frac{dT}{T} = \frac{1}{\frac{\gamma}{2} - \kappa + \frac{1}{4} - \beta} \int\limits_0^{+\infty} \frac{w_p(f,t)}{t^\beta} \frac{dt}{t}. \tag{4.3}$$

The case $r = 1$ follows from (4.2) and (4.3).

Case $1 < r < +\infty$. In this time we set $\frac{1}{r} + \frac{1}{r'} = 1$, then from the Fubini–Tonelli theorem and the Hölder inequality, we get

$$
\begin{aligned}
\int\limits_0^{+\infty} (G(T))^r \frac{dT}{T} &= \int\limits_0^{+\infty} G(T)^{r-1} T^\beta \left(\int\limits_0^1 w_p\left(f, \frac{y}{T}\right) u^{2\kappa-1} du \right) \frac{dT}{T} \\
&= \int\limits_0^1 \left(\int\limits_0^{+\infty} (G(T))^{r-1} T^\beta w_p\left(f, \frac{u}{T}\right) \frac{dT}{T} \right) u^{2\kappa-1} du \\
&\leq \int\limits_0^1 \left(\int\limits_0^{+\infty} (G(T))^{(r-1)r'} \frac{dT}{T} \right)^{\frac{1}{r'}} \left(\int\limits_0^{+\infty} \left(T^\beta w_p(f, \frac{u}{T}) \right)^r \frac{dT}{T} \right)^{\frac{1}{r}} u^{2\kappa-1} du.
\end{aligned}
$$

Thus

$$\left(\int\limits_0^{+\infty} (G(t))^r \frac{dT}{T} \right)^{\frac{1}{r}} \leq \frac{1}{2\kappa + \beta} \left(\int\limits_0^{+\infty} \left(\frac{w_p(f,t)}{t^\beta} \right)^r \frac{dt}{t} \right)^{\frac{1}{r}}, \tag{4.4}$$

and

$$\left(\int\limits_0^{+\infty} (H(T))^r \frac{dT}{T} \right)^{\frac{1}{r}} \leq \frac{1}{\frac{\gamma}{2} - \kappa - \beta + \frac{1}{4}} \left(\int\limits_0^{+\infty} \left(\frac{w_p(f,t)}{t^\beta} \right)^r \frac{dt}{t} \right)^{\frac{1}{r}}. \tag{4.5}$$

Relations (4.4) and (4.5) imply the case $1 < r < +\infty$.

Case $r = +\infty$. Let us set

$$M = \operatorname*{ess\,sup}_{t>0} \left(\frac{w_p(f,t)}{t^\beta} \right).$$

Then for every $A > M$, we have

$$w_p(f,t) \le At^\beta \quad a.e.$$

Thus for a.e. $T > 0$,

$$T^\beta \|\sigma_T^\gamma f - f\|_{\kappa,p} \le AC \left(T^{2\kappa+\beta} \int_0^{\frac{1}{T}} t^{\beta+2\kappa-1} dt + T^{\kappa+\beta-\frac{\gamma}{2}-\frac{1}{4}} \int_{\frac{1}{T}}^{+\infty} t^{\beta+\kappa-\frac{\gamma}{2}-\frac{5}{4}} dt \right) \tag{4.6}$$

$$\le AC \left(\frac{1}{\beta+2\kappa} - \frac{1}{\beta+\kappa-\frac{\gamma}{2}-\frac{1}{4}} \right).$$

Relation (4.6) implies the case $r = \infty$.

Case $p = \infty$. By assumption, $f - a \in L^\gamma$ for some a and $1 \le \gamma < +\infty$, so the properties (i), (ii), (iii) are equivalent, and for (i)$\Rightarrow$(iii) it suffices to take into account that

$$\Delta(f-a, x, t) = \Delta(f, x, t), \quad \sigma_T^\gamma(f-a)(x) = \sigma_T^\gamma(f)(x) - a.$$

In the particular case $p = r = \infty$, we get a characterization of the Lipschitz classes. According to [9, Lemma 6] and [3, Lemma 2.2], it follows that if $0 < \beta < 1$, and $\gamma > 2\kappa + \beta - \frac{1}{2}$, we get

$$\left\{ \int_0^{+\infty} \left(T^\beta \|\sigma_T^\gamma f - f\|_{\kappa,p} \right)^r \frac{dT}{T} \right\}^{\frac{1}{r}} \le C \left\{ \int_0^{+\infty} \left(\frac{w_p(f,y)}{y^\beta} \right)^r \frac{dy}{y} \right\}^{\frac{1}{r}} < +\infty.$$

Thus, (ii) is established.

The implication (ii)$\Rightarrow$(iii) is clear.

