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An extension of Klamka's method to positive descriptor discrete-time linear systems with bounded inputs

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The minimum energy control problem for the positive descriptor discrete-time linear systems with bounded inputs by the use of Weierstrass-Kronecker decomposition is formulated and solved. Necessary and sufficient conditions for the positivity and reachability of descriptor discrete-time linear systems are given. Conditions for the existence of solution and procedure for computation of optimal input and the minimal value of the performance index is proposed and illustrated by a numerical example.

Key words: descriptor, positive, discrete-time, linear, system, Weierstrass-Kronecker decomposition, minimum energy control

1. Introduction

A dynamical system is called positive if its trajectory starting from any nonnegative initial condition state remains forever in the positive orthant for all nonnegative inputs. An overview of state of the art in positive system theory is given in the monographs [8, 19] and in the papers [11, 20–24]. Models having positive behavior can be found in engineering, economics, social sciences, biology and medicine, etc.

Descriptor (singular) linear systems were considered in many papers and books [1–7, 9, 21, 31–33]. The positive standard and descriptor systems and their stability have been analyzed in [19, 23]. Descriptor positive discrete-time and continuous-time nonlinear systems have been analyzed in [11].

The minimum energy control problem for standard linear systems has been formulated and solved by J. Klamka [27–29] and for 2D linear systems with variable coefficients in [27]. The relative controllability and minimum energy control problem of linear systems with distributed delays in control has been investigated by Klamka in [30]. The minimum energy control of fractional positive

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linear systems has been addressed in [14, 15] and for positive discrete-time linear systems in [10, 13, 16, 18, 26]. The minimum energy control problem for positive electrical circuits has been investigated in [17].

In this paper the minimum energy control problem for positive descriptor discrete-time linear systems with bounded inputs by the use of Weierstrass-Kronecker decomposition will be formulated and solved.

The paper is organized as follows. In section 2 Weierstrass-Kronecker decomposition theorem for regular pencil and conditions for the reachability of positive systems are recalled. Necessary and sufficient conditions for the positivity of the descriptor linear systems are given in section 3. The minimum energy control problem for positive descriptor systems with bounded inputs is formulated and solved in section 4. Concluding remarks are given in section 5.

The following notation will be used: $\Re$ – the set of real numbers, $\Re^{n\times m}$ – the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ – the set of $n\times m$ matrices with nonnegative entries and $\Re_+^n = \Re_+^{n\times 1}$, I_n – the $n\times n$ identity matrix, Z_+ – the set of nonnegative integers.

2. Preliminaries

Consider the descriptor discrete-time linear system

$$Ex_{i+1} = Ax_i + Bu_i, \quad i\in Z_+ = \{0,1,\dots\}, \tag{1}$$

where $x_i\in\Re^n$, $u_i\in\Re^m$ are the state and input vectors and $A\in\Re^{n\times n}$, $B\in\Re^{n\times m}$.

It is assumed that $\det E = 0$ and

$$\det[Ez-A]\neq 0 \ \text{ for some } \ z\in C \ \text{ (the field of complex numbers).} \tag{2}$$

It is well-known [12, 25] that if (2) holds then there exist nonsingular matrices $P_1, P_2\in\Re^{n\times n}$ such that

$$P_1[Ez-A]P_2 = \begin{bmatrix} I_{n_1}z-A_1 & 0 \\ 0 & Nz-I_{n_2}\end{bmatrix}, \qquad A_1\in\Re^{n_1\times n_1}, \quad N\in\Re^{n_2\times n_2}, \tag{3}$$

where $n_1 = \deg\{\det[Ez-A]\}$, $n_2 = n-n_1$ and N is the nilpotent matrix with the index μ, i.e. $N^{\mu-1}\neq 0$, $N^\mu = 0$.

The matrices P_1 and P_2 can be computed using procedures given in [12, 25, 32].

