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NORDHAUS–GADDUM BOUNDS FOR UPPER TOTAL DOMINATION

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Abstract. A set S of vertices in an isolate-free graph G is a total dominating set if every vertex in G is adjacent to a vertex in S. A total dominating set of G is minimal if it contains no total dominating set of G as a proper subset. The upper total domination number $\Gamma_t(G)$ of G is the maximum cardinality of a minimal total dominating set in G. We establish Nordhaus–Gaddum bounds involving the upper total domination numbers of a graph G and its complement $\overline{G}$. We prove that if G is a graph of order n such that both G and $\overline{G}$ are isolate-free, then $\Gamma_t(G) + \Gamma_t(\overline{G}) \leq n + 2$ and $\Gamma_t(G)\Gamma_t(\overline{G}) \leq \frac{1}{4}(n + 2)^2$, and these bounds are tight.

Keywords: upper total domination, Nordhaus–Gaddum bounds.

Mathematics Subject Classification: 05C69.

1. INTRODUCTION

A set S of vertices in a graph G is a *dominating set* if every vertex not in S is adjacent to a vertex in S. Further if every vertex in G is adjacent to some other vertex in S, then S is a *total dominating set*, abbreviated TD-set of G. An *independent dominating set*, abbreviated ID-set, of G is a dominating set S with the added property that S is an independent set. Equivalently, an ID-set is a maximal independent set of G. The *domination number* $\gamma(G)$ of G is the minimum cardinality of a dominating set of G, while the *total domination number* $\gamma_t(G)$ of G is the minimum cardinality of a TD-set of G. The *independent domination number* $i(G)$ of G is the minimum cardinality of an ID-set of G. For recent books on domination in graphs, we refer the reader to [10, 11].

The *upper domination number* $\Gamma(G)$ of G is the maximum cardinality of a minimal dominating set in G, and the *upper total domination number* $\Gamma_t(G)$ of G is the maximum cardinality of a minimal TD-set in G. A minimal TD-set of cardinality $\Gamma_t(G)$ is called a *Γ_t-set of G*. The *independence number* $\alpha(G)$ of G is the maximum cardinality of an independent set in G. By definition of the core domination parameters, we have

$$\gamma(G) \le i(G) \le \alpha(G) \le \Gamma(G) \quad \text{and} \quad \gamma(G) \le \gamma_t(G) \le \Gamma_t(G).$$

In 1956 Nordhaus and Gaddum [16] established bounds for the sum and product of the chromatic numbers of a graph G and its complement $\overline{G}$. Subsequently, (tight) lower or upper bounds on the sum or product of a parameter of a graph and its complement were coined *Nordhaus–Gaddum bounds*. We refer the reader to the excellent 2013 survey paper on Nordhaus–Gaddum bounds by Aouchiche and Hansen [1].

Nordhaus–Gaddum bounds for the domination number $\gamma(G)$, the independent domination number $i(G)$, the independence number $\alpha(G)$, the upper domination number $\Gamma(G)$, and the total domination number $\gamma_t(G)$ are known, and are presented in the following papers [3, 4, 6–9, 14, 15, 17, 18]. In this paper, we complete the study by presenting Nordhaus–Gaddum bounds for the upper domination number $\Gamma_t(G)$.

We proceed as follows. In Section 1.1, we present the graph theory notation and terminology we adopt in this paper. Thereafter, we present known Nordhaus–Gaddum bounds for $\gamma(G)$, $i(G)$, $\alpha(G)$, $\Gamma(G)$, and $\gamma_t(G)$ for a graph G in Section 2. Our main result, which determines tight Nordhaus–Gaddum bounds for the sum $\Gamma_t(G) + \Gamma_t(\overline{G})$ and the product $\Gamma_t(G)\Gamma_t(\overline{G})$ of the upper total numbers of a graph G and its complement $\overline{G}$, both of which are isolate-free, is given in Section 3.

