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Optimal velocity in the race over variable slope trace

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The minimum-time running problem is reconsidered. The time of covering a given distance is minimized. The function that should be found is the runner's velocity that varies with the distance. The Hill-Keller model of motion is employed. It is based on the Newton second law and an equation of power balance. The new element of the current approach is that the trace slope angle varies with the distance. The problem is formulated and solved in optimal control applying the Chebyshev direct pseudospectral method. The essential finding is that the optimal velocity during the cruise is constant regardless of the local slope of the terrain. Such result is valid if the inequality constraints imposed on the propulsive force or the energy are not active.

Key words: *minimum-time running, variable slope trace*

1. Introduction

The fundamental question referring to competitive middle- and long-distance running is: how should a runner vary his speed to minimize the time of covering a given distance? It is obvious that the runner should accelerate just after the start to the race. His strategy during the finish is not so obvious – should he accelerate or decelerate before passing the finish line? If the runner moves on the flat track the intuition indicates that optimal cruising velocity should be constant with the distance. The specific value of this velocity is not known further. The problem complicates as the athlete moves over variable slope trace. Here the consumption of the energy is different at each point of the trace. The case under investigation is a typical optimal control problem.

The minimum-time running problem using the formalism of calculus of variations or optimal control has been considered by many authors [1], [3], [4], [7], [8], [10], [11], [13], [22]. The Keller theory of competitive running is the work of fundamental importance [7], [8]. The problem is solved equating the first variation of the functional to zero. His essential finding is that the middle- and long-distance run may be divided into three sections: acceleration, cruise with constant velocity, and finish with decreasing velocity. Behncke [3] uses more general model of resistive force. He extends considerations to swimming. The problem is solved by employing Pontryagin's maximum principle. Cooper [4] applies Behncke's model for wheelchair races. Woodside [22] computes the switching times separating three segments of the distance mentioned above assuming that the cruising velocity is constant. Maroński [10], [11], for the model of Keller with modified resistive force, solves the problem using the extremization method of linear integrals via Green's theorem. The shortcoming of the solutions quoted above is that such a way is ineffective for more complicated models of motion, especially for the motion over variable slope trace. Maroński and Rogowski [13] show that a direct numerical method may be effective. The Chebyshev pseudospectral method is tested for the motion on the flat track. The present approach is an extension of earlier investigations for the motion over variable slope trace.

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Formulation of the problem

The Hill–Keller model of running is adopted in this paper [17]. Its fundamental assumptions are:

(1) The runner is represented as a particle located in his centre of mass. It means that his body dimensions are neglected comparing with the distance.
(2) The displacements of the centre of mass associated with the start to the race and the cyclic nature of the stride pattern are low, so the centre of mass moves exactly along the trace.
(3) The motion is in the vertical plane. The racer does not reduce his speed in turnings.
(4) The athlete is alone. Other competitors do not disturb his motion.
(5) The profile of the track is described by known smooth function and its local slope varies with the distance. This is the fundamental difference in comparison with earlier investigations.

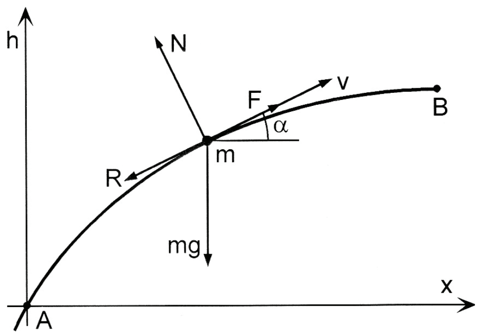

Fig. 1. Forces exerted on the competitor. F – the propulsive force variable with the distance, $F = mf_{\max}(v)\eta$, R – the resistive force, $R = mr(v)$, N – the normal reaction of the ground, mg – the weighting force, α – the local slope angle, v – the velocity

The mathematical model of running is described by differential equations. The first one results from the Newton second law

$$\frac{dv}{dt} = f_{\max}\eta - g\sin\alpha - r(v)\,, \tag{1}$$

where v – the runner's velocity, t – the time (independent variable), $f_{\max}$ – the maximal propulsive force per unit mass, η – the propulsive force setting varying with time (control variable), g – the gravitational acceleration, α – the local slope of the trace, $r(v)$ – the resistance per unit mass

$$r(v) = \frac{v}{\tau}\,. \tag{2}$$

The resistance $r(v)$ linearly depends on the velocity v, τ is an inverse of damping coefficient in the Hill–Keller model of running [7], [8].

