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Positivity of a class of fractional descriptor continuous-time nonlinear systems

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Abstract. The positivity of a class of fractional descriptor continuous-time nonlinear systems is addressed by the use of the Weierstrass-Kronecker decomposition of the pencil of linear part of nonlinear system. Sufficient conditions for the positivity are established and illustrated by an example of fractional continuous-time descriptor nonlinear systems.

Key words: fractional, descriptor, nonlinear, continuous-time, positive system.

1. Introduction

In positive systems inputs, state variables and outputs take only non-negative values. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear behavior can be found in engineering, management science, economics, social sciences, biology and medicine. An overview of state of art in positive systems theory is given in [1–3].

Descriptor (singular) linear systems have been considered in many papers and books [4–13]. The eigenvalues and invariants assignment by state and output feedbacks have been investigated in [8, 9, 12, 14]. The positive linear systems with different fractional orders have been addressed in [15–17]. Selected problems in theory of fractional linear systems has been given in monograph [16].

Descriptor standard positive linear systems by the use of Drazin inverse has been addressed in [4–6, 17, 18]. The checking of the positivity of descriptor linear systems is addressed in [5, 11, 15]. The stability of positive descriptor systems has been investigated in [14, 19] and the stability of fractional linear systems with delays of the retarded type has been investigated in [20] and of linear systems consisting of n subsystems with different fractional orders in [21]. The practical stability of positive fractional discrete-time systems has been analyzed in [22]. The descriptor standard and positive discrete-time nonlinear systems have been considered in [23] and the stability of a class of positive nonlinear systems in [24].

In this paper sufficient conditions for the positivity of the fractional descriptor continuous-time nonlinear systems will be presented.

The paper is organized as follows. In Sec. 2 sufficient conditions for the positivity of the fractional descriptor nonlinear systems are established by the use of the Weierstrass-Kronecker decomposition of the pencil of linear part of nonlinear system. An illustrating example of positive fractional descriptor nonlinear system is presented in Sec. 3. Concluding remarks are given in Sec. 4.

The following notation will be used: $\Re$ – the set of real numbers, $\Re^{n\times m}$ – the set of $n\times m$ real matrices, Z_+ – the set of nonnegative integers, $\Re_+^{n\times m}$ – the set of $n\times m$ matrices with nonnegative entries and $\Re_+^n = \Re_+^{n\times 1}$, M_n – the set of $n\times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), I_n – the $n\times n$ identity matrix.

2. Positivity of fractional descriptor nonlinear systems

The following Caputo definition of the fractional derivative is used [16]

$$ {}_0D_t^\alpha f(t) = \frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\dot f(\tau)}{(t-\tau)^\alpha}d\tau, \quad 0<\alpha<1, \tag{1}$$

where $\alpha\in\Re$ is the order of the derivative $\dot f(\tau) = \frac{df(\tau)}{d\tau}$ and $\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt$ is the gamma function.

Consider the fractional descriptor nonlinear system

$$E\frac{d^\alpha x(t)}{dt^\alpha} = Ax(t) + f(x(t)), \tag{2}$$

where $x(t)\in\Re^n$ is the state vector, $f(x(t))\in\Re^n$ is the continuous vector function of $x(t)$ and $E,\ A\in\Re^{n\times n}$. For properties of linear approximations of the system (2) see e.g. [25].

It is assumed that $\det E = 0$ and the

$$\det[E\lambda - A] \neq 0 \text{ for some } \lambda\in C \text{ (the field of complex numbers)}. \tag{3}$$

It is well-known [13, 23] that if (3) holds then there exists a pair of nonsingular matrices $P, Q\in\Re^{n\times n}$ such that

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$$P[E\lambda - A]Q = \begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix}\lambda - \begin{bmatrix} A_1 & 0 \\ 0 & I_{n_2} \end{bmatrix} \quad (n_1 + n_2 = n), \tag{4}$$

where n_1 is equal to the degree of the polynomial $\det[E\lambda - A]$, $A_1 \in \Re^{n_1\times n_1}$, $N \in \Re^{n_2\times n_2}$ is a nilpotent matrix with the index μ (i.e. $N^\mu = 0$ and $N^{\mu-1} \neq 0$).

It is assumed that the matrix Q is a monomial matrix (in each row and each column only one entry is positive and the remaining entries are zero).

