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Controllability of a class of infinite dimensional systems with age structure*

by

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Dedicated to Günter Leugering for his 65th birthday with friendship, gratitude and admiration

Abstract: Given a linear dynamical system, we investigate the linear infinite dimensional system obtained by grafting an age structure. Such systems appear essentially in population dynamics with age structure when phenomena like spatial diffusion or transport are also taken into consideration. We first show that the new system preserves some of the wellposedness properties of the initial one. Our main result asserts that if the initial system is null controllable in a time small enough then the structured system is also null controllable in a time depending on the various involved parameters.

Keywords: infinite dimensional linear system, age structure, admissible control operator, null controllability, population dynamics

1. Introduction

Infinite dimensional dynamical systems coupling age structuring with diffusion or transport phenomena appear naturally in population dynamics, medicine or epidemiology (see, for instance, Brikci et al., 2008; Webb, 1985,1988; Magal and Ruan, 2018). A by now classical example is the Lotka-McKendrick system with

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spatial diffusion (Gurtin, 1973). For the convenience of the reader, we describe below the type of systems to be considered using a simplified example. To this aim, let X (the state space) and U (the input space) be finite dimensional inner product spaces. Our departure point is the linear time invariant control system described by

$$\dot{p}(t) = Ap(t) + Bu(t), \tag{1}$$

where $A : X \to X$ and $B : U \to X$ are linear operators. The system (1) is supposed to describe the evolution of a certain population density (particles, individuals,...) and it is possibly obtained by approximating a partial differential system. Adding an age structure to the system described by (1) means that we assume that p depends not only on t, but also on the age parameter a, which lies in some bounded interval $[0, a_\dagger]$. Moreover, we assume that individuals can die (with a certain probability) before the limit age $a_\dagger$ or be born at a certain fertility rate. In this situation, the original system (1) becomes

$$\dot{p}(t,a) + \frac{\partial p}{\partial a}(t,a) = Ap(t,a) - \mu(a)p(t,a) + \chi(a)Bu(t,a), \tag{2}$$

$$p(t,0) = \int_0^{a_\dagger} \beta(a)p(t,a)\,\mathrm{d}a, \tag{3}$$

where μ and β are the mortality and fertility rates, respectively, and χ is the characteristic function of some subinterval of $[0, a_\dagger]$.

For $X = U = \mathbb{C}$, $A = 0$ and $B = 1$ in the original system (1), the corresponding age structure system (2) becomes the classical Lotka-McKendrick system, which has been first studied, from the controllability perspective, in Barbu et al. (2001). This problem was recently revisited by Hegoburu et al. (2018), Maity (2019), and by Hegoburu and Aniţa (2019). One of the consequences of our main results improves the above mentioned ones, in the sense that for every $n,\ m \in \mathbb{N}$, $X = \mathbb{C}^n$, $U = \mathbb{C}^m$, such that the original system (1) is controllable, then, under appropriate assumptions on $\mu,\ \beta$ and χ, the same property holds for the corresponding age structured system (2) (see Subsection 4.1 further on).

The main focus in this work is on the more complicated situation, where X and U are possibly infinite dimensional spaces, with the operators A and B possibly unbounded. We think, in particular, of the case when $X = L^2(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is an open bounded set, A is an advection-diffusion operator and B describes a boundary or internal control. From the controllability viewpoint, particular cases of such systems have been studied in several papers. The first ones are probably Aniseba and Aniţa (2001, 2004) (see also Aniseba, 2012; Hegoburu and Tucsnak, 2018; and Maity, Tucsnak and ZuaZua, 2019) .

The main results in this article assert that in the infinite dimensional case (namely when (1) is a PDE system with distributed or boundary control), the wellposedness and null controllability of the system described by (1) are inherited by the corresponding age structured system (2). One of the advantages of this approach is that it allows for obtaining in a unified manner a variety of results existing in the literature, such as those corresponding to an operator A, describing diffusion (possibly with singular coefficients) or transport phenomena, with an operator B, corresponding to a distributed control. Moreover, we obtain controllability results, which seem new, in the case of an unbounded control operator B (corresponding to boundary control problems).