1.1. GRAPH THEORY NOTATION AND TERMINOLOGY

We generally adopt the notation and graph theory terminology in the book [13]. The *order* of a graph G with vertex set $V(G)$ and edge set $E(G)$ is denoted by $n(G) = |V(G)|$ and its *size* by $m(G) = |E(G)|$. The *open neighborhood* of a vertex v in G, denoted $N(v)$ (or $N_G(v)$ to refer to G), is the set $\{u \in V(G) \colon uv \in E(G)\}$, and the *closed neighborhood* of v is the set $N[v] = N_G[v] = N_G(v) \cup \{v\}$. The *degree* of a vertex v in G equals $|N_G(v)|$. An *isolated vertex* is a vertex of degree 0, and a graph is *isolate-free* if it contains no isolated vertex. For a set $S \subseteq V(G)$, the subgraph induced by S in G is denoted $G[S]$. If $S \subseteq V(G)$, then the *open S-private neighborhood* of a vertex v in S is the set

$$\mathrm{pn}(v, S) = \{w \in V(G) \colon N_G(w) \cap S = \{v\}\}.$$

The set $\mathrm{pn}(v, S) \setminus S$ is the *open S-external private neighborhood* of v, abbreviated $\mathrm{epn}(v, S)$, while the *open S-internal private neighborhood* of v is the set $\mathrm{ipn}(v, S) = \mathrm{pn}(v, S) \cap S$. Thus,

$$\mathrm{epn}(v, S) = \mathrm{pn}(v, S) \setminus S,$$
$$\mathrm{ipn}(v, S) = \mathrm{pn}(v, S) \cap S,$$

and

$$\mathrm{pn}(v, S) = \mathrm{ipn}(v, S) \cup \mathrm{epn}(v, S).$$

Our focus in this paper is on total domination in graphs, which deals with open neighborhoods. Therefore, for notational simplicity we omit the term "open" in the above definitions. For example, we refer to the *open S-private neighborhood* as the *S-private neighborhood.* We call a vertex in $\mathrm{epn}(v, S)$ an *S-external private neighbor* of v, and a vertex in $\mathrm{ipn}(v, S)$ we call an *S-internal private neighbor* of v (where again we omit the term "open").

2. KNOWN RESULTS

The following fundamental property of minimal TD-sets was established in 1980 by Cockayne, Dawes and Hedetniemi [2].

Lemma 2.1 ([2])**.** *A TD-set S in a graph G is a minimal TD-set in G if and only if* $|\mathrm{epn}(v, S)| \geq 1$ *or* $|\mathrm{ipn}(v, S)| \geq 1$ *for every vertex $v \in S$.*

In 1972 Jaeger and Payan [15] presented the first Nordhaus–Gaddum bounds involving the domination number.

Theorem 2.2 ([15])**.** *If G is a graph of order $n \geq 2$, then the following inequalities hold:*

a) $\gamma(G) + \gamma(\overline{G}) \leq n + 1$,
b) $\gamma(G)\gamma(\overline{G}) \leq n$.

Nordhaus–Gaddum bounds involving the sum of the independent domination numbers of a graph and its complement can be established using a similar proof of Theorem 2.2 as given in [15]. However, the situation is considerably more complex for the product of the independent domination numbers. In 2003 Goddard and Henning [7] determined exactly the maximum possible value for the product $i(G)i(\overline{G})$.

Theorem 2.3 ([7, 15])**.** *If G is a graph of order $n \geq 2$, then the following inequalities hold:*

a) $i(G) + i(\overline{G}) \leq n + 1$ (see [15]),
b) $i(G)i(\overline{G}) \leq \frac{1}{16}(n + 4)^2$ (see [7]).

In 2003 Goddard and Henning [7] also determined exactly the maximum possible value for the sum $i(G) + i(\overline{G})$ when both G and $\overline{G}$ are isolate-free.