(iii)$\Rightarrow$(i) We define the operator Δ on $L^p(\mathbb{R}, d\mu_\kappa)$ as follows. For $p \in [1, +\infty[$, we set

$$\Delta(f, x, t) = T_t^\kappa f(x) + T_{-t}^\kappa f(x) - 2f(x), \quad x, t \in \mathbb{R}.$$

By Lemma 3.5 and (2.12), we can write, for all $t \in \mathbb{R}$ and almost every $x \in \mathbb{R}$, that

$$\Delta(f, x, t) \log(2) = \int_0^{+\infty} [\sigma_{2T}^\gamma \Delta(f, \cdot, t)(x) - \sigma_T^\gamma \Delta(f, \cdot, t)(x)] \frac{dT}{T}$$

$$= \int_0^{+\infty} (\Phi_{2T,\gamma} - \Phi_{T,\gamma}) *_\kappa \left(T_t^\kappa f + T_{-t}^\kappa f - 2f \right)(x) \frac{dT}{T} \tag{4.7}$$

$$= \int_0^{+\infty} \Delta(\sigma_{2T}^\gamma f - \sigma_T^\gamma f, x, t) \frac{dT}{T}.$$

From Theorem 2.13 (ii) we deduce that

$$\|\Delta(\sigma_{2T}^\gamma f - \sigma_T^\gamma f, \cdot, t)\|_{\kappa,p} \leq (2A_\kappa + 2)\|\sigma_{2T}^\gamma f - \sigma_T^\gamma f\|_{\kappa,p}, \quad t \in \mathbb{R}, T > 0. \qquad (4.8)$$

On the other hand, since $\mathcal{C}_c(\mathbb{R})$ is a dense subset of $L^p(\mathbb{R}, d\mu_\kappa)$, there exists a sequence $(f_n)_{n\geq 1}$ in $\mathcal{C}_c(\mathbb{R})$ such that $f_n \to f$ as $n \to +\infty$, in $L^p(\mathbb{R}, d\mu_\kappa)$. By Theorem 2.13 (iv), $\Phi_{T,\gamma} *_\kappa f_n \to \Phi_{T,\gamma} *_\kappa f$, as $n \to +\infty$, in $L^p(\mathbb{R}, d\mu_\kappa)$ for every $T > 0$. Then we infer from Theorem 2.13 (ii) that $T_t^\kappa(\Phi_{T,\gamma} *_\kappa f_n) \to T_t^\kappa(\Phi_{T,\gamma} *_\kappa f)$, as $n \to +\infty$, in $L^p(\mathbb{R}, d\mu_\kappa)$ for every $t \in \mathbb{R}$ and $T > 0$. Consequently, we have

$$\|\Delta(\sigma_{2T}^\gamma f - \sigma_T^\gamma f, \cdot, t)\|_{\kappa,p} = \lim_{n \to +\infty}\|\Delta(\sigma_{2T}^\gamma f_n - \sigma_T^\gamma f_n, \cdot, t)\|_{\kappa,p}. \qquad (4.9)$$

We chose a smooth function φ on $\mathbb{R}$, such that $\varphi(y) = 1$ if $|y| \leq 1$, and $\varphi(y) = 0$ if $|y| \geq 2$. Put $\psi = \mathcal{F}_\kappa\varphi$ and $\psi_\epsilon(y) = \epsilon^{4\kappa}\psi(\epsilon y)$ for every $\epsilon > 0$ and $y \in \mathbb{R}$. One verifies that $\mathcal{F}_\kappa\psi_\epsilon(y) = \varphi\left(\frac{y}{\epsilon}\right)$, for every $y \in \mathbb{R}$ and $\epsilon > 0$. So $\mathcal{F}_\kappa\psi_\epsilon(y) = 1$ if $|y| \leq \epsilon$. From Theorem 2.13 (iii) and (v), (3.3) and [18, p. 411], we are able to write

$$\psi_{2T} *_\kappa \Delta(\sigma_{2T}^\gamma f_n - \sigma_T^\gamma f_n, \cdot, t) = \Delta(\sigma_{2T}^\gamma f_n - \sigma_T^\gamma f_n, \cdot, t).$$

Thus, we obtain

$$\Delta(\sigma_{2T}^\gamma f_n - \sigma_T^\gamma f_n, \cdot, t) = (T_t^\kappa\psi_{2T} + T_{-t}^\kappa\psi_{2T} - 2\psi_{2T}) *_\kappa (\sigma_{2T}^\gamma f_n - \sigma_T^\gamma f_n).$$

Theorem 2.13 (iv) and equality (4.9) imply that

$$\begin{aligned}
\|\Delta(\sigma_{2T}^\gamma f - \sigma_T^\gamma f, \cdot, t)\|_{\kappa,p} &= \|(T_t^\kappa\psi_{2T} + T_{-t}^\kappa\psi_{2T} - 2\psi_{2T}) *_\kappa (\sigma_{2T}^\gamma f - \sigma_T^\gamma f)\|_{\kappa,p} \\
&\leq A_\kappa\|T_t^\kappa\psi_{2T} + T_{-t}^\kappa\psi_{2T} - 2\psi_{2T}\|_{\kappa,1}\|\sigma_{2T}^\gamma f - \sigma_T^\gamma f\|_{\kappa,p}.
\end{aligned}$$

As in [10, Theorem 2.1, Corollary 2.2], we get

$$\|\Delta(\sigma_{2T}^\gamma f - \sigma_T^\gamma f, \cdot, t)\|_{\kappa,p} \leq CtT\|\sigma_{2T}^\gamma f - \sigma_T^\gamma f\|_{\kappa,p}, \quad t > 0, T > 0. \qquad (4.10)$$

Combining (4.7), (4.8) and (4.10) and the generalized Minkowski inequality, we obtain for $t > 0$,

$$w_p(f, t) \leq C \left\{ t\int_0^{\frac{1}{t}} \|\sigma_{2T}^\gamma f - \sigma_T^\gamma f\|_{\kappa,p}dT + \int_{\frac{1}{t}}^{+\infty} \|\sigma_{2T}^\gamma f - \sigma_T^\gamma f\|_{\kappa,p}\frac{dT}{T} \right\}.$$