The energy conversion in the athlete's body is described by the equation (cf. [3], [7], [8])

$$\frac{dE}{dt} = \sigma - \frac{vf_{\max}\eta}{e} \tag{3}$$

where E – the actual reserves of chemical energy per unit mass in excess of the non-running metabolism, σ – the recovery rate of chemical energy per unit mass, e – the efficiency of converting the chemical energy into mechanical one (for the Hill–Keller model, $e = 1$).

The kinematic equations should be added

$$\frac{dx}{dt} = v\cos\alpha\,, \tag{4}$$

$$\frac{ds}{dt} = v\,, \tag{5}$$

where x – the horizontal coordinate, s – the actual distance measured along the path $h = h(x)$. Symbol h denotes the known smooth function of the profile of the trace. The equations are valid

$$\sin\alpha = \frac{dh/dx}{\sqrt{1+(dh/dx)^2}} \tag{6}$$

and

$$\cos\alpha = \frac{1}{\sqrt{1+(dh/dx)^2}} \tag{7}$$

where dh/dx – the derivative of the shape function. The state equations (1), (3), (4) and (5) should be solved with boundary conditions

$$v(0) = 0,\quad E(0) = E_0,\quad x(0) = 0,\quad s(0) = 0,\quad s(t_f) = s_f\,, \tag{8}$$

where s_f – the given distance to be covered, t_f – the unknown time of covering the distance that is minimized

$$t_f = \int_0^{t_f} dt \Rightarrow \text{MIN}\,. \tag{9}$$

Inequality constraints are imposed on the control variable

$$0 \le \eta(t) \le 1\,, \tag{10}$$

and on the state variables

$$v(t) \ge 0, \quad E(t) \ge 0\,. \tag{11}$$

This means that the propulsive force $f = f_{\max}\eta$ cannot surpass the maximum propulsive force $f_{\max}$ and the energy E cannot be negative.

2. Materials and methods

The method of Fahroo and Ross [6] is applied for the problem solution. It employs the *N*-th degree Lagrange polynomial approximations for the state variables (v, E, x, s) and the control variable (η). The values of these variables at the Chebyshev–Gauss–Lobatto (CGL) points are the expansions coefficients. The state and control variables at the CGL points are unknown parameters and they are optimized in nonlinear programming problem (NLP). The function *fmincon* from *Optimization Toolbox* of MATLAB is used to solve NLP problem. *Fmincon* is an implementation of the sequential quadratic programming method. It allows mimicking Newton's method for constrained optimization. At each major iteration an approximation of the Hessian of the Lagrangian function is made using a quasi-Newton updating method and then it is used to form a search direction for a line search procedure. The details can be found in [20]

3. Results

The numerical method has been tested using the data given by Keller [7], [8]: E_0 = 2409.25 m^2 s^{-2}, $e = 1$, f_{max} = 12.2 m s^{-2}, τ = 0.892 s, σ = 41.61 m^2 s^{-3}, v_0 = 0 m s^{-1}, g = 9.81 m s^{-2}.

Two distances have been considered: 400 and 800 m. The diagrams for 400 m run seem to be more spectacular. Computations show that the distance may be divided into three sections: acceleration, cruise, and final slowing down, for different shapes of the trace and gentle slopes of the terrain. The optimal velocity during the cruise is constant regardless of the local slope of the terrain. The athlete should run holding the same velocity uphill and downhill. This result is out of accordance with intuition. It has been known earlier that the optimal cruising velocity is constant for a flat track. The optimal cruising velocity varies as the constraint imposed on the propulsive force setting is active, η = 1. The runner has no reserves as he moves uphill, so his velocity decreases. Typical solutions for different slopes of the terrain are depicted in Figs. 2–5.