Theorem 2.4 ([7])**.** *If G and its complement $\overline{G}$ are isolate-free graphs of order n, then*

$$i(G) + i(\overline{G}) \leq n + 4 - 2\sqrt{n},$$

and this is best possible (*if one rounds down*) *for all n* (*necessarily, $n \geq 4$*)*.*

Nordhaus–Gaddum bounds on the independence number and upper domination number were given by Cockayne and Mynhardt [4] in 1989. (We remark that they proved a stronger result and showed that the same result holds for the upper irredundance number $\mathrm{IR}(G)$, but we do not define this well studied parameter here.)

Theorem 2.5 ([4])**.** *If G is a graph of order n, then the following inequalities hold:*

a) $\alpha(G) + \alpha(\overline{G}) \leq \Gamma(G) + \Gamma(\overline{G}) \leq n + 1$,
b) $\alpha(G)\alpha(\overline{G}) \leq \Gamma(G)\Gamma(\overline{G}) \leq \frac{1}{4}(n + 1)^2$.

Cockayne, Dawes, and Hedetniemi [2] in 1980 provided a Nordhaus–Gaddum upper bound on the sum of the total domination numbers of a graph and its complement, while Henning, Joubert, and Southey [14] in 2011 established a Nordhaus–Gaddum upper bound on the product of the total domination numbers of a graph and its complement.

Theorem 2.6 ([2, 14])**.** *If G and its complement $\overline{G}$ are isolate-free graphs of order n, then following inequalities hold:*

a) $\gamma_t(G) + \gamma_t(\overline{G}) \le n + 2$ (see [2]),
b) $\gamma_t(G)\gamma_t(\overline{G}) \le 2n$ (see [14]).

We remark that the bounds given in Theorems 2.2–2.6 are all best possible. For example, equality holds in the bounds in Theorem 2.6(a) and (b) if and only if G or $\overline{G}$ consists of disjoint copies of K_2. We summarize these known Nordhaus–Gaddum bounds for $\gamma(G)$, $i(G)$, $\alpha(G)$, $\Gamma(G)$, and $\gamma_t(G)$ in Table 1, according to their publication date.

Table 1
A summary of Nordhaus–Gaddum bounds for $\gamma(G)$, $i(G)$, $\alpha(G)$, $\Gamma(G)$, and $\gamma_t(G)$

Year	μ	$\mu(G) + \mu(\overline{G})$	$\mu(G)\mu(\overline{G})$	Reference
1972	$\gamma(G)$	$n+1$	n	[15]
1972	$i(G)$	$n+1$	–	[15]
1980	$\gamma_t(G)$	$n+2$	–	[2]
1989	$\alpha(G)$	$n+1$	$\frac{1}{4}(n+1)^2$	[4]
1989	$\Gamma(G)$	$n+1$	$\frac{1}{4}(n+1)^2$	[4]
2003	$i(G)$	–	$\frac{1}{16}(n+4)^2$	[7]
2011	$\gamma_t(G)$	–	$2n$	[14]

Nordhaus–Gaddum upper bounds on the sum and product of the upper total domination numbers of a graph and its complement have not, to the best of the authors' knowledge, been presented in the literature before. Our immediate objective in this paper is therefore to fill this gap, and to complete the study of Nordhaus–Gaddum bounds for the core domination parameters $\gamma(G)$, $i(G)$, $\alpha(G)$, $\Gamma(G)$, $\gamma_t(G)$, and $\Gamma_t(G)$.

We observe that in 2008 Dorbec, Henning, and Rall [5] showed that the parameters $\Gamma(G)$ and $\Gamma_t(G)$ are incomparable for isolate-free graphs G, and hence Nordhaus–Gaddum upper bounds for $\Gamma_t(G)+\Gamma_t(\overline{G})$ and $\Gamma_t(G)\Gamma_t(\overline{G})$ cannot be deduced from the known Nordhaus–Gaddum upper bounds for $\Gamma(G) + \Gamma(\overline{G})$ and $\Gamma(G)\Gamma(\overline{G})$. Indeed, the following relation between the upper domination and upper total domination numbers was established in [5], where both the lower and upper bounds are tight.