So

$$\left\{\int_0^{+\infty} \left(\frac{w_p(f, t)}{t^\beta}\right)^r \frac{dt}{t}\right\}^{\frac{1}{r}} \leq \left\{\int_0^{+\infty} (G_1(t))^r \frac{dt}{t}\right\}^{\frac{1}{r}} + \left\{\int_0^{+\infty} (H_1(t))^r \frac{dt}{t}\right\}^{\frac{1}{r}}, \qquad (4.11)$$

where

$$G_1(t) = t^{1-\beta} \int_0^{\frac{1}{t}} \|\sigma_{2T}^\gamma f - \sigma_T^\gamma f\|_{\kappa,p} dT,$$

$$H_1(t) = t^{-\beta} \int_{\frac{1}{t}}^{+\infty} \|\sigma_{2T}^\gamma f - \sigma_T^\gamma f\|_{\kappa,p} \frac{dT}{T}.$$

Using the same method as in the first case, we obtain

$$\int_0^\infty (G_1(t))^r \frac{dt}{t} \le \frac{1}{1-\beta} \int_0^{+\infty} \left(T^\beta \|\sigma_{2T}^\gamma f - \sigma_T^\gamma f\|_{\kappa,p}\right)^r \frac{dT}{T}, \tag{4.12}$$

$$\int_0^\infty (H_1(t))^r \frac{dt}{t} \le \frac{1}{1-\beta} \int_0^{+\infty} \left(T^\beta \|\sigma_{2T}^\gamma f - \sigma_T^\gamma f\|_{\kappa,p}\right)^r \frac{dT}{T}. \tag{4.13}$$

The assertion (i) follows from (4.11), (4.12) and (4.13).   □

4.2. CHARACTERIZATION IN TERMS OF THE DEFORMED PARTIAL HANKEL INTEGRAL

In the following, we characterize the deformed Besov–Hankel spaces through the deformed partial Hankel integrals.

Theorem 4.2. *Let* $0 < \beta < 1$, $\frac{8\kappa}{4\kappa+1} < p < \frac{8\kappa}{4\kappa-1}$, $1 \le r < +\infty$ *and* $f \in L^p(\mathbb{R}, d\mu_\kappa)$. *The following three properties are equivalent.*

(i) $f \in BH_{\kappa,\beta}^{p,r}$.
(ii) $T^\beta \|S_T f - f\|_{\kappa,p} \in L^r\left((0,+\infty), \frac{dT}{T}\right)$.
(iii) $T^\beta \|S_{2T} f - S_T f\|_{\kappa,p} \in L^r\left((0,+\infty), \frac{dT}{T}\right)$.

Proof. Let $\gamma > 2\kappa + \beta - \frac{1}{2}$.

(i)⇒(ii) For $T > 0$ and $g \in \mathcal{C}_c(\mathbb{R})$, according to relation (3.3) and Theorem 2.13 (v), we can write

$$\mathcal{F}_\kappa(\sigma_T^\gamma g) = \mathcal{F}_\kappa(\Phi_{T,\gamma})\mathcal{F}_\kappa(g).$$

Since $j_\alpha, \alpha = 2\kappa - 1 > -\frac{1}{2}$ is an even function, by using formula in [18, p. 411], we obtain

$$\mathcal{F}_\kappa(\Phi_{T,\gamma})(\lambda) = \chi_{[-T,T]}(\lambda) \left(1 - \frac{|\lambda|}{T}\right)^\gamma,$$

where $\chi_{[-T,T]}$ is the indicator function of the interval $[-T,T]$. Hence, it follows that

$$S_T(\sigma_T^\gamma g) = \sigma_T^\gamma g. \tag{4.14}$$

Moreover, both members of the last equality define bounded linear operators from $L^p(\mathbb{R}, d\mu_\kappa)$ into itself, provided that $\frac{8\kappa}{4\kappa+1} < p < \frac{8\kappa}{4\kappa-1}$. Since $\mathcal{C}_c(\mathbb{R})$ is a dense subset of $L^p(\mathbb{R}, d\mu_\kappa)$, we deduce that

$$S_T(\sigma_T^\gamma f) = \sigma_T^\gamma f.$$

By using Proposition 3.7, we can assert that if $\frac{8\kappa}{4\kappa+1} < p < \frac{8\kappa}{4\kappa-1}$, we have

$$\|S_T f - f\|_{\kappa,p} \leq \|S_T(\sigma_T^\gamma f - f)\|_{\kappa,p} + \|\sigma_T^\gamma f - f\|_{\kappa,p} \leq C\|\sigma_T^\gamma f - f\|_{\kappa,p}.$$

So

$$\left\{ \int_0^{+\infty} (T^\beta \|S_T f - f\|_{\kappa,p})^r \frac{dT}{T} \right\}^{\frac{1}{r}} \leq C \left\{ \int_0^{+\infty} (T^\beta \|\sigma_T^\gamma f - f\|_{\kappa,p})^r \frac{dT}{T} \right\}^{\frac{1}{r}},$$

where $C > 0$ is a constant not depending on T. Hence (ii) can be deduced from Theorem 4.1.

The implication (ii)$\Rightarrow$(iii) is clear.