To give a precise description of our results, we introduce some notation. Let $A : \mathcal{D}(A) \to X$ be the generator of the C^0 semigroup $\mathbb{S}$ on the Hilbert space X and let U be another Hilbert space. Both X and U will be identified with their duals. Let B be a (possibly unbounded) linear operator from U to X, which is supposed to be an admissible control operator for $\mathbb{S}$ (see Section 2 for the precise definition of this concept). In the examples we have in mind, the above spaces and operators describe the dynamics of a system without age structure. In particular, X is the state space and U is the control space. The corresponding age structured system is obtained by first extending these spaces to

$$\mathcal{X} = L^2(0, a_\dagger; X), \tag{4}$$

$$\mathcal{U} = L^2(0, a_\dagger; U), \tag{5}$$

where $a_\dagger > 0$ denotes the maximal age individuals can attain. Let $p(t) \in \mathcal{X}$ be the distribution density of the individuals with respect to age $a \geqslant 0$ and at some time $t \geqslant 0$. Then, the abstract version of the Lotka-McKendrick system to be considered in this paper writes:

$$\begin{cases} \dfrac{\partial p}{\partial t} + \dfrac{\partial p}{\partial a} - Ap + \mu(a)p = \mathbb{1}_{(a_1,a_2)}Bu, & t \geqslant 0, a \in (0, a_\dagger), \\ p(t,0) = \displaystyle\int_0^{a_\dagger} \beta(s)p(t,s)\,\mathrm{d}s, & t \geqslant 0, \\ p(0,a) = p_0, \end{cases} \tag{6}$$

where $\mathbb{1}$ is the characteristic function of the interval (a_1, a_2) with $0 \leqslant a_1 < a_2 \leqslant a_\dagger$ and p_0 is the initial population density. In the above system, the positive function $\mu : [0, a_\dagger] \to \mathbb{R}_+$ denotes the natural mortality rate of individuals of age a. We denote by $\beta : [0, a_\dagger] \to \mathbb{R}_+$ the positive function, describing the fertility rate at age a. We assume that the fertility rate β and the mortality rate μ satisfy the conditions

(H1) $\beta \in L^\infty(0, a_\dagger)$, $\beta \geqslant 0$ for almost every $a \in (0, a_\dagger)$.

(H2) $\mu \in L^\infty[0, a^*]$ for every $a^* \in (0, a_\dagger)$, $\mu \geqslant 0$ for almost every $a \in (0, a_\dagger)$.

(H3) $\displaystyle\int_0^{a_\dagger} \mu(a)\,\mathrm{d}a = +\infty$.

For more details about the modelling of such system and the biological significance of the hypotheses, we refer to Webb (1985).

Before we state our main result, let us introduce the notion of null controllability of the pair (A, B).

Definition 1.1 *We say that a pair (A, B) is null-controllable in time τ, if for every $z_0 \in X$ there exists a control $u \in L^2(0, \tau, U)$ such that the solution of the system*

$$\dot{z}(t) = Az(t) + Bu(t) \quad t \in [0, \tau], \qquad z(0) = z_0,$$

satisfies $z(\tau) = 0$.

The main result of this paper is:

Theorem 1.1 *Assume that β and μ satisfy the conditions (H1)-(H3) above. Moreover, suppose that the fertility rate β is such that*

$$\beta(a) = 0 \text{ for all } a \in (0, a_b), \tag{7}$$

for some $a_b \in (0, a_\dagger)$ and that $a_1 < a_b$. Let us assume that the pair (A, B) is null controllable in any time $\tau > \tau_0$, with

$$0 \leqslant \tau_0 < \overline{\tau}, \quad \overline{\tau} = \min\{a_2 - a_1, a_b - a_1\}. \tag{8}$$