Theorem 2.7. ([5]) *If G is an isolate-free graph, then* $(\frac{2}{n-1})\Gamma(G) \le \Gamma_t(G) \le 2\Gamma(G)$.

3. MAIN RESULT

In this section, we present Nordhaus–Gaddum upper bounds on the sum $\Gamma_t(G)+\Gamma_t(\overline{G})$ and the product $\Gamma_t(G)\Gamma_t(\overline{G})$, where G and $\overline{G}$ are isolate-free. We shall prove the following result.

Theorem 3.1. *If G and its complement $\overline{G}$ are isolate-free graphs of order n, then the following inequalities hold:*

a) $\Gamma_t(G) + \Gamma_t(\overline{G}) \leq n + 2$,
b) $\Gamma_t(G)\Gamma_t(\overline{G}) \leq \lfloor \frac{1}{4}(n + 2)^2 \rfloor$.

Both bounds are sharp.

Proof. Let X be a Γ_t-set of G, and let Y be a Γ_t-set of $\overline{G}$. Thus, X is a minimal TD-set of G satisfying $\Gamma_t(G) = |X|$, and Y is a minimal TD-set of $\overline{G}$ satisfying $\Gamma_t(\overline{G}) = |Y|$. Let $|X| = x$ and $|Y| = y$. Let $S = X \cap Y$ and $T = V(G) \setminus (X \cup Y)$. Further, let $|S| = s$ and $|T| = t$. We note that $X \cup Y = V(G) \setminus T$, and so $x + y - s = n - t$. Thus,

$$x + y = n + (s - t) \tag{3.1}$$

and

$$xy = x(n - x + s - t). \tag{3.2}$$

We proceed further with the following claim.

Claim 3.2. $s \leq t + 2$.

Proof. Suppose, to the contrary, that $s \geq t+3$. Let $S = \{v_1, v_2, \ldots, v_s\}$. By supposition, $s \geq 3$. At least one of $G[S]$ and $\overline{G}[S]$ is connected. We may assume, without loss of generality, that $G[S]$ is connected. Further, if $\overline{G}[S]$ is connected, then we may assume, without loss of generality, that the number of vertices of degree at least 2 in $G[S]$ is at least as large as the number of vertices of degree at least 2 in $\overline{G}[S]$. Since X is a minimal TD-set of G, by Lemma 2.1, we have $|\mathrm{epn}_G(v, X)| \geq 1$ or $|\mathrm{ipn}_G(v, X)| \geq 1$ for every vertex $v \in X$. □

Claim 3.3. *If $|\mathrm{epn}_G(v, X)| \geq 1$ for some vertex $v \in X$, then $\mathrm{epn}_G(v, X) \subseteq T$.*

Proof. Let $|\mathrm{epn}_G(v, X)| \geq 1$ for some vertex $v \in X$ and suppose, to the contrary, that $\mathrm{epn}_G(v, X) \nsubseteq T$. Thus, there exists a vertex $y \notin T$ such that $y \in \mathrm{epn}_G(v, X)$. Necessarily, $y \in Y \setminus S$. We now consider the complement $\overline{G}$. The vertex y is adjacent to every vertex of $X \setminus \{v\}$ in $\overline{G}$, and the vertex v is the only vertex in X that is not adjacent to y in $\overline{G}$.