Premultiplying (2) by the matrix $P \in \Re^{n\times n}$, introducing the new state vector

$$\overline{x}(t) = Q^{-1}x(t) = \begin{bmatrix} \overline{x}_1(t) \\ \overline{x}_2(t) \end{bmatrix},$$

$$\overline{x}_1(t) = \begin{bmatrix} \overline{x}_{11}(t) \\ \overline{x}_{12}(t) \\ \vdots \\ \overline{x}_{1n_1}(t) \end{bmatrix}, \tag{5}$$

$$\overline{x}_2(t) = \begin{bmatrix} \overline{x}_{21}(t) \\ \overline{x}_{22}(t) \\ \vdots \\ \overline{x}_{2n_2}(t) \end{bmatrix}$$

and using (4) we obtain

$$\frac{d^\alpha \overline{x}_1(t)}{dt^\alpha} = A_1\overline{x}_1 + f_1(\overline{x}(t)), \tag{6a}$$

$$N\frac{d^\alpha \overline{x}_2(t)}{dt^\alpha} = \overline{x}_2 + f_2(\overline{x}(t)), \tag{6b}$$

where

$$Pf(x(t)) = Pf[Q\overline{x}(t)] = \begin{bmatrix} f_1(\overline{x}(t)) \\ f_2(\overline{x}(t)) \end{bmatrix}, \quad f_1(\overline{x}(t)) \in \Re^{n_1}, \qquad f_2(\overline{x}(t)) \in \Re^{n_2}. \tag{6c}$$

From (5) it follows that $\overline{x}(t) \in \Re^n_+$, $t \geq 0$ for $x(t) \in \Re^n_+$, $t \geq 0$ if and only if the matrix $Q \in \Re^{n\times n}_+$ is monomial and $Q^{-1} \in \Re^{n\times n}_+$.

If $f_1(\overline{x}(t))$ is given then the solution of Eq. (6a) has the form [16]

$$\overline{x}_1(t) = \Phi_0(t)x_{10} + \int_0^t \Phi(t-\tau)f_1(\overline{x}(\tau))d\tau, \tag{7a}$$

where

$$\Phi_0(t) = \sum_{k=0}^{\infty} \frac{A_1^k t^{k\alpha}}{\Gamma(k\alpha + 1)}, \tag{7b}$$

$$\Phi(t) = \sum_{k=0}^{\infty} \frac{A_1^k t^{(k+1)\alpha - 1}}{\Gamma[(k+1)\alpha]}. \tag{7c}$$

Note that if $A_1 \in M_{n_1}$ than $\phi_0(t) \in \Re^{n_1\times n_1}_+$ for $t \geq 0$. Using Picard method in similar way as in [24] it can be shown that $\overline{x}_1(t) \in \Re^{n_1}_+$, $t \geq 0$ if $f_1(\overline{x}(t)) \in \Re^{n_1}_+$, $\overline{x}(t) \in \Re^n_+$, $t \geq 0$ and $x_{10} \in \Re^{n_1}_+$.

To simplify the notation it is assumed that the matrix N in (6b) has the form

$$N = \begin{bmatrix} 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & ... & 1 \\ 0 & 0 & 0 & ... & 0 \end{bmatrix} \in \Re^{n_2\times n_2} \tag{8}$$

and

$$f_2(\overline{x}(t)) = \begin{bmatrix} f_{21}(\overline{x}(t)) \\ f_{22}(\overline{x}(t)) \\ \vdots \\ f_{2n_2}(\overline{x}(t)) \end{bmatrix} \in \Re^{n_2}_+ \text{ for } \overline{x}(t) \in \Re^n_+,$$

$$f_{2n_2}(\overline{x}(t)) \text{ – arbitrary}$$

$$t \geq 0, \quad f_{2n_2-1}(\overline{x}(t)) = f_{2n_2-1}(\overline{x}_1(t), \overline{x}_{2n_2}(t)) \tag{9}$$

$$\vdots$$

$$f_{22}(\overline{x}(t)) = f_{21}(\overline{x}_1(t), \overline{x}_{2n_2}(t), ..., \overline{x}_{23}(t)).$$

From (6b), (8) and (9) we have

$$\begin{bmatrix} 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & ... & 1 \\ 0 & 0 & 0 & ... & 0 \end{bmatrix} \frac{d^\alpha}{dt^\alpha} \begin{bmatrix} \overline{x}_{21}(t) \\ \overline{x}_{22}(t) \\ \vdots \\ \overline{x}_{2n_2}(t) \end{bmatrix}$$

$$= \begin{bmatrix} \overline{x}_{21}(t) \\ \overline{x}_{22}(t) \\ \vdots \\ \overline{x}_{2n_2}(t) \end{bmatrix} - \begin{bmatrix} f_{21}(\overline{x}_1(t), \overline{x}_{2n_2}(t), ..., \overline{x}_{22}(t)) \\ f_{22}(\overline{x}_1(t), \overline{x}_{2n_2}(t), ..., \overline{x}_{23}(t)) \\ \vdots \\ f_{2n_2}(\overline{x}(t)) \end{bmatrix} \tag{10}$$