Then, for every $\tau > a_1 + a_\dagger - a_2 + 2\tau_0$ and for every $p_0 \in \mathcal{X}$ there exists a control $v \in L^2(0, \tau; \mathcal{U})$ such that the solution p of (6) satisfies

$$p(\tau, a) = 0 \text{ for all } a \in (0, a_\dagger). \tag{9}$$

This result can be seen as a generalization of those obtained in Aniseba and Aniţa (2001, 2004); Aniseba (2012); Hegoburu and Tucsnak (2018); Maity, Tucsnak and Zuazua (2019) in the case when A is an elliptic operator with Neumann or Dirichlet homogeneous boundary conditions, or in Aniseba et al. (2013), Boutaayamou and Echarroudi (2017), or Fragnelli (2018), when A is a degenerate elliptic operator. As shown in Section 4, our approach applies, besides the above mentioned examples, to operators A such that the systems without age structure describe fractional diffusion, transport phenomena or even Schrödinger type dynamics, with internal or boundary control.

The proof of the above theorem relies on final state observability of its adjoint system. This consists of combining the characteristics method with final state observability of the pair (A^*, B^*), with no reference to the methodology employed to prove this observability result for the system without age structure. This idea was already used in Maity, Tucsnak and Zuazua (2019) where A was second order elliptic differential operator and B was interior control operator.

The remaining part of this work is organized as follows: In Section 2, we study the wellposedness of the system (6) and we determine its adjoint. Section 3 is devoted to the proof of Theorem 1.1. In Section 4, we give several applications of our main theorem. In Section 5 we study controllability of the system (6) with regular controls.

2. Wellposedness of the system (6)

In this section, we rewrite (6) as an abstract control system. Next, we study the wellposedness of this system and we determine the adjoint of the corresponding semigroup generator.

Let us remind that if A generates a C^0-semigroup $\mathbb{S}$ on X, then there exist $M \geqslant 1$ and ω such that

$$\|\mathbb{S}_t\| \leqslant Me^{\omega t}, \text{ for all } t \geqslant 0. \tag{10}$$

We denote by A^* the adjoint of A. Then A^* generates a C^0-semigroup $\mathbb{S}^* = (\mathbb{S}^*_t)_{t\geqslant 0}$ on X. Moreover,

$$\|\mathbb{S}^*_t\| \leqslant Me^{\omega t}, \text{ for all } t \geqslant 0. \tag{11}$$

We define $X_1^d = \mathcal{D}(A^*)$ equipped with the graph norm. Let X_{-1} be the dual of X_1^d with respect to the pivot space X. In particular,

$$X_1^d \subset X \subset X_{-1},$$

with continuous and dense embeddings. It is known (see, for instance, Tucsnak and Weiss, 2009, Section 2.10) that $\mathbb{S}$ extends to a C^0 semigroup on X_{-1}, whose generator, which is an extension of A, has the domain X.

Let $B \in \mathcal{L}(U, X_{-1})$ and $\tau > 0$. We define $\Phi^A_\tau \in \mathcal{L}(L^2(0,\infty;U), X_{-1})$ by

$$\Phi^A_\tau u = \int_0^\tau \mathbb{S}_{\tau-s} Bu(s)\,\mathrm{d}s. \tag{12}$$

We introduce *admissible control operators:*

Definition 2.1 *(Tucsnak and Weiss, 2009, Definition 4.2.1) The operator $B \in \mathcal{L}(U, X_{-1})$ is called an admissible control operator for $\mathbb{S}$ if for some $\tau > 0$,* Ran $\Phi^A_\tau \subset X$.

The above admissibility condition can also be reformulated in terms of the adjoint of the operators (see Tucsnak and Weiss, 2009, Proposition 4.4.1). The operator $B \in \mathcal{L}(U, X_{-1})$ is an admissible control operator for $\mathbb{S}$, if and only if, for all $\tau > 0$, there exists a constant $C_\tau > 0$ such that

$$\int_0^\tau \|B^*\mathbb{S}^*_t z\|^2_U \mathrm{d}t \leqslant C_\tau \|z\|^2_X, \qquad \forall z \in \mathcal{D}(A^*). \tag{13}$$