Suppose firstly that $v \in S$. In this case, the vertex y is adjacent to every vertex of $S \setminus \{v\}$ and $|S \setminus \{v\}| = s - 1 \geq 2$. Since Y is a minimal TD-set of $\overline{G}$, by Lemma 2.1, we have $|\mathrm{epn}_{\overline{G}}(u, Y)| \geq 1$ or $|\mathrm{ipn}_{\overline{G}}(u, Y)| \geq 1$ for every vertex $u \in Y$. In particular, $|\mathrm{epn}_{\overline{G}}(u, Y)| \geq 1$ or $|\mathrm{ipn}_{\overline{G}}(u, Y)| \geq 1$ for every vertex $u \in S$. If $|\mathrm{epn}_{\overline{G}}(u, Y)| \geq 1$ for some vertex $u \in S$, then $\mathrm{epn}_{\overline{G}}(u, Y) \subseteq T$ since y dominates the set $X \setminus S$ in $\overline{G}$. If $|\mathrm{ipn}_{\overline{G}}(u, Y)| \geq 1$ for some vertex $u \in S$, then either $v \in \mathrm{ipn}_{\overline{G}}(u, Y)$ or $\mathrm{ipn}_{\overline{G}}(u, Y) \subseteq Y \setminus \{y\}$ since $s \geq 3$ and y dominates the set $S \setminus \{v\}$ in $\overline{G}$. Moreover, if $z \in \mathrm{ipn}_{\overline{G}}(u, Y)$ and $z \in Y \setminus \{y\}$, then $|\mathrm{epn}_{\overline{G}}(z, Y)| \geq 1$ and we can uniquely associate the set $\mathrm{epn}_{\overline{G}}(z, Y)$ with the vertex u. As observed earlier, since y dominates the set $X \setminus S$, in this case, we must have $\mathrm{epn}_{\overline{G}}(z, Y) \subseteq T$. We note that at most one vertex in $S \setminus \{v\}$ has the vertex v as a Y-internal private neighbor in $\overline{G}$. Further, we can uniquely associate with each vertex in $S \setminus \{v\}$ at least one vertex in T. These observations imply that we can associate at least $s - 2$ vertices of T with the vertices in the set S, implying that $t = |T| \geq s - 2$, a contradiction.

Suppose secondly that $v \in X \setminus S$. In this case, the vertex y is adjacent to every vertex of $X \setminus \{v\}$. In particular, the vertex y is adjacent to every vertex of S. As in the previous paragraph, $|\mathrm{epn}_{\overline{G}}(u, Y)| \geq 1$ or $|\mathrm{ipn}_{\overline{G}}(u, Y)| \geq 1$ for every vertex $u \in S$. At most one vertex in the set Y has the vertex v as a Y-external private neighbor in $\overline{G}$. For all other vertices w that belong to Y and satisfy $|\mathrm{epn}_{\overline{G}}(w, Y)| \geq 1$, we have $\mathrm{epn}_{\overline{G}}(w, Y) \subseteq T$. If $w \in S$ and $|\mathrm{ipn}_{\overline{G}}(w, Y)| \geq 1$, then since y dominates the set S in $\overline{G}$, we note that $\mathrm{ipn}_{\overline{G}}(w, Y) \subseteq Y \setminus S$. In this case, if $w' \in \mathrm{ipn}_{\overline{G}}(w, Y)$, then $|\mathrm{epn}_{\overline{G}}(w', Y)| \geq 1$. Furthermore, if $v \notin \mathrm{epn}_{\overline{G}}(w', Y)$, then $\mathrm{epn}_{\overline{G}}(w', Y) \subseteq T$ and we can uniquely associate the set $\mathrm{epn}_{\overline{G}}(w', Y)$ with the vertex $w \in S$. These observations imply that we can associate at least $s - 1$ vertices of T with the vertices in the set S, implying that $t = |T| \geq s - 1$, a contradiction. □

By Claim 3.3, if $|\mathrm{epn}_G(v, X)| \geq 1$ for some vertex $v \in X$, then $\mathrm{epn}_G(v, X) \subseteq T$. As observed earlier, $|\mathrm{epn}_G(v, X)| \geq 1$ or $|\mathrm{ipn}_G(v, X)| \geq 1$ for every vertex $v \in X$. Let

$$S_1 = \{v \in S \colon |\mathrm{epn}_G(v, X)| \geq 1\} \quad \text{and} \quad S_2 = S \setminus S_1.$$