Premultiplying (1) by the matrix P_1 and introducing the new state vector

$$\bar{x}_i = \begin{bmatrix}\bar{x}_{1,i} \\ \bar{x}_{2,i}\end{bmatrix} = P_2^{-1}x_i, \qquad \bar{x}_{1,i}\in\Re^{n_1}, \quad \bar{x}_{2,i}\in\Re^{n_2} \tag{4}$$

and using (4) we obtain

$$P_1EP_2P_2^{-1}x_{i+1} = P_1AP_2P_2^{-1}x_i + P_1Bu_i \tag{5}$$

and

$$\bar{x}_{1,i+1} = A_1\bar{x}_{1,i} + B_1u_i\,, \tag{6a}$$

$$N\bar{x}_{2,i+1} = \bar{x}_{2,i} + B_2u_i\,, \tag{6b}$$

where

$$P_1B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \qquad B_1 \in \Re^{n_1\times m}, \quad B_2 \in \Re^{n_2\times m}. \tag{6c}$$

Theorem 1 *The solution $\bar{x}_{1,i}$ of the equation (6a) has the form*

$$\bar{x}_{1,i} = A_1^i\bar{x}_{10} + \sum_{k=1}^{i-1} A_1^{i-k-1}B_1u_k\,. \tag{7}$$

Proof. The proof is given in [12]. □

Theorem 2 *The solution $\bar{x}_{2,i}$ of the equation (6b) for zero initial conditions $\bar{x}_{20} = 0$ has the form*

$$\bar{x}_{2,i} = -\sum_{k=0}^{\mu-1} N^kB_2u_{i+k}\,. \tag{8}$$

Proof. The proof is given in [12]. □

Definition 1 *The positive descriptor discrete-time linear system (1) is called reachable in q steps $(q \leqslant n)$ if for every given final state $x_f \in \Re^n_+$ there exists an input sequence $u_k \in \Re^m_+$, $k = 0, 1, \ldots, q-1$ which steers the state of the system from zero initial condition $x_0 = 0$ to x_f.*

A real square matrix is called monomial if each its row and each its column contains only one positive entry and the remaining entries are zero.

Theorem 3 *The descriptor discrete-time linear system (1) is reachable in q steps if and only if the matrices*

$$[B_1 \quad A_1B_1 \quad \cdots \quad A_1^{n_1-1}B_1], \tag{9}$$

$$[B_2 \quad NB_2 \quad \cdots \quad N^{\mu-1}B_2] \tag{10}$$

contain full rank monomial matrices.

Proof. The proof is given in [12]. □

3. Positivity of the descriptor systems

Definition 2 *The descriptor system (1) is called (internally) positive if $x_i \in \Re^n_+$, $i \in Z_+$ for any $x_0 \in \Re^n_+$ and all inputs $u_i \in \Re^m_+$, $i \in Z_+$.*

Theorem 4 *The descriptor system (1) is positive if and only if the following conditions are satisfied*

(1) $P_2 \in \Re^{n\times n}_+$ is monomial,

(2) $A_1 \in M_{n_1}$, $B_1 \in \Re^{n_1\times m}_+$,

(3) $-B_2 \in \Re^{n_2\times m}_+$.

Proof. . It is well-known [19] that $P_2^{-1} \in \Re^{n\times n}_+$ if and only if $P_2 \in \Re^{n\times n}_+$ is monomial matrix. From (4) we have $\bar{x}_i = P_2^{-1}x_i \in \Re^n_+$ if and only if $x_i \in \Re^n_+$ for $i \in Z_+$. The standard subsystem (6a) is positive if and only if the condition 2) is satisfied [19]. From (8) it follows that $\bar{x}_{2,i} \in \Re^{n_2}_+$ if and only if $-B_2 \in \Re^{n_2\times m}_+$ since $N \in \Re^{n_2\times n_2}_+$ and $u_i \in \Re^m_+$ for $i \in Z_+$. Therefore, the descriptor system (1) is positive if and only if the three conditions are satisfied. □

Example 1 Consider the descriptor linear system (1) with the matrices

$$E = \begin{bmatrix} -0.5 & 0 & 1 & 0 \\ 0.25 & 0 & 0 & 1 \\ -0.5 & 0 & 1 & 0.5 \\ 0 & 0 & 0 & 0.5 \end{bmatrix}, \quad A = \begin{bmatrix} 1.5 & 0 & -2 & 0 \\ 0 & 0.2 & 1 & 0 \\ 1.5 & 0.1 & -2 & -0.5 \\ 0 & 0.1 & 0 & 0.5 \end{bmatrix}, \quad B = \begin{bmatrix} -1 \\ 0.5 \\ -0.5 \\ -0.5 \end{bmatrix}. \tag{11}$$