(iii)$\Rightarrow$(i) Firstly, we will prove the following equality: for every $f \in L^p(\mathbb{R}, d\mu_\kappa)$,

$$\frac{\gamma}{T^\gamma} \int_0^T (T-t)^{\gamma-1} S_t f(x) dt = \sigma_T^\gamma f(x), \quad T > 0, \text{ and } a.e \ x \in \mathbb{R}. \qquad (4.15)$$

Indeed, if $g \in \mathcal{C}_c(\mathbb{R})$, then by Fubini's theorem we have

$$\frac{\gamma}{T^\gamma} \int_0^T (T-t)^{\gamma-1} S_t g(x) dt$$

$$= \frac{\gamma}{T^\gamma} \int_0^T (T-t)^{\gamma-1} \left(\int_{-t}^t B_\kappa(\lambda, x) \mathcal{F}_\kappa(g)(\lambda) d\mu_\kappa(\lambda) \right) dt$$

$$= \frac{\gamma}{T^\gamma} \int_{-T}^T \left(\int_{|\lambda|}^T (T-t)^{\gamma-1} dt \right) B_\kappa(\lambda, x) \mathcal{F}_\kappa(g)(\lambda) d\mu_\kappa(\lambda)$$

$$= \int_{-T}^T B_\kappa(\lambda, x) \left(1 - \frac{|\lambda|}{T} \right)^\gamma \mathcal{F}_\kappa(g)(\lambda) d\mu_\kappa(\lambda) = \sigma_T^\gamma g(x), \quad x \in \mathbb{R}, T > 0.$$

From the generalized Minkowski inequality and again the uniform boundedness of the family $\{S_T\}_{T>0}$ we obtain

$$\left\| \frac{\gamma}{T^\gamma} \int_0^T (T-t)^{\gamma-1} S_t f(\cdot) dt \right\|_{\kappa,p} \leq \frac{\gamma}{T^\gamma} \int_0^T (T-t)^{\gamma-1} \|S_t f\|_{\kappa,p} dt \leq C\|f\|_{\kappa,p}.$$

Hence the left hand of (4.15) defines a bounded operator from $L^p(\mathbb{R}, d\mu_\kappa)$ into itself, provided that $\frac{8\kappa}{4\kappa+1} < p < \frac{8\kappa}{4\kappa-1}$. Consequently, since $\mathcal{C}_c(\mathbb{R})$ is a dense subset of $L^p(\mathbb{R}, d\mu_\kappa)$, the relationship (4.15) holds. According to (4.15) and by using the generalized Minkowski inequality and [9, Lemma 5] it follows that

$$
\left\{ \int\limits_0^{+\infty} \left[T^\beta \| \sigma_{2T}^\gamma f - \sigma_T^\gamma f \|_{\kappa,p} \right]^r \frac{dT}{T} \right\}^{\frac{1}{r}}
$$

$$
\leq C \left\{ \int\limits_0^{+\infty} \left[T^{\beta-\gamma} \int\limits_0^T (T-t)^{\gamma-1} \| S_{2t} f - S_t f \|_{\kappa,p} dt \right]^r \frac{dT}{T} \right\}^{\frac{1}{r}}
$$

$$
\leq C \left\{ \int\limits_0^{+\infty} \left[K(T) \right]^r \frac{dT}{T} \right\}^{\frac{1}{r}},
$$

where

$$
K(T) = T^{\beta-1} \int\limits_0^T \| S_{2t} f - S_t f \|_{\kappa,p} dt.
$$

Using the same method as in the proof of Theorem 4.1, we obtain

$$
\int\limits_0^\infty (K(t))^r \frac{dt}{t} \leq \frac{1}{1-\beta} \int\limits_0^{+\infty} \left(T^\beta \| S_{2T} f - S_T f \|_{\kappa,p} \right)^r \frac{dT}{T}.
$$

This gives

$$
\left\{ \int\limits_0^{+\infty} \left[T^\beta \| \sigma_{2T}^\gamma f - \sigma_T^\gamma f \|_{\kappa,p} \right]^r \frac{dT}{T} \right\}^{\frac{1}{r}} \leq C \left\{ \int\limits_0^{+\infty} \left[T^\beta \| S_{2T} f - S_T f \|_{\kappa,p} \right]^r \frac{dT}{T} \right\}^{\frac{1}{r}}.
$$

The assertion (i) holds by Theorem 4.1. $\qquad\square$

4.3. CHARACTERIZATION OF THE DEFORMED LIPSCHITZ HANKEL SPACES

Theorem 4.3. *Let $f \in L^2(\mathbb{R}, d\mu_\kappa)$, $0 < \beta < 1$ and γ be such that $\gamma > 2\kappa + \beta - \frac{1}{2}$. Then $f \in \Lambda_{\kappa,\beta}$ if and only if $T^\beta \| \sigma_T^\gamma f - f \|_{\kappa,\infty}$ is bounded on $(0, +\infty)$.*

Proof. Assume that $f \in \Lambda_{\kappa,\beta}$. According to Proposition 3.2 we can write

$$
\sigma_T^\gamma f(x) - f(x) = \int\limits_\mathbb{R} \Phi_{t,\gamma}(y) \left(T_y^\kappa f(x) + T_{-y}^\kappa f(x) - 2f(x) \right) d\mu_\kappa(y), \quad x \in \mathbb{R},
$$

where $\Phi_{T,\gamma}$ is given by the relation (3.1). Thus for all $x \in \mathbb{R}$

$$
\sigma_T^\gamma f(x) - f(x)
$$
$$
= \frac{T^{2\kappa}\Gamma(\gamma+1)}{\Gamma(2\kappa)\Gamma(2\kappa+\gamma+1)} \int\limits_0^{+\infty} \left[T_y^\kappa f(x) + T_{-y}^\kappa f(x) - 2f(x) \right] j_{2\kappa+\gamma}(2\sqrt{|y|T}) y^{2\kappa-1} dy. \tag{4.16}
$$