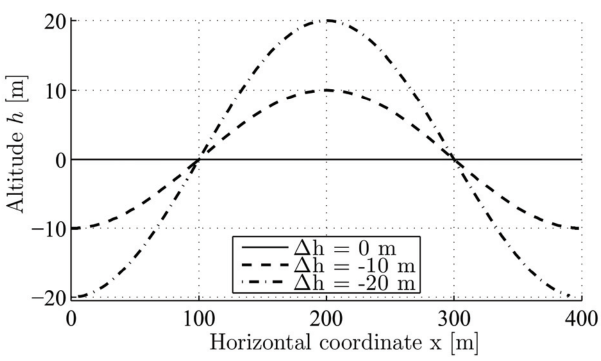

Fig. 2. The shape of the route profile $h(x)$

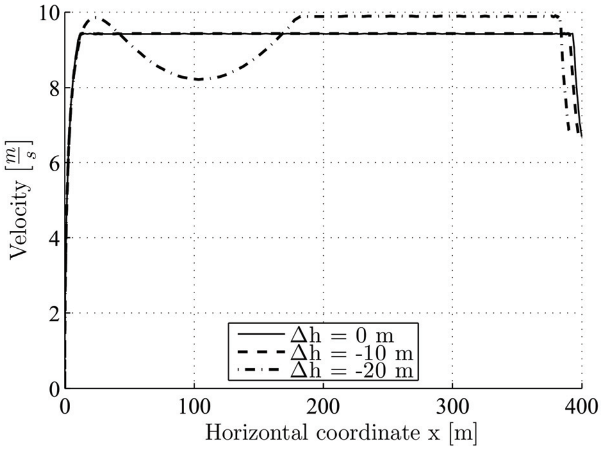

Fig. 3. Optimal velocities for 400 m run

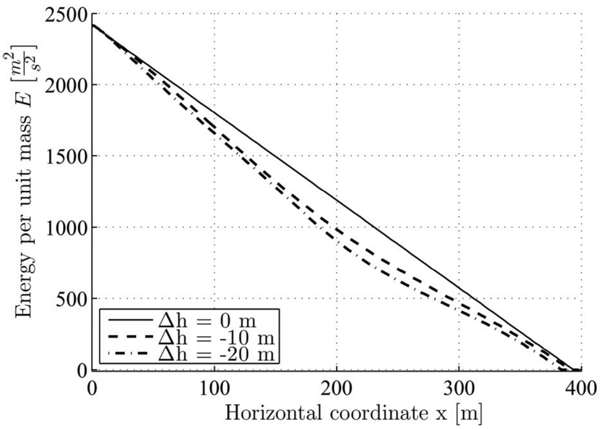

Fig. 4. The actual reserves of chemical energy per unit mass E for the 400 m run

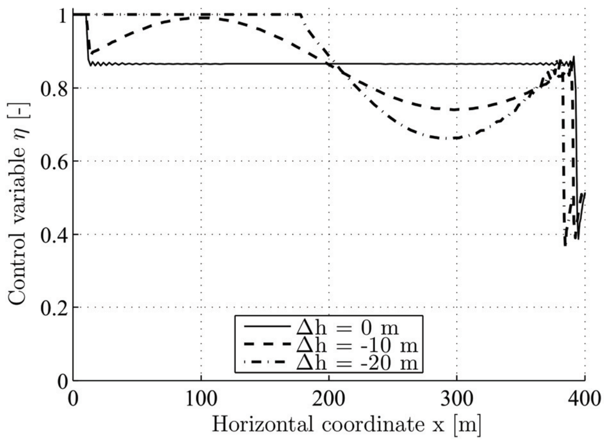

Fig. 5. The propulsive force setting η (a control variable) for 400 m run

4. Discussion

The Chebyshev direct pseudospectral method applied in this paper is effective for different models and different inequality constraints imposed on state and control variables [15], [18], [19]. The algorithm should be programmed on a computer. It is much more sophisticated than application of the optimization algorithm for the race over a flat trace [10]. The result of investigations is easy for practical applications. The runner's cruising velocity is constant, regardless of the local slope of terrain. The specific value of the optimal velocity should be computed earlier, however.