and

$$\overline{x}_{2n_2-1}(t) = \frac{d^\alpha \overline{x}_{2n_2}(t)}{dt^\alpha} + f_{2n_2-1}(\overline{x}_1(t), \overline{x}_{2n_2}(t))$$

$$\overline{x}_{2n_2-2}(t) = \frac{d^\alpha \overline{x}_{2n_2-1}(t)}{dt^\alpha} + f_{2n_2-2}(\overline{x}_1(t), \overline{x}_{2n_2}(t), \overline{x}_{2n_2-1}(t))$$

$$\vdots$$

$$\overline{x}_{21}(t) = \frac{d^\alpha \overline{x}_{22}(t)}{dt^\alpha} + f_{21}(\overline{x}_1(t), \overline{x}_{2n_2}(t), ..., \overline{x}_{22}(t)) \tag{11}$$

Note that $\overline{x}_{2n_2}(t)$ and $f_{2n_2}(\overline{x}(t))$ are arbitrary.

Therefore, the following theorem has been proved.

Theorem 1. The fractional descriptor nonlinear system (2) is positive if the following conditions are satisfied

1) the matrix $Q \in \Re_+^{n\times n}$ is monomial, $A_1 \in M_{n_1}$ and $f_1(\overline{x}(t)) \in \Re_+^{n_1}$ for $\overline{x}(t) \in \Re_+^n$,

2) the vector function $f_2(\overline{x}(t))$ satisfies the condition (9).

The considerations can be easily extended to the case when the matrix N in (6b) has the form

$$N = \text{block diag}[N_1, ..., N_q], \quad q \geq 1 \tag{12}$$

and N_k for $k = 1, ..., q$ has the form (8).

3. Example

Consider the fractional descriptor nonlinear system (2) with

$$E = \begin{bmatrix} 0 & 2 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix},$$

$$A = \begin{bmatrix} 0 & 2 & 0 & 0.5 \\ 2 & 1 & 2 & 0 \\ 2 & 1 & 0 & 0.5 \\ 0 & 0 & 4 & 1 \end{bmatrix}, \tag{13}$$

$$f(x(t)) = \begin{bmatrix} x_1(t)x_2(t) + 2x_3^2(t) \\ x_1^2(t) + 4x_2^2(t) \\ x_1(t)x_2(t) + 2x_3^2(t) \\ 2x_1(t)x_2(t) + 2x_1^2(t) + 8x_2^2(t) + 4x_3^2(t) \end{bmatrix}.$$

We shall show that this system is positive one.

The pencil of the system is regular since

$$\det[E\lambda - A] = \begin{vmatrix} 0 & 2\lambda-2 & \lambda & -0.5 \\ \lambda-2 & \lambda-1 & -2 & 0 \\ \lambda-2 & \lambda-1 & \lambda & -0.5 \\ 0 & 0 & 2\lambda-4 & -1 \end{vmatrix} = 8(\lambda-1)(\lambda-2). \tag{14}$$

In this case the matrices P and Q have the form

$$P = \begin{bmatrix} 1 & 0.5 & -0.5 & -0.25 \\ -0.5 & 0.25 & 0.75 & -0.125 \\ 0 & -1 & 1 & 0.5 \\ 0 & 0.5 & -0.5 & 0.25 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{15}$$

and

$$[\overline{E}\lambda - \overline{A}] = P[E\lambda - A]Q = \begin{bmatrix} 1 & 0.5 & -0.5 & -0.25 \\ -0.5 & 0.25 & 0.75 & -0.125 \\ 0 & -1 & 1 & 0.5 \\ 0 & 0.5 & -0.5 & 0.25 \end{bmatrix} \cdot \begin{bmatrix} 0 & 2\lambda-2 & \lambda & -0.5 \\ \lambda-2 & \lambda-1 & -2 & 0 \\ \lambda-2 & \lambda-1 & \lambda & -0.5 \\ 0 & 0 & 2\lambda-4 & -1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \lambda - \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad n_1 = n_2 = 2, \tag{16}$$

$$\begin{bmatrix} f_1(\overline{x}(t)) \\ f_2(\overline{x}(t)) \end{bmatrix} = \begin{bmatrix} 1 & 0.5 & -0.5 & -0.25 \\ -0.5 & 0.25 & 0.75 & -0.125 \\ 0 & -1 & 1 & 0.5 \\ 0 & 0.5 & -0.5 & 0.25 \end{bmatrix} \cdot \begin{bmatrix} x_1(t)x_2(t) + 2x_3^2(t) \\ x_1^2(t) + 4x_2^2(t) \\ x_1(t)x_2(t) + 2x_3^2(t) \\ 2x_1(t)x_2(t) + 2x_1^2(t) + 8x_2^2(t) + 4x_3^2(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \overline{x}_{11}(t)\overline{x}_{12}(t) + \overline{x}_{22}^2(t) \\ \overline{x}_{11}^2(t) + \overline{x}_{12}^2(t) \end{bmatrix}. \tag{17}$$