Using the fact that $f \in \Lambda_{\kappa,\beta}$ and by a change of variable, we obtain

$$
|\sigma_T^\gamma f(x) - f(x)| \leq C T^{-\beta} \int\limits_0^{+\infty} |j_{2\kappa+\gamma}(t)| \, t^{4\kappa+2\beta-1} dt.
$$

From (2.1) and (2.2) we deduce that $j_{2\kappa+\gamma}$ and $y^{\kappa+\frac{\gamma}{2}+\frac{1}{2}} j_{2\kappa+\gamma}$ are bounded on $(0, +\infty)$. Therefore, we obtain

$$
T^\beta \|\sigma_T^\gamma f - f\|_{\kappa,\infty} \leq C \left[\int\limits_0^1 t^{4\kappa+2\beta-1} dt + \int\limits_1^{+\infty} t^{2\kappa+2\beta-\gamma-\frac{3}{2}} dt \right] < \infty.
$$

Hence $T^\beta \|\sigma_T^\gamma f - f\|_{\kappa,\infty}$ is bounded on $(0, +\infty)$.

Conversely, suppose that $T^\beta \|\sigma_T^\gamma f - f\|_{\kappa,\infty}$ is bounded on $(0, +\infty)$. According to (3.3), Theorem 2.13 (ii) and the Hölder inequality, we get for all $T > 0$

$$
\|\sigma_T^\gamma f\|_{\kappa,\infty} \leq \|\Phi_{T,\gamma}\|_{\kappa,2} \|f\|_{\kappa,2}.
$$

Hence,

$$
\|f\|_{\kappa,\infty} \leq \|\sigma_T^\gamma f - f\|_{\kappa,\infty} + \|\sigma_T^\gamma f\|_{\kappa,\infty} \leq C T^{-\beta} + \|\Phi_{T,\gamma}\|_{\kappa,2} \|f\|_{\kappa,2}.
$$

This means that f in $L^\infty(\mathbb{R}, d\mu_\kappa)$. We define the operator Δ on $L^2(\mathbb{R}, d\mu_\kappa)$ as follows

$$
\Delta(f, x, t) = T_t^\kappa f(x) + T_{-t}^\kappa f(x) - 2f(x), \quad t > 0, x \in \mathbb{R}.
$$

Under the condition $\gamma > 2\kappa - \frac{1}{2}$, we obtain from Lemma 3.5 and relation (2.10) of [12], for almost every $t \in \mathbb{R}$, $x > 0$,

$$
\Delta(f, x, t) \log 2 = \int\limits_0^{+\infty} \Delta(\sigma_{2T}^\gamma f - \sigma_T^\gamma f, x, t) \frac{dT}{T}. \tag{4.17}
$$

It follows from Theorem 2.13 (ii) that

$$
\|\Delta(\sigma_{2T}^\gamma f - \sigma_T^\gamma f, x, t)\|_{\kappa,\infty} \leq (2A_\kappa + 2)\|\sigma_{2T}^\gamma f - \sigma_T^\gamma f\|_{\kappa,\infty}, \quad x > 0, T > 0. \tag{4.18}
$$

We choose a smooth function φ on $\mathbb{R}$, such that $\varphi(y) = 1$ if $|y| \leq 1$, and $\varphi(y) = 0$ if $|y| \geq 2$. Put $\psi = \mathcal{F}_\kappa \varphi$ and $\psi_\epsilon(y) = \epsilon^{4\kappa}\psi(\epsilon y)$ for every $\epsilon > 0$ and $y \in \mathbb{R}$. One can

verify that $\mathcal{F}_\kappa \psi_\epsilon(y) = \varphi\left(\frac{y}{\epsilon}\right)$ for every $y \in \mathbb{R}$ and $\epsilon > 0$. So $\mathcal{F}_\kappa \psi_\epsilon(y) = 1$ if $|y| \leq \epsilon$. By combining Theorem 2.13 (iii) and (v) with (3.3), we can write, for all $T > 0, t > 0$,

$$\Delta(\sigma_{2T}^\gamma f - \sigma_T^\gamma f, x, t) = (T_t^\kappa \psi_{2T} + T_{-t}^\kappa \psi_{2T} - 2\psi_{2T}) *_\kappa (\sigma_{2T}^\gamma f - \sigma_T^\gamma f).$$

Therefore, from Theorem 2.13 (iv) we get

$$\|\Delta(\sigma_{2T}^\gamma f - \sigma_T^\gamma f, x, t)\|_{\kappa,\infty} \leq \|T_t^\kappa \psi_{2T} + T_{-t}^\kappa \psi_{2T} - 2\psi_{2T}\|_{\kappa,1} \|\sigma_{2T}^\gamma f - \sigma_T^\gamma f\|_{\kappa,\infty}, \quad (4.19)$$

where $T > 0$, $t > 0$. Then, acting in the same manner as in [10, Corollary 2.2], relation (4.19) yields

$$\|\Delta(\sigma_{2T}^\gamma f - \sigma_T^\gamma f, x, t)\|_{\kappa,\infty} \leq CtT \|\sigma_{2T}^\gamma f - \sigma_T^\gamma f\|_{\kappa,\infty}, \quad t > 0, T > 0. \quad (4.20)$$