We note that

$$S = S_1 \cup S_2.$$

If $v \in S_2$, then $\mathrm{epn}_G(v, X) = \emptyset$, and so by Lemma 2.1, we have $\mathrm{ipn}_G(v, X) \neq \emptyset$. Further, we note that if $u \in \mathrm{ipn}_G(v, X)$ for some vertex $v \in X$, then the vertex v is the unique neighbor of u in $G[X]$, that is, the vertex u has degree 1 in $G[X]$.

Claim 3.4. *If $v \in S_2$, then $\mathrm{ipn}_G(v, X) \cap S_2 = \emptyset$.*

Proof. Let $v \in S_2$ and suppose, to the contrary, that $\mathrm{ipn}_G(v, X) \cap S_2 \neq \emptyset$. Thus, $\mathrm{epn}_G(v, X) = \emptyset$ and there is an X-internal private neighbor of v in G that belongs to the set S_2. Let u be such a neighbor of v. Thus, $u \in \mathrm{ipn}_G(v, X)$ and $u \in S_2$, and so the vertex u has degree 1 in $G[X]$ with v as its only neighbor in $G[X]$. Since $\mathrm{epn}_G(u, X) = \emptyset$, by the minimality of the set X, we have $\mathrm{ipn}_G(u, X) \neq \emptyset$, implying that $\mathrm{ipn}_G(u, X) = \{v\}$ and that the vertex v has degree 1 in $G[X]$. Thus, the vertices u and

v form a K_2-component in the subgraph $G[X]$, and therefore also in the subgraph $G[S]$. Since $|S| = s \geq 3$, the subgraph $G[S]$ is therefore disconnected, a contradiction. $\square$

Let

$$S_{2,1} = \{v \in S_2 \colon |\mathrm{ipn}_G(v, X) \cap S_1| \geq 1\},$$
$$S_{2,2} = \{v \in S_2 \colon \mathrm{ipn}_G(v, X) \subseteq X \setminus S\}.$$

By Claim 3.4, if $v \in S_2$, then $\mathrm{ipn}_G(v, X) \cap S_2 = \emptyset$, and so

$$S_2 = S_{2,1} \cup S_{2,2}.$$

For each vertex $v \in S_1$, we have $|\mathrm{epn}_G(v, X)| \geq 1$. By Claim 3.3, $\mathrm{epn}_G(v, X) \subseteq T$. We uniquely associate the set $\mathrm{epn}_G(v, X)$ with the vertex v.

Suppose that $S_{2,2} \neq \emptyset$, and consider a vertex $v \in S_{2,2}$. Let $u \in \mathrm{ipn}_G(v, X)$, and so $u \in X \setminus S$. By Lemma 2.1, we have $|\mathrm{epn}_G(u, X)| \geq 1$ or $|\mathrm{ipn}_G(u, X)| \geq 1$. If $\mathrm{epn}_G(u, X) = \emptyset$, then it follows that $\mathrm{ipn}_G(u, X) = \{v\}$ and that the vertex v has degree 1 in $G[X]$. But this implies that v is an isolated vertex in $G[S]$, a contradiction since $|S| \geq 3$ and $G[S]$ is connected. Hence, $\mathrm{epn}_G(u, X) \neq \emptyset$. By Claim 3.3, $\mathrm{epn}_G(u, X) \subseteq T$. We now uniquely associate the set $\mathrm{epn}_G(u, X)$ with the vertex $v \in S_{2,2}$.