The parameters E_0 and σ in equation (3) representing the energy balance of the runner have not been estimated during an experiment, therefore the mechanical power generated by a runner following the Hill–Keller model is quite high. Keller [7], [8] assumes that the runner has some initial reserves of the energy E_0 which are depleted at the rate of working of the actual propulsive force $f_{\max}\eta$. The losses of the energy are mitigated by the recovery rate σ. Keller assumes σ to be constant. It corresponds to the maximum oxygen uptake of the athlete. Both constants E_0 and σ relate to longer distance events where aerobic energy conversion plays an essential role in the energy balance. The Hill–Keller model of running is simplified one but it gives a good representation of the world track records. Keller determined the best values of E_0 and σ by minimizing the sum of squares of the relative errors for 14 races from 400 to 10 000 m. With these values the model predictions agreed with observations to within ±3%. Behncke [3] gives similar values: E_0 = 2750 m^2 s^{-2}, σ = 28.2 m^2 s^{-3} (for men); E_0 = 2300 m^2 s^{-2}, σ = 22.5 m^2 s^{-3} (for women).

In the Hill–Keller model of running it is somewhat unrealistic to assume that an athlete can actually apply a constant maximum propulsive force $f_{\max}$ for the duration of a race. It seems logical to assume the force $f_{\max}$ decreases with time. Mureika [14] assumes a linear decrease of the force $f_{\max}$. Such weakness of the model refers to the short distance running. During the cruise the maximum limit of the propulsive force is usually not active.

In the paper, the runner is modelled as a particle located in his centre of mass moving in the vertical plane (assumptions (1) and (3)). Schultz and Mombaur [21] consider physics-based running motions for complex models of human-like running in three dimensions. Running is modelled as multiphase periodic motion with discontinuities, based on multi-body system models of the locomotor system with actuators and spring-damper elements at each joint. Such complicated model is not necessary for strategy optimization during the middle-distance running.

The only physical difference between straight running considered in the paper (assumption (3)) and curve running is the effect of centrifugal forces on the sprinter. The sprinter's racing spikes provide ample traction to stop outward translational motion. To compensate for the rotational effects, the sprinter leans into the turn. Greater speed requires greater lean. Probably the maximum propulsive force would not be generated at its limit. A curve model is not a trivial one to construct. Some attempts are given by Mureika [14].

In the Hill–Keller model, the athlete is running alone (assumption (4)). Pitcher [16] investigates a mathematical model of a two-runner race and examines optimal strategies. One of the reasons for this is so the runner behind can take advantage of the slipstream of the runner in front. Keller [9] considers a similar problem.

In the paper presented, the variable slope track is considered (assumption (5)). Ardigo et al. [2] investigate whether cycling is more economical than running and walking when moving on a gradient. Ebben [5] using an experimental method considers a variety of downhill slopes to determine the optimal slope for overspeed running. The slope angles are constant, however. Andreeva and Behncke [1] consider the model of running similar to the Hill–Keller model extended by introducing separated glycogen compartment. In addition, the resistance term may depend on the variable slope of the track. The analysis of the optimal running strategy is based on the qualitative study of the switching function of the optimal control problem. The slope gradient is assumed to be small (lower than 12°) therefore the horizontal coordinate x is identical as the actual distance s measured along the path. Such assumption is not necessary in this paper. The result of qualitative considerations is similar to that presented in the paper – the runner's velocity v is almost constant during the cruise. No numerical example is given, however.

The fundamental finding of this paper, the optimal cruising velocity is constant, has been obtained for constant efficiency of transforming the chemical energy into the mechanical one. It is not clear to the authors what will happen for variable efficiency. The efficiency is a function of rider's velocity in the problem of minimization of energy expenditure during cycling considered by Maroński [12]. There, the optimal cruising velocity is not constant but it varies in a relatively narrow range. The problem formulation and the applied method are different, however.

It is obvious that there are many reasons for further investigation of the minimum-time running problem investigated in this paper.

5. Conclusions

The optimal strategy of the minimum-time run is reconsidered in the paper. The modified Hill–Keller model of the run is used. The new element of the model is a variable slope trace. The essential finding is that the optimal cruising velocity is constant for gentle slopes of the trace. It means that the rate of the exhaustion of the energy varies compensating the variations of the resistive force resulting from a negative component of the gravitational force on the direction of motion. This work confirms that the Chebyshev direct pseudospectral method may be an efficient tool for investigations of optimal strategies during sports events like skating, swimming, rowing or cycling.

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