It is easy to check that the pencil is regular since

$$\det[Ez - A] = 0.025z^2 - 0.15 \neq 0. \tag{12}$$

In this case

$$P_1 = \begin{bmatrix} 3 & 2 & -2 & -2 \\ 2 & 2 & -2 & -2 \\ -1 & 0 & 1 & 1 \\ 1 & 0 & -1 & 1 \end{bmatrix}, \qquad P_2 = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{13}$$

and

$$P_1EP_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad P_1AP_2 = \begin{bmatrix} 0 & 3 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad P_1B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix},$$
$$N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 3 \\ 2 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad -B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{14}$$
$$n_1 = n_2 = 2, \quad m = 1.$$

Therefore, by Theorem 4 the descriptor system (1) with (11) is positive.

Using (9) and (10) for (14) we obtain the monomial matrices

$$[B_1 \quad A_1B_1] = \begin{bmatrix} 0 & 3 \\ 1 & 0 \end{bmatrix}, \tag{15}$$

$$[B_2 \quad NB_2] = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}. \tag{16}$$

Therefore, the descriptor system (1) with (11) is reachable in $q = 2$ steps.

4. Minimum energy control of the descriptor systems

Consider the positive reachable in q steps descriptor system (1) and the performance index

$$I(u) = \sum_{k=0}^{q-1} u_k^{\mathrm{T}} Q u_k\,, \tag{17}$$

where $u_k \in \Re_+^m$ and $Q \in \Re_+^{m\times m}$ is a symmetric positive defined matrix.

The minimum energy control problem can be stated as follows: Given the matrices E, A, B of (1), the final state $x_f \in \Re_+^n$ and the matrix Q of the performance index (17), find an input sequence $u_k \in \Re_+^m$, $k = 0,1,\ldots,q-1$ satisfying the condition

$$u_k < U \quad (U \in \Re_+^m \;\text{ is given) for }\; k = 0,1,\ldots,q-1 \tag{18}$$

that steers the state vector of the system from $x_0 = 0$ to $x_f \in \Re_+^n$ and minimizes the performance index (17).

From the block-diagonal structure of matrices P_1EP_2 and P_1AP_2 it follows that minimum energy control problem can be applied to both subsystems (6) separately (Fig. 1). The minimum energy control problem can be stated as follows.

Given the matrices $A_1 \in M_{n_1}$, $B_1 \in \Re_+^{n_1\times m}$, $B_2 \in \Re_+^{n_2\times m}$, $N \in \Re_+^{n_2\times n_2}$, $Q_k \in \Re_+^{m\times m}$ of the performance matrix (17) and $x_f \in \Re_+^n$, find an input sequence $u_i = \begin{bmatrix} u_{1,i} \\ u_{2,i} \end{bmatrix} \in \Re_+^{(l+\mu)m}$, where $\max(l+\mu) = q$, $u_{1,k} \in \Re_+^{lm}$, $k = 0,1,\ldots,l-1$ and $u_{2,j} \in \Re_+^{\mu m}$, $j = 0,1,\ldots,\mu-1$, that steers the state vector from $x_0 = 0$ to $x_f \in \Re_+^n$ and minimizes performance index (17).

Let us first consider the subsystem (6a). To solve the problem we define the matrix

$$W_l = R_l \widetilde{Q}_1^{-1} R_l^{\mathrm{T}} \in \Re_+^{n_1\times n_1}, \tag{19}$$

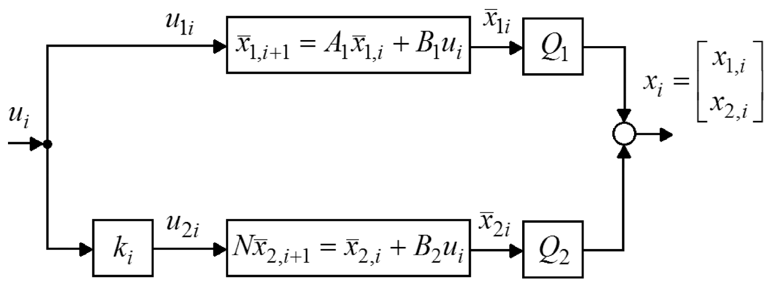

Figure 1: Descriptor discrete-time linear system

where R_l is the reachability matrix defined by

$$R_l = [B_1 \quad A_1B_1 \quad \cdots \quad A_1^{n_1-1}B_1] \tag{20}$$

and

$$Q_1 = blockdiag[Q_1^{-1} \quad \cdots \quad Q_1^{-1}] \in \Re_+^{lm\times lm}. \tag{21}$$