By using the relations (4.17), (4.18), (4.20), we obtain, for $t > 0$,

$$\|T_t^\kappa f + T_{-t}^\kappa f - 2f\|_{\kappa,\infty} \leq C \left\{ t \int_0^{\frac{1}{t}} \|\sigma_{2T}^\gamma f - \sigma_T^\gamma f\|_{\kappa,\infty} dT + \int_{\frac{1}{t}}^{+\infty} \|\sigma_{2T}^\gamma f - \sigma_T^\gamma f\|_{\kappa,\infty} \frac{dT}{T} \right\}.$$

Thus by taking into account of the boundedness of $T^\beta \|\sigma_T^\gamma f - f\|_{\kappa,\infty}$ on $(0, +\infty)$ we get for all $t > 0$,

$$\|T_t^\kappa f + T_{-t}^\kappa f - 2f\|_{\kappa,\infty} \leq C \left\{ t \int_0^{\frac{1}{t}} T^{-\beta} dT + \int_{\frac{1}{t}}^{+\infty} T^{-\beta-1} dT \right\} \leq Ct^\beta.$$

This means that $f \in \Lambda_{\kappa,\beta}$. $\qquad \square$

4.4. THE CASE OF DEFORMED LIPSCHITZ–HANKEL SPACES $\Lambda_{\kappa,1}$

Now we deal with the Lipschitz–Hankel space $\Lambda_{\kappa,1}$. In this case we must modify Theorem 4.3 into the following weaker form.

Theorem 4.4. *Let f be a function in $\Lambda_{\kappa,1} \cap L^2(\mathbb{R}, d\mu_\kappa)$ and $\gamma > 2\kappa - \frac{1}{2}$ Then, as $T \to +\infty$, we have*

$$\|\sigma_T^\gamma f - f\|_{\kappa,\infty} = \begin{cases} O(\frac{1}{T}) & \text{if} \quad \gamma > 2\kappa + \frac{1}{2}, \\ O\left(T^{2\kappa-\gamma+\frac{1}{2}}\right) & \text{if} \quad 2\kappa - \frac{1}{2} < \gamma \leq 2\kappa + \frac{1}{2}. \end{cases}$$

Proof. Let $T \in (1, +\infty)$. From the relations (2.1) and (2.2) we can assert that $\Phi_{T,\gamma} \in L^2(\mathbb{R}, d\mu_\kappa)$. Then from (4.16), for $x \in \mathbb{R}$, we get

$$\sigma_T^\gamma f(x) - f(x) = C(I_1 + I_2 + I_3),$$

where C does not depend on T and

$$I_1 = T^{2\kappa} \int\limits_0^{\frac{1}{T}} [T_y^\kappa f(x) + T_{-y}^\kappa f(x) - 2f(x)] j_{2\kappa+\gamma}(2\sqrt{|y|T}) y^{2\kappa-1} dy,$$

$$I_2 = T^{2\kappa} \int\limits_{\frac{1}{T}}^{T} [T_y^\kappa f(x) + T_{-y}^\kappa f(x) - 2f(x)] j_{2\kappa+\gamma}(2\sqrt{|y|T}) y^{2\kappa-1} dy,$$

$$I_3 = T^{2\kappa} \int\limits_{T}^{+\infty} [T_y^\kappa f(x) + T_{-y}^\kappa f(x) - 2f(x)] j_{2\kappa+\gamma}(2\sqrt{|y|T}) y^{2\kappa-1} dy.$$

Since $j_{2\kappa+\gamma}$ and $y^{\kappa+\frac{\gamma}{2}+\frac{1}{2}} j_{2\kappa+\gamma}$ are bounded on $(0,+\infty)$, we obtain

$$|I_1| = \frac{1}{T}, \tag{4.21}$$

$$|I_2| \le CT^{2\kappa-\gamma+\frac{1}{2}}, \tag{4.22}$$

and

$$|I_3| \le CT^{2\kappa-\gamma-\frac{1}{2}}. \tag{4.23}$$

The result follows from (4.21), (4.22) and (4.23). $\qquad\square$

4.5. APPROXIMATION RESULT
IN DEFORMED LIPSCHITZ–HANKEL SPACES

Finally, we prove an approximation result for the functions belonging to $\Lambda_{\kappa,\beta}$, $0 < \beta < 1$, involving deformed partial Hankel integrals. To this end, we first prove the following lemma.

Lemma 4.5. *Let $T > 0$ and $f \in L^2(\mathbb{R}, d\mu_\kappa)$. For $1 < p < +\infty$ such that $\kappa < 1-|\frac{1}{2}-\frac{1}{p}|$, we have*

$$\left(\frac{1}{T} \int\limits_0^T |S_\nu f(x)|^p d\nu \right)^{\frac{1}{p}} \le C\|f\|_{\kappa,\infty}, \quad x \in \mathbb{R}, \tag{4.24}$$

where C does not depend on T and x.