Reminding that the input space $\mathcal{X}$ and the control space $\mathcal{U}$ for the corresponding age structured system are defined in (4) and (5), respectively, we introduce the operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \to \mathcal{X}$, defined by

$$\mathcal{D}(\mathcal{A}) = \left\{ \varphi \in C([0,a_\dagger]; X) \mid \varphi(0) = \int_0^{a_\dagger} \beta(a)\varphi(a)\mathrm{d}a, -\frac{\partial \varphi}{\partial a} + A\varphi - \mu\varphi \in \mathcal{X} \right\},$$

$$\mathcal{A}\varphi = -\frac{\partial \varphi}{\partial a} + A\varphi - \mu\varphi. \tag{14}$$

Let us set

$$\mathcal{X}_{-1} = L^2(0, a_\dagger; X_{-1}) \tag{15}$$

and we introduce the control operator $\mathcal{B} \in \mathcal{L}(\mathcal{U}, \mathcal{X}_{-1})$, defined by

$$\mathcal{B}u = \mathbb{1}_{(a_1,a_2)}Bu \qquad\qquad (u \in \mathcal{U}). \tag{16}$$

With the above notation, we rewrite the system (6) as

$$\dot{p} = \mathcal{A}p + \mathcal{B}u, \qquad p(0) = p_0. \tag{17}$$

We now show that $\mathcal{A}$ generates a C^0-semigroup on $\mathcal{X}$ under the assumption that A generates a C^0 semigroup on X. More precisely:

THEOREM 2.1 *Assume A generates a C^0 semigroup on X. Then $\mathcal{A}$, defined in (2), generates a C^0 semigroup on $\mathcal{X}$.*

The proof of this theorem is divided into several parts. We are going to follow the approach of Webb (2008) and Walker (2013). Upon integrating along the characteristic lines, the solution of (17) with $u = 0$, at least formally, can be written as

$$p(t,a) = \begin{cases} \dfrac{\pi(a)}{\pi(a-t)}\mathbb{S}_t p_0(a-t), & t < a, \\ \pi(a)\mathbb{S}_a b_{p_0}(t-a) & t \geqslant a, \end{cases} \tag{18}$$

where

$$\pi(a) = e^{-\displaystyle\int_0^a \mu(s)\mathrm{d}s}$$

is the probability of survival of an individual from age 0 to a and $b_\varphi(t)$ is the unique continuous solution of the following linear Volterra integral equation in X :

$$b_\varphi(t) = \int_0^t \beta(a)\pi(a)\mathbb{S}_a b_\varphi(t-a) + \mathbb{S}_t \int_0^{a_\dagger - t} \beta(a+t)\frac{\pi(a+t)}{\pi(a)}\varphi(a)\,\mathrm{d}a, \tag{19}$$

where the last integral is 0 if $t \geqslant a_\dagger$. This motivates us to define a semigroup $\mathbb{T}$ on $\mathcal{X}$ as follows:

$$\mathbb{T}_t\varphi = \begin{cases} \dfrac{\pi(a)}{\pi(a-t)}\mathbb{S}_t\varphi(a-t), & t < a, \\ \pi(a)\mathbb{S}_a b_\varphi(t-a) & t \geqslant a. \end{cases} \tag{20}$$

Note that

$$b_\varphi(t) = \int_0^{a_\dagger} \beta(a)\mathbb{T}_t\varphi(a)\ \mathrm{d}a. \tag{21}$$

The following result can be obtained along the lines of Webb (2008, Theorem 4) (see also Walker, 2013, Theorem 2.2):

Proposition 2.1 *The family of operators $\mathbb{T}$ defined in (20) is a C^0-semigroup on $\mathcal{X}$.*

Let $\mathbb{A}$ denote the generator of the semigroup $\mathbb{T}$. Therefore, to prove Theorem 2.1 we only need to show $\mathbb{A} = \mathcal{A}$, where $\mathcal{A}$ is defined in (2). To this aim, we first prove the following result :

Lemma 2.1 *Let $\mathcal{A}$ be the unbounded operator defined in (2). Then, $\lambda I - \mathcal{A}$ is onto for λ large enough.*

Proof Given $\lambda > 0$, $f \in \mathcal{X}$ and $\psi \in X$, we consider the following problem

$$\lambda\varphi + \frac{\partial\varphi}{\partial a} - A\varphi + \mu\varphi = f, \quad \varphi(0) = \psi. \tag{22}$$