If the system (1) is reachable in q steps then the input sequence

$$u_l = \begin{bmatrix} u_{l-1} \\ u_{l-2} \\ \vdots \\ u_0 \end{bmatrix} = Q_1^{-1}R_l^{\mathrm{T}}W_l^{-1}\bar{x}_{1,f} \in \Re_+^{lm} \tag{22}$$

steers the subsystem (6a) from $\bar{x}_{10} = 0$ to $\bar{x}_{1,f}$ since

$$\bar{x}_{1,l} = R_lu_l = R_lQ_1^{-1}R_l^{\mathrm{T}}W_l^{-1}\bar{x}_{1,f} = W_lW_l^{-1}\bar{x}_{1,f} = \bar{x}_{1,f}\,. \tag{23}$$

Now let us consider the subsystem (6b). To solve the problem for the subsystem (6b) we define the matrix

$$W_\mu = R_\mu Q_2^{-1}R_\mu^{\mathrm{T}} \in \Re_+^{n_2\times n_2}, \tag{24}$$

where R_μ is the reachability matrix defined by

$$R_\mu = [B_2 \quad NB_2 \quad \cdots \quad N^{\mu-1}B_2] \tag{25}$$

and

$$Q_2 = blockdiag[Q_2^{-1} \quad \cdots \quad Q_2^{-1}] \in \Re_+^{\mu m\times \mu m}. \tag{26}$$

If the system (1) is reachable in q steps then the input sequence

$$u_\mu = \begin{bmatrix} u_{\mu-1} \\ u_{\mu-2} \\ \vdots \\ u_0 \end{bmatrix} = Q_2^{-1}R_\mu^{\mathrm{T}}W_\mu^{-1}\bar{x}_{2,f} \in \Re_+^{\mu m} \tag{27}$$

steers the subsystem (6b) from $\bar{x}_{20} = 0$ to $\bar{x}_{2,f}$ since

$$\bar{x}_{2,\mu} = R_\mu \hat{u}_\mu = R_\mu Q_2^{-1} R_\mu^{\mathrm{T}} W_\mu^{-1} \bar{x}_{2,f} = W_\mu W_\mu^{-1} \bar{x}_{2,f} = \bar{x}_{2,f}\,. \tag{28}$$

Finally, we define the matrices

$$Q = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}, \quad R_q = \begin{bmatrix} R_l & 0 \\ 0 & R_\mu \end{bmatrix}, \quad W_q = \begin{bmatrix} W_l & 0 \\ 0 & W_\mu \end{bmatrix} \tag{29}$$

and the input sequence can be computed from

$$\hat{u}_q = \begin{bmatrix} u_l \\ u_\mu \end{bmatrix} = Q^{-1} v R_q^{\mathrm{T}} W_q^{-1} \bar{x}_f \in \Re_+^{qm}. \tag{30}$$

The vector

$$\bar{x}_f = \begin{bmatrix} \bar{x}_{1,f} \\ \bar{x}_{2,f} \end{bmatrix}, \quad \bar{x}_{1,f} \in \Re_+^{n_1}, \quad \bar{x}_{2,f} \in \Re_+^{n_2} \tag{31}$$

is related with $x_f \in \Re_+^n$ by (4).

Theorem 5 *Let the positive system (1) be reachable in q steps and $\bar{u}_i \in \Re_+^{qm}$, $i = 0, 1, \ldots, q-1$ be an input sequence satisfying (18) that steers the state of the system (1) from $x_0 = 0$ to $x_f \in \Re_+^n$. Then the input sequence (30) satisfying (18) also steers the state of the system from $x_0 = 0$ to $x_f \in \Re_+^n$ and minimizes the performance index (17), i.e. $I(\hat{u}) \leqslant I(\bar{u})$. The minimal value of the performance index (11) is given by*

$$I(\hat{u}) = x_f^{\mathrm{T}} W_q^{-1} x_f\,, \tag{32}$$

Proof. The proof is similar to the proof in [13, 19].