Proof. For $\nu > 0$ and $x \in \mathbb{R}$, the deformed partial Hankel integral $S_\nu f(x)$ can be written as

$$S_\nu f(x) = \int\limits_\mathbb{R} T_x^\kappa f(z) \Phi_\nu(z) d\mu_\kappa(z)$$

$$= \frac{\nu^{2\kappa}}{\Gamma(2\kappa+1)} \int\limits_0^{+\infty} T_x^\kappa f(z) j_{2\kappa}(2\sqrt{z\nu}) z^{2\kappa-1} dz.$$

So we obtain

$$\frac{1}{T} \int_0^T |S_\nu f(x)|^p d\nu = \frac{1}{(\Gamma(2\kappa+1))^p} \left(J_1 + J_2 \right),$$

where

$$J_1 = \frac{1}{T} \int_0^T \left| \int_0^{\frac{1}{T}} T_x^\kappa f(z) j_{2\kappa}(2\sqrt{z\nu}) z^{2\kappa-1} dz \right|^p \nu^{2\kappa p} d\nu,$$

$$J_2 = \frac{1}{T} \int_0^T \left| \int_{\frac{1}{T}}^{+\infty} T_x^\kappa f(z) j_{2\kappa}(2\sqrt{z\lambda}) z^{2\kappa-1} dy \right|^p \nu^{2\kappa p} d\nu.$$

Using the elementary inequality $|j_{2\kappa}(t)| \leq 1$, $t \geq 0$, and Theorem 2.13 (ii) we get

$$J_1 \leq \left(\frac{A_\kappa \|f\|_{\kappa,\infty}}{2\kappa} \right)^p \int_0^T \left(\frac{\nu}{T} \right)^{2\kappa p} d\nu \leq C \|f\|_{\kappa,\infty}^p.$$

To estimate J_2 we assume that $p \geq 2$ and take q such that $\frac{1}{p} + \frac{1}{q} = 1$. From the asymptotic behavior of the Bessel function (see [18]), we have

$$(z\nu)^{\frac{1}{4}} J_{2\kappa}(2\sqrt{z\nu}) = \sqrt{\frac{2}{\pi}} \cos\left(2\sqrt{z\nu} - \frac{\pi}{4} - \kappa\pi \right) + \theta(2\sqrt{z\nu})$$

$$= \cos\left(2\sqrt{z\nu} \right) \cos\left(\frac{\pi}{4} + \kappa\pi \right) + \sin\left(2\sqrt{z\nu} \right) \sin\left(\frac{\pi}{4} + \kappa\pi \right) + \theta(2\sqrt{z\nu}),$$

where

$$|O(2\sqrt{z\nu})| \leq \begin{cases} A & \text{if } z \leq \frac{1}{4\nu}, \\ \frac{B}{2\sqrt{z\nu}} & \text{if } z > \frac{1}{4\nu} \end{cases}$$

and A and B are two positive constants. Thus,

$$J_2 \leq C \left(\frac{1}{T} \int_0^{2\sqrt{T}} [|\phi_1(\nu)|^p + |\phi_2(\nu)|^p + |\phi_3(\nu)|^p + |\phi_4(\nu)|^p] d\nu \right),$$

where

$$\phi_1(\nu) = \nu^{\kappa - \frac{1}{4}} \int\limits_{T}^{+\infty} T_x^{\kappa} f(z) z^{\kappa - \frac{5}{4}} \cos\left(2\sqrt{z\nu}\right) dz,$$

$$\phi_2(\nu) = \nu^{\kappa - \frac{1}{4}} \int\limits_{T}^{+\infty} T_x^{\kappa} f(z) z^{\kappa - \frac{5}{4}} \sin\left(2\sqrt{z\nu}\right) dz,$$

$$\phi_3(\nu) = \nu^{\kappa - \frac{1}{4}} \int\limits_{\frac{1}{T}}^{\frac{1}{\nu}} T_x^{\kappa} f(z) z^{\kappa - \frac{5}{4}} dz,$$

$$\phi_4(\nu) = \nu^{\kappa - \frac{3}{4}} \int\limits_{\frac{1}{\nu}}^{+\infty} T_x^{\kappa} f(z) z^{\kappa - \frac{7}{4}} dz.$$

By a change of variable, we obtain

$$\phi_1(\nu) = 2\nu^{\kappa - \frac{1}{4}} \int\limits_{\sqrt{T}}^{+\infty} T_x^{\kappa} f(y^2) y^{2\kappa - \frac{3}{2}} \cos\left(2y\sqrt{\nu}\right) dy = 2\nu^{\kappa - \frac{1}{4}} \mathcal{F}_c(\varphi_x)(2\sqrt{\nu}),$$

and

$$\phi_2(\nu) = 2\nu^{\kappa - \frac{1}{4}} \mathcal{F}_s(\varphi_x)(2\sqrt{\nu}),$$

where

$$\varphi_x(y) = T_x^{\kappa} f(y^2) y^{2\kappa - \frac{3}{2}} \mathbf{1}_{]T, +\infty[}(y),$$

and $\mathcal{F}_c$, $\mathcal{F}_s$ are the cosine and sine Fourier transforms.

By a change of variable and using the Hausdorff–Young inequality we get

$$\frac{1}{T} \int\limits_{0}^{T} |\phi_1(\nu)|^p d\nu \leq CT^{(\kappa - \frac{1}{4})p - \frac{1}{2}} \int\limits_{0}^{2\sqrt{T}} |\mathcal{F}_c(\varphi_x)(\lambda)|^p \lambda d\lambda$$

$$\leq CT^{(\kappa - \frac{1}{4})p - \frac{1}{2}} \left(\int\limits_{\frac{1}{\sqrt{T}}}^{+\infty} \left| T_x^{\kappa} f(y^2) y^{2\kappa - \frac{3}{4}} \right|^q dy \right)^{\frac{p}{q}}$$

$$\leq C\|f\|_{\kappa,\infty}^p T^{(\kappa - \frac{1}{4})p - \frac{1}{2}} \left(\int\limits_{\frac{1}{\sqrt{T}}}^{+\infty} y^{(2\kappa - \frac{3}{2})q} dz \right)^{\frac{p}{q}}$$

$$= C\|f\|_{\kappa,\infty}^p,$$

provided that $2\kappa < \frac{1}{2} + \frac{1}{p} = 1 - |\frac{1}{2} - \frac{1}{p}|$.