Let W_S be the set of all X-external private neighbors in G that are associated with vertices that belong to the set $S_1 \cup S_{2,2}$. Hence, if $w \in W_S$, then either $w \in \mathrm{epn}_G(v, X)$ for some vertex $x \in S_1$ or $w \in \mathrm{epn}_G(u, X)$ for some vertex $u \in X \setminus S$ that is adjacent to the vertex $v \in S_{2,2}$. We note that $W_S \subseteq T$ and

$$|W_S| \geq |S_1| + |S_{2,2}| = s - |S_{2,1}|.$$

By supposition, $s \geq t + 3$, and so

$$s - 3 \geq t = |T| \geq |W_S| \geq s - |S_{2,1}|,$$

implying that $|S_{2,1}| \geq 3$. Let $\{u_1, u_2, u_3\} \subseteq S_{2,1}$ and let $v_i \in \mathrm{ipn}_G(v, X) \cap S_1$ for $i \in [3]$. By our earlier observations, the vertex v_i has degree 1 in $G[X]$ with the vertex u_i as its unique neighbor in $G[X]$ for $i \in [3]$. We now consider the complement $\overline{G}$. The vertex v_i is adjacent to every vertex of $X \setminus \{u_i\}$ in $\overline{G}$ for all $i \in [3]$. These observations imply that $\overline{G}[S]$ is a connected graph and all s vertices in the set S have degree at least 2 in $\overline{G}[S]$. However, $G[S]$ is a connected graph with at least three vertices of degree 1 in $G[S]$, and therefore at most $s - 3$ vertices of degree at least 2 in $G[S]$. This contradicts our assumption that if both $G[S]$ and $\overline{G}[S]$ are connected graphs, then there are at least as many vertices of degree at least 2 in $G[S]$ as in $\overline{G}[S]$. We deduce, therefore, that our supposition that $s \geq t + 3$ is incorrect, that is, $s \leq t + 2$, which completes the proof of Claim 3.2.

By Claim 3.2, $s \leq t + 2$. By Equation (3.1), we have

$$\Gamma_t(G) + \Gamma_t(\overline{G}) = x + y = n + (s - t) \leq n + 2.$$

This proves Part a). To prove Part b), by Part a) and Equation (3.2), we have

$$xy = x(n - x + s - t) \leq x(n - x + 2).$$

Using elementary calculus, the function $x(n - x + 2)$ is maximized when $x = \frac{1}{2}(n + 2)$, yielding

$$\Gamma_t(G)\Gamma_t(\overline{G}) = xy \leq \frac{1}{4}(n + 2)^2. \tag{3.3}$$

Since $\Gamma_t(G)$ and $\Gamma_t(\overline{G})$ are integral, the upper bound in Part (b) now follows.

To show that the bounds are sharp, we note that if G is the disjoint union of $k \geq 2$ copies of K_2, that is, if $G = kK_2$, then G has order $n = 2k$ and $\Gamma_t(G) = 2k$ and $\Gamma_t(\overline{G}) = 2$, and so $\Gamma_t(G) + \Gamma_t(\overline{G}) = n + 2$. Thus, the bound in Part (a) on the sum of the upper total domination numbers of a graph and its complement is sharp.

To construct an extremal graph for the product of the upper total domination numbers of a graph and its complement that achieves the bound in Part (b), let $k \geq 2$ be an integer and let G be a graph of order $n = 4k + 2$ with $V(G)$ partitioned into sets V_1 and V_2, where $|V_1| = 2k + 2$ and $|V_2| = 2k$. The edge set of G is defined as follows. Let $G[V_1] \cong K_{2k+2} - M_1$, where M_1 is a perfect matching in the complete graph K_{2k+2} with vertex set V_1, and let $G[V_2] = kK_2$. We note that the vertices in V_2 have degree 1 in G. Equivalently, in the complement $\overline{G}$ of G, we have $\overline{G}[V_1] = (k + 1)K_2$ and $\overline{G}[V_2] \cong K_{2k} - M_2$, where M_2 is a perfect matching in the complete graph K_{2k} with vertex set V_2. Further, all edges between the vertices V_1 and the vertices V_2 are present in $\overline{G}$. The set $D_2 = V_2 \cup \{u_1, v_1\}$ is a Γ_t-set in G, where u_1 and v_1 are any two vertices in V_1 that are adjacent in G. We note that $G[D_2] = (k + 1)K_2$. Moreover, the set V_1 is a Γ_t-set in $\overline{G}$. Thus,