Since A generates a C^0-semigroup on X, the above problem admits a unique solution $\varphi \in C([0, a_\dagger]; X)$, given by

$$\varphi(a) = e^{-\lambda a}\pi(a)\mathbb{S}_a\psi + \int_0^a e^{-\lambda(a-s)}\pi(a-s)\ \mathbb{S}_{a-s}f(s)\ \mathrm{d}s. \tag{23}$$

From the above formula, we obtain

$$\begin{aligned}
&\varphi(0) - \int_0^{a_\dagger} \beta(a)\varphi(a)\mathrm{d}a \\
&= \psi - \int_0^{a_\dagger} e^{-\lambda a}\pi(a)\beta(a)\mathbb{S}_a\psi\ \mathrm{d}a - \int_0^{a_\dagger} \beta(a)\int_0^a e^{-\lambda(a-s)}\pi(a)\mathbb{S}_{a-s}f(s)\ \mathrm{d}s\mathrm{d}a.
\end{aligned} \tag{24}$$

Now, consider the operator $\mathcal{F}(\lambda) \in \mathcal{L}(X)$ defined by

$$\mathcal{F}(\lambda)\psi = \int_0^{a_\dagger} e^{-\lambda a}\pi(a)\beta(a)\mathbb{S}_a\psi\ \mathrm{d}a. \tag{25}$$

Using (10), we have

$$\|\mathcal{F}(\lambda)\psi\|_X \leqslant M\|\beta\|_{L^\infty(0,a_\dagger)}\frac{1}{\lambda-\omega}\|\psi\|_X.$$

Thus, $\lim\limits_{\lambda\to\infty} \|\mathcal{F}(\lambda)\|_{\mathcal{L}(X)} = 0$, and we clearly have that $I - \mathcal{F}(\lambda)$ is invertible for large λ. Let us take

$$\psi = (I - \mathcal{F}(\lambda))^{-1}\int_0^{a_\dagger} \beta(a)\int_0^a e^{-\lambda(a-s)}\pi(a-s)\mathbb{S}_{a-s}f(s)\ \mathrm{d}s\mathrm{d}a.$$

Then, using (2) it is easy to see that φ, defined by (23), with the above choice of ψ satisfies the following system

$$\lambda\varphi + \frac{\partial\varphi}{\partial a} - A\varphi = f, \quad \varphi(0) = \int_0^{a_\dagger} \beta(a)\varphi(a)\, da.$$

Thus, $\lambda I - \mathcal{A}$ is onto. Moreover, the unique solution of the above system is given by

$$\varphi(a) = e^{-\lambda a}\pi(a)\mathbb{S}_a(I-\mathcal{F}(\lambda))^{-1}\left(\int_0^{a_\dagger}\beta(a)\int_0^a e^{-\lambda(a-s)}\pi(a-s)\mathbb{S}_{a-s}f(s)\mathrm{d}s\mathrm{d}a\right) + \int_0^a e^{-\lambda(a-s)}\pi(a-s)\mathbb{S}_{a-s}f(s)\mathrm{d}s. \tag{26}$$

□

Now we show that the generator of the semigroup $\mathbb{T}$ coincides with $\mathcal{A}$.

Proposition 2.2 *Let $\widetilde{\mathcal{A}}$ be the generator of the semigroup $\mathbb{T}$ and let $\mathcal{A}$ be defined in (2). Then, $\widetilde{\mathcal{A}} = \mathcal{A}$.*

Proof Let $\varphi \in \mathcal{D}(\mathbb{A})$. Let $\lambda > 0$ sufficiently large and we set $f := \lambda\varphi - \widetilde{\mathcal{A}}\varphi$. Then, using (31), we have

$$\begin{aligned}
&\varphi(a) = \int_0^\infty e^{-\lambda t}\mathbb{T}_t f(a)\mathrm{d}t = \\
&\int_0^a e^{-\lambda t}\frac{\pi(a)}{\pi(a-t)}\mathbb{S}_t f(a-t)\mathrm{d}t + \int_a^\infty e^{-\lambda t}\pi(a)\mathbb{S}_a b_f(t-a)\mathrm{d}t \\
&= \int_0^a e^{-\lambda(a-s)}\pi(a-s)\mathbb{S}_{a-s}f(s)\mathrm{d}s + e^{-\lambda a}\pi(a)\mathbb{S}_a\int_0^\infty e^{-\lambda t}b_f(t)\mathrm{d}t.
\end{aligned} \tag{27}$$