In the same manner we get

$$\frac{1}{T}\int_0^T |\phi_2(\nu)|^p d\nu \le C\|f\|_{\kappa,\infty}^p,$$

provided that $2\kappa < \frac{1}{2} + \frac{1}{p} = 1 - |\frac{1}{2} - \frac{1}{p}|$.

In addition, we have

$$\frac{1}{T}\int_0^T |\phi_3(\nu)|^p d\nu \le \frac{C\|f\|_{\kappa,\infty}^p}{T}\int_0^T \left(1 - \frac{\nu}{T}\right)^{(\kappa-\frac{1}{4})p} d\nu \le C\|f\|_{\kappa,\infty}^p.$$

On the other hand, under the hypothesis $2\kappa < \frac{1}{2} + \frac{1}{p}$, we obtain that $\kappa < \frac{3}{4}$ and

$$\frac{1}{T}\int_0^T |\phi_4(\nu)|^p d\nu \le \frac{C}{T}\|f\|_{\kappa,\infty}^p \int_0^T \nu^{\kappa-\frac{3}{4}} \int_{\frac{1}{\nu}}^\infty z^{\kappa-\frac{7}{4}} d\nu \le C\|f\|_{\kappa,\infty}^p.$$

Hence (4.24) holds in this case.

Assume now that $1 < p \le 2$. Let q the real number verifying $\frac{1}{p} + \frac{1}{q} = 1$, then $q \ge 2$ and the Hölder inequality yields

$$\left(\frac{1}{T}\int_0^T |S_\nu f(x)|^p d\nu\right)^{\frac{1}{p}} \le \left(\frac{1}{T}\int_0^T |S_\nu f(x)|^q d\nu\right)^{\frac{1}{q}}.$$

Applying the same method for q instead of p, it follows that (4.24) holds provided that $2\kappa < \frac{1}{2} + \frac{1}{q} = \frac{3}{2} - \frac{1}{p} = 1 - |\frac{1}{2} - \frac{1}{p}|$.   □

Theorem 4.6. *Let* $0 < \beta < 1$ *and* $f \in \Lambda_{\kappa,\beta} \cap L^2(\mathbb{R}, d\mu_k)$. *If* $1 < p < \frac{1}{\beta}$ *and* $\gamma < \frac{1}{2} - \left|\frac{1}{2} - \frac{1}{p}\right|$, *then*

$$\left\|\left(\frac{1}{T}\int_0^T |S_\nu f - f|^p d\nu\right)^{\frac{1}{p}}\right\|_{\kappa,\infty} \le CT^{-\beta}, \quad T \in (0, +\infty).$$

Proof. Choose $\gamma > \max\{2\kappa + 2\beta - \frac{1}{2}, 1\}$. According to (4.14), we are able to write for every $U > 0$ and $x \in \mathbb{R}$,

$$\left(\frac{1}{U}\int_U^{2U} |S_\nu f(x) - f(x)|^p d\nu\right)^{\frac{1}{p}} = \left(\frac{1}{U}\int_U^{2U} |S_\nu(f - \sigma_U^\gamma f)(x) + \sigma_U^\gamma f(x) - f(x)|^p d\nu\right)^{\frac{1}{p}}$$

$$\le \left(\frac{1}{U}\int_U^{2U} |S_\nu(f - \sigma_U^\gamma f)(x)|^p d\nu\right)^{\frac{1}{p}} + \|\sigma_U^\gamma f - f\|_{\kappa,\infty}.$$

By virtue of Lemma 4.5, it follows that

$$\left(\frac{1}{U}\int_{U}^{2U}|S_\nu(f-\sigma_U^\gamma f)(x)|^p d\nu\right)^{\frac{1}{p}} \leq \left(\frac{1}{U}\int_{0}^{2U}|S_\nu(f-\sigma_U^\gamma f)(x)|^p d\nu\right)^{\frac{1}{p}}$$

$$\leq C\|\sigma_U^\gamma f - f\|_{\kappa,\infty}.$$

Therefore, Theorem 4.3 yields

$$\left(\frac{1}{U}\int_{U}^{2U}|S_\nu(f-\sigma_U^\gamma f)(x)|^p d\nu\right)^{\frac{1}{p}} \leq CU^{-\beta}, \quad U > 0, x \in \mathbb{R}.$$

So

$$\left(\frac{1}{U}\int_{U}^{2U}|S_\nu f(x) - f(x)|^p d\nu\right)^{\frac{1}{p}} \leq CU^{-\beta}.$$

It follows that, for every $n \in \mathbb{N}$, $x \in \mathbb{R}$ and $T > 0$, we have

$$\frac{1}{\frac{T}{2^{n+1}}}\int_{\frac{T}{2^{n+1}}}^{\frac{T}{2^n}}|S_\nu f(x) - f(x)|^p d\nu \leq CT^{-p\beta}2^{(p\beta-1)(n+1)}.$$

Consequently, for every $x \in \mathbb{R}$ and $T > 0$, we obtain

$$\frac{1}{T}\int_{0}^{T}|S_\nu f(x) - f(x)|^p d\nu \leq C\frac{2^{(p\beta-1)}}{1 - 2^{(p\beta-1)}}T^{-p\beta}.$$

This completes the proof. □

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