If $m = 1$ then the matrix

$$R_\mu = -[B_2 \quad NB_2 \quad \cdots \quad N^{\mu-1}B_2] \in \Re_+^{n_2 \times n_2} \tag{33}$$

is monomial. From (8) we have

$$\bar{x}_{21} = R_\mu u_\mu \tag{34}$$

and

$$u_\mu = R_\mu^{-1} \bar{x}_{2i} \in \Re_+^{n_2}. \tag{35}$$

Note that (35) should satisfy the condition (18). Therefore, the problem has a solution if and only if $u_\mu < U$.

In general case when $m > 1$ the matrix R_μ has μm monomial columns and the equation (34) has the solution

$$u_\mu = \bar{R}_\mu \bar{x}_{2i} \in \Re_+^{n_2}, \tag{36}$$

where

$$\bar{R}_\mu = R_\mu^{\mathrm{T}}\left[R_\mu R_\mu^{\mathrm{T}}\right]^{-1} + R_\mu\left\{I_{n_2} - R_\mu^{\mathrm{T}}\left[R_\mu R_\mu^{\mathrm{T}}\right]^{-1} R_\mu\right\} K \tag{37}$$

and $K \in \Re_+^{n_2 \times n_2}$ is arbitrary matrix.

Therefore, the problem has a solution if and only if $u_\mu < U$ and the following theorem has been proved. □

Theorem 6 *The minimum energy control problem for the positive descriptor system with bounded inputs has a solution only if $u_\mu < U$, where u_μ is defined by (36).*

The optimal input sequence (30) and the minimal value of the performance index (32) can be computed by the use of the following procedure.

Procedure 1

Step 1. Knowing $E, A \in \Re^{n \times n}$, $B \in \Re^{n \times n}$ find matrices $P_1, P_2 \in \Re^{n \times n}$ and using (3), (6c) compute $A_1 \in M_{n_1}$, $B_1 \in \Re_+^{n_1 \times m}$, $B_2 \in \Re_+^{n_2 \times m}$, $N \in \Re_+^{n_2 \times n_2}$.

Step 2. Knowing the matrix Q_1 and using (19)–(20) compute the matrices R_l and W_l.

Step 3. Knowing the matrix Q_2 and using (24)–(25) compute the matrices R_μ and W_μ.

Step 4. Using (4) find the vector $\bar{x}_f$ for given x_f.

Step 5. Using (29) and (30) find the desired input sequence $u_i \in \Re_+^{qm}$, $i = 0, 1, \ldots, q-1$.

Step 6. Using (32) compute the minimal value of the performance index.

Example 2 (Continuation of Example 1)

For the positive system (1) with (11) find an input sequence $u_k \in \Re_+^m$, $k = 0, 1, \ldots$ satisfying the condition (18) with

$$u_k < \frac{1}{3}, \quad k = 0, 1, \ldots \tag{38}$$

that steers the state of the system from $x_0 = 0$ to $x_f = [2 \;\; 1 \;\; 1 \;\; 2]^{\mathrm{T}}$ and minimizes the performance index (17) for

$$Q = diag[2 \;\; 2 \;\; 2 \;\; 2]. \tag{39}$$

Using Procedure 1 we obtain for the subsystem (A_1, B_1) the following.

Step 1. In this case the matrices A_1, B_1, B_2 and N are given by (13) and

$$\bar{x}_i = P_2^{-1} x_i = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0.2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0.2 \\ 2 \end{bmatrix}. \tag{40}$$

Step 2. Taking into account that the reachability matrix

$$R_{1l} = [B_1 \quad A_1 B_1 \quad \cdots \quad A_1^{l-1} B_1] = \begin{bmatrix} 0 & 3 & 0 & 18 & \cdots \\ 1 & 0 & 6 & 0 & \cdots \end{bmatrix} \tag{41}$$

has only monomial columns and using (19) we obtain

$$W_{1l} = R_{1l} Q_{1l}^{-1} R_{1l}^{\mathrm{T}} = \begin{bmatrix} 0 & 3 & 0 & 18 & \cdots \\ 1 & 0 & 6 & 0 & \cdots \end{bmatrix} diag[0.5 \;\; 0.5 \;\; 0.5 \;\; \cdots] \begin{bmatrix} 0 & 1 \\ 3 & 0 \\ 0 & 6 \\ 18 & 0 \\ \vdots & \vdots \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 9 + 18^2 + \ldots & 0 \\ 0 & 1 + 6^2 + \ldots \end{bmatrix} \in \Re_+^{2\times 2}. \tag{42}$$