$$\Gamma_t(G) = |D_2| = 2(k + 1) \quad \text{and} \quad \Gamma_t(\overline{G}) = |V_1| = 2(k + 1),$$

and so, in this example,

$$\Gamma_t(G)\Gamma_t(G) = 4(k + 1)^2 = \frac{1}{4}(n + 2)^2. \qquad \square$$

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REFERENCES

[1] M. Aouchiche, P. Hansen, *A survey of Nordhaus–Gaddum type relations*, Discrete Appl. Math. **161** (2013), 466–546.

[2] E.J. Cockayne, R.M. Dawes, S.T. Hedetniemi, *Total domination in graphs*, Networks **10** (1980), no. 3, 211–219.

[3] E.J. Cockayne, G. Fricke, C.M. Mynhardt, *On a Nordhaus–Gaddum type problem for independent domination*, Discrete Math. **138** (1995), 199–205.

[4] E.J. Cockayne, C.M. Mynhardt, *On the product of upper irredundance numbers of a graph and its complement*, Discrete Math. **76** (1989), 117–121.

[5] P. Dorbec, M.A. Henning, D.F. Rall, *On the upper total domination number of Cartesian products of graphs*, J. Comb. Optim. **16** (2008), 68–80.

[6] J.E. Dunbar, T.W. Haynes, S.T. Hedetniemi, *Nordhaus–Gaddum bounds for domination sums in graphs with specified minimum degree*, Util. Math. **67** (2005), 97–105.

[7] W. Goddard, M.A. Henning, *Nordhaus–Gaddum bounds for independent domination*, Discrete Math. **268** (2003), 299–302.

[8] W. Goddard, M.A. Henning, H.C. Swart, *Some Nordhaus–Gaddum type results*, J. Graph Theory **16** (1992), 221–231.

[9] F. Harary, T.W. Haynes, *Nordhaus–Gaddum inequalities for domination in graphs*, Discrete Math. **155** (1996), 99–105.

[10] T.W. Haynes, S.T. Hedetniemi, M.A. Henning (eds), *Topics in Domination in Graphs*, Developments in Mathematics, vol. 64, Springer, Cham, 2020.

[11] T.W. Haynes, S.T. Hedetniemi, M.A. Henning (eds), *Structures of Domination in Graphs*, Developments in Mathematics, vol. 66, Springer, Cham, 2021.

[12] T.W. Haynes, S.T. Hedetniemi, M.A. Henning (eds), *Domination in Graphs: Core Concepts*, Developments in Mathematics, Springer, Cham, 2022.

[13] M.A. Henning, A. Yeo, *Total Domination in Graphs*, Springer Monographs in Mathematics, Springer, New York, 2013.

[14] M.A. Henning, E.J. Joubert, J. Southey, *Nordhaus–Gaddum bounds for total domination*, Appl. Math. Lett. **24** (2011), 987–990.

[15] F. Jaeger, C. Payan, *Relations du type Nordhaus–Gaddum pour le nombre d'absorption d'un graphe simple*, C.R. Acad. Sci. Paris A, **274** (1972), 728–730.

[16] E.A. Nordhaus, J.W. Gaddum, *On complementary graphs*, Amer. Math. Monthly **63** (1956), 175–177.

[17] L. Volkmann, *Nordhaus–Gaddum bounds for domination sums of graphs with minimum degree at least two or three*, Util. Math. **82** (2010), 3–9.

[18] L. Volkmann, *Nordhaus–Gaddum type results for domination sums in graphs with minimum degree at least four, five or six*, Util. Math. **85** (2011), 113–127.

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