Using the new state vector

$$\overline{x}(t) = \begin{bmatrix} \overline{x}_1(t) \\ \overline{x}_2(t) \\ \overline{x}_3(t) \\ \overline{x}_4(t) \end{bmatrix} = Q^{-1}x(t) = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} 2x_2(t) \\ x_1(t) \\ x_4(t) \\ 2x_3(t) \end{bmatrix} \tag{18}$$

and (6) we obtain

$$\frac{d^\alpha}{dt^\alpha} \begin{bmatrix} \overline{x}_{11}(t) \\ \overline{x}_{12}(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \overline{x}_{11}(t) \\ \overline{x}_{12}(t) \end{bmatrix} \tag{19}$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \frac{d^\alpha}{dt^\alpha} \begin{bmatrix} \overline{x}_{21}(t) \\ \overline{x}_{22}(t) \end{bmatrix} = \begin{bmatrix} \overline{x}_{21}(t) \\ \overline{x}_{22}(t) \end{bmatrix} - \begin{bmatrix} \overline{x}_{11}(t)\overline{x}_{12}(t) + \overline{x}_{22}^2(t) \\ \overline{x}_{11}^2(t) + \overline{x}_{12}^2(t) \end{bmatrix}. \tag{20}$$

Taking into account that in this case $f_1(\overline{x}(\tau)) = 0$ from (7a) and (7b) we obtain

$$\begin{bmatrix} \overline{x}_{11}(t) \\ \overline{x}_{12}(t) \end{bmatrix} = \sum_{k=0}^{\infty} \frac{t^{k\alpha}}{\Gamma(k\alpha+1)} \cdot \begin{bmatrix} 1^{k\alpha} & 0 \\ 0 & 2^{k\alpha} \end{bmatrix} \begin{bmatrix} \overline{x}_{11}(0) \\ \overline{x}_{12}(0) \end{bmatrix}, \tag{21}$$

where

$$\begin{bmatrix} \overline{x}_{11}(0) \\ \overline{x}_{12}(0) \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 2x_2(0) \\ x_1(0) \end{bmatrix} \tag{22}$$

for given initial conditions

$$x(0) = [x_1(0),\ x_2(0),\ x_3(0),\ x_4(0)]^T$$

and T denotes the transpose.

From (21) and (22) it follows that

$$\begin{bmatrix} \overline{x}_{11}(t) \\ \overline{x}_{12}(t) \end{bmatrix} \in \Re_+^2 \quad \text{for} \quad t \geq 0$$

$$\text{and} \quad \begin{bmatrix} \overline{x}_{11}(0) \\ \overline{x}_{12}(0) \end{bmatrix} \in \Re_+^2.$$

From (20) and (13) we have

$$\overline{x}_{22}(t) = \overline{x}_{11}^2(t) + \overline{x}_{12}^2(t),$$

$$\overline{x}_{21}(t) = \frac{d^\alpha}{dt^\alpha}\overline{x}_{22}(t) + \overline{x}_{11}(t)\overline{x}_{12}(t) + \overline{x}_{22}^2(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\dot{x}_{22}(\tau)}{(t-\tau)^\alpha} d\tau + \overline{x}_{11}(t)\overline{x}_{12}(t) + \overline{x}_{22}^2(t). \tag{23}$$

From (23) it follows that also $\overline{x}_{22}(t) \in \Re_+$ and $\overline{x}_{21}(t) \in \Re_+$ for $t \geq 0$.

Therefore, from (18) we have $x(t) = Q\overline{x}(t) \in \Re_+^4$ for $t \geq 0$, $x(0) \in \Re_+^4$ and the fractional descriptor nonlinear system (2) with (13) is positive one.

Note that the system satisfies the conditions of Theorem 1.

4. Concluding remarks

The positivity of a class of fractional descriptor continuous-time nonlinear systems has been addressed. Sufficient conditions for the positivity of the class of fractional descriptor nonlinear systems has been established. The conditions has been illustrated on an example of fractional descriptor nonlinear systems. An open problem is an extension of the considerations to the fractional descriptor continuous-time nonlinear systems described by the equation

$$E\frac{d^\alpha x(t)}{dt^\alpha} = Ax(t) + f(x(t)) + g(x(t))u(t),$$

where

$$x(t) \in \Re^n, \qquad u(t) \in \Re^n,$$

$$t \geq 0, \qquad E, A \in \Re^{n\times n},$$

$$\det[E\lambda - A] \neq 0, \qquad f(x(t)) \in \Re^n,$$

$$g(x(t)) \in \Re^n, \qquad t \geq 0.$$

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