Step 3. Using (22) and (41)-(42) we obtain the input

$$\hat{u}_2 = \begin{bmatrix} u_1 \\ u_0 \end{bmatrix} = Q_{12}^{-1} R_{12}^{\mathrm{T}} W_{12}^{-1} \bar{x}_{1,f}$$

$$= \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 2/9 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{2}{3} \end{bmatrix} \tag{43}$$

which does not satisfy the condition (38). Therefore, we compute

$$\hat{u}_3 = \begin{bmatrix} u_2 \\ u_1 \\ u_0 \end{bmatrix} = Q_{13}^{-1} R_{13}^{\mathrm{T}} W_{13}^{-1} \bar{x}_{1,f}$$

$$= \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.5 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 3 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} \frac{2}{9} & 0 \\ 0 & \frac{2}{37} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{37} \\ \frac{1}{3} \\ \frac{6}{37} \end{bmatrix}. \tag{44}$$

The input (44) does not satisfy the condition (38) and we continue the procedure

$$\hat{u}_4 = \begin{bmatrix} u_3 \\ u_2 \\ u_1 \\ u_0 \end{bmatrix} = Q_{14}^{-1} R_{14}^{\mathrm{T}} W_{14}^{-1} \bar{x}_{1,f}$$

$$= \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 3 & 0 \\ 0 & 6 \\ 18 & 0 \end{bmatrix} \begin{bmatrix} \frac{2}{9+18^2} & 0 \\ 0 & \frac{2}{37} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{37} \\ \frac{3}{9+18^2} \\ \frac{6}{37} \\ \frac{18}{9+18^2} \end{bmatrix}. \tag{45}$$

The input (45) satisfies the condition (38) and by Theorem 5 is the optimal one for the subsystem (6a).

Step 4. The minimal value of the performance index (17) for the subsystem (6a) is

$$I_1(\hat{u}_4) = \bar{x}_{1,f}^{\mathrm{T}} W_{14}^{-1} \bar{x}_{1,f} = [1 \quad 1] \begin{bmatrix} \frac{2}{9+18^2} & 0 \\ 0 & \frac{2}{37} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{2}{9+18^2} + \frac{2}{37}. \tag{46}$$

Now using Procedure 1 for the subsystem (6b) we obtain the following.

Step 2. Using (14) and (24)–(25) we obtain

$$R_\mu = -[B_2 \quad NB_2 \quad \cdots \quad N^{\mu-1}B_2] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{47}$$

and

$$W_\mu = Q_2^{-1} R_\mu^{\mathrm{T}} W_\mu^{-1} \bar{x}_{2,f} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}. \tag{48}$$

Step 3. Taking into account (47) and (48) we obtain the input

$$u_\mu = Q_2^{-1} R_\mu^{\mathrm{T}} W_\mu^{-1} \bar{x}_{2,f} = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix} \tag{49}$$

which satisfy the condition (38).

Therefore, in this case the optimal input for system (1) with (11) is given by

$$u_i = \begin{cases} u_{1i} & \text{for } i = 0,1,2,3, \\ u_{2i} = \begin{cases} k_i u_i & \text{for } i = 0,1 \text{ and} \\ 0 & \text{for } i = 2,3, \end{cases} & k_0 = \dfrac{0.2(9+18^2)}{18}, \ \dfrac{0.2 \cdot 37}{6}. \end{cases} \tag{50}$$

The minimal value of the performance index (32) of the system is

$$I(\hat{u}) = I_1(\hat{u}_4) + I_2(u_\mu), \tag{51}$$

where $I_1(\hat{u}_4)$ is given by (46) and

$$I_2(u_\mu) = \bar{x}_{2,f}^{\mathrm{T}} W_\mu^{-1} \bar{x}_{2,f} = [0.2 \ \ 0.2] \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix} = 0.16. \tag{52}$$

5. Concluding remarks

Necessary and sufficient conditions for the reachability of the positive descriptor discrete-time linear systems have been given (Theorem 4). The minimum energy control problem for the descriptor discrete-time linear systems by the use of Weierstrass-Kronecker decomposition has been formulated and solved (Theorems 5 and 6). A procedure for computation of the optimal input and the minimal value of the performance index has been proposed. The effectiveness of the procedure has been demonstrated on the example of positive descriptor discrete-time linear system.

The presented method can be extended to positive fractional descriptor continuous-time linear systems with bounded inputs.

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