Now, using (21) and (31), we get

$$\begin{aligned}
&\int_0^\infty e^{-\lambda t}b_f(t)\ \mathrm{d}t = \int_0^{a_\dagger}\beta(a)\int_0^\infty e^{-\lambda t}\mathbb{T}_t f(a)\ \mathrm{d}t\mathrm{d}a \\
&= \int_0^{a_\dagger}\beta(a)\int_0^a e^{-\lambda t}\frac{\pi(a)}{\pi(a-t)}\mathbb{S}_t f(a-t)\ \mathrm{d}t\mathrm{d}a \\
&\quad + \int_0^{a_\dagger}\beta(a)\int_a^\infty e^{-\lambda t}\pi(a)\mathbb{S}_a b_f(t-a)\ \mathrm{d}t\mathrm{d}a \\
&= \int_0^{a_\dagger}\beta(a)\int_0^a e^{-\lambda(a-s)}\pi(a-s)\mathbb{S}_{a-s}f(s)\ \mathrm{d}s\mathrm{d}a \\
&\quad + \int_0^{a_\dagger} e^{-\lambda a}\beta(a)\pi(a)\mathbb{S}_a\int_0^\infty e^{-\lambda t}b_f(t)\ \mathrm{d}t\mathrm{d}a.
\end{aligned}$$

Therefore,

$$\int_0^\infty e^{-\lambda t}b_f(t)\ \mathrm{d}t = (I-\mathcal{F}(\lambda))^{-1}\int_0^{a_\dagger}\beta(a)\int_0^a e^{-\lambda(a-s)}\pi(a-s)\mathbb{S}_{a-s}f(s)\ \mathrm{d}s\mathrm{d}a,$$

where $\mathcal{F}(\lambda)$ is defined in (25). Using the above relation in (27) and comparing this expression with (26) one can easily see that $\varphi \in \mathcal{D}(\mathcal{A})$. We have thus proven that $\mathcal{D}(\widetilde{\mathcal{A}}) \subset \mathcal{D}(\mathcal{A})$ and

$$\widetilde{\mathcal{A}}\varphi = -\frac{\partial \varphi}{\partial a} + A\varphi - \mu\varphi = \mathcal{A}\varphi \qquad (\varphi \in \mathcal{D}(\widetilde{\mathcal{A}})). \tag{28}$$

Conversely, let us assume that $\varphi \in \mathcal{D}(\mathcal{A})$. For λ sufficiently large, we define $f := \lambda\varphi + \frac{\partial \varphi}{\partial a} - A\varphi + \mu\varphi$. Then, $f \in \mathcal{X}$. Set $\psi = (\lambda I - \mathbb{A})^{-1} f \in \mathcal{D}(\widetilde{\mathcal{A}})$. Therefore, using (28) we have that

$$\lambda(\varphi - \psi) + \frac{\partial}{\partial a}(\varphi - \psi) - A(\varphi - \psi) + \mu(\varphi - \psi) = 0.$$

Thus,

$$\varphi - \psi = e^{-\lambda a}\pi(a)\mathbb{S}_a(\varphi - \psi)(0).$$

Using the definition of $\mathcal{F}(\lambda)$ in (25), it is easy to see that the above relation is equivalent to

$$(I - \mathcal{F}(\lambda))(\varphi - \psi)(0) = 0.$$

Hence, for λ sufficiently large, $\varphi(0) = \psi(0)$ and therefore $\varphi = \psi \in \mathcal{D}(\widetilde{\mathcal{A}})$. This completes the proof of the proposition. □

Proof of Theorem 2.1 The proof of this theorem follows from Proposition 2.1 and Proposition 2.2. □

Remark 2.1 *An alternative proof of Theorem 2.1 can be obtained by combining the results in Magal and Rual (2018, Section 3.8) with a perturbation result of Desch-Schappcher type (see, for instance, Tucsnak and Weiss 2009, Section 5.4).*

Next we show that $\mathcal{B}$ defined in (16) is an admissible control operator:

Lemma 2.2 *Let us assume that $B \in \mathcal{L}(U, X_{-1})$ is an admissible control operator for $\mathbb{S}$. Then, the operator $\mathcal{B} \in \mathcal{L}(\mathcal{U}, \mathcal{X}_{-1})$ defined in (16) is an admissible control operator for the semigroup $\mathbb{T}$ generated by $\mathcal{A}$.*

Proof The proof follows easily from Definition 2.1 and the fact that B is an admissible control operator. □

Using Theorem 2.1 and Lemma 2.2, we have the following wellposedness result of the system (17) (see, for instance Tucsnak and Weiss, Proposition 4.2.5):

Theorem 2.2 *For every $p_0 \in \mathcal{X}$ and for every $u \in L^2(0, a_\dagger; \mathcal{U})$ the system (17) admits a unique solution*

$$p \in C([0, a_\dagger]; \mathcal{X}).$$

With the above notation our main result in Theorem 1.1 can be restated as: If the pair (A, B) is null controllable in time τ_0, then the pair $(\mathcal{A}, \mathcal{B})$ is null controllable in time $\tau > a_1 + a_\dagger - a_2 + 2\tau_0$. To prove this assertion, we are going to use the fact that null controllability of the pair $(\mathcal{A}, \mathcal{B})$ at time τ is equivalent to final state observability in time τ of the pair $(\mathcal{A}^*, \mathcal{B}^*)$. In the following theorem we determine the adjoint of $\mathcal{A}$ and $\mathcal{B}$. To this aim, we first consider an auxiliary operator $\mathcal{A}_0$, defined by

$$\mathcal{D}(\mathcal{A}_0) = \left\{ \psi \in \mathcal{X} \mid q(t, a_\dagger) = 0, \quad \frac{\partial \psi}{\partial a} - \mu\psi + A^*\psi \in \mathcal{X} \right\}, \tag{29}$$
$$\mathcal{A}_0 \psi = \frac{\partial \psi}{\partial a} - \mu\psi + A^*\psi.$$

We have the following proposition:

Proposition 2.3 *The operator $\mathcal{A}_0$ is the infinitesimal generator of a C^0-semigroup $\mathbb{T}^0$ on $\mathcal{X}$. Moreover,*

$$\|\mathbb{T}^0_t\| \leqslant M e^{\omega t}, \tag{30}$$

where M and ω are defined in (11).

Proof The proof of this proposition is similar to that of Theorem 2.1. We briefly sketch the idea. By integrating along the characteristic lines, we define the semigroup $\mathbb{T}^0$ on $\mathcal{X}$ as follows:

$$\mathbb{T}^0_t \varphi = \begin{cases} \dfrac{\pi(a)}{\pi(a+t)} \mathbb{S}^*_t \varphi(a+t), & t < a_\dagger - a, \\ 0 & t \geqslant a_\dagger - a. \end{cases} \tag{31}$$

As $\mathbb{S}^*_t$ is a C^0-semigroup, it follows that $\mathbb{T}_t$ is also a C^0-semigroup (see Proposition 2.1). Moreover, proceeding as in Proposition 2.2 we can show that the domain of the semigroup $\mathbb{T}^0_t$ is $\mathcal{A}_0$. The estimate (30) is easy to obtain from the expression of the semigroup $\mathbb{T}^0_t$. □

The result below gives the adjoint operators of $\mathcal{A}$ and $\mathcal{B}$. We skip its proof since it is fully similar to the one given for Maity et al. (2019, Proposition 2.3).

Proposition 2.4 *The adjoint of $\mathcal{A}$ in $\mathcal{X}$ is defined by*

$$\mathcal{D}(\mathcal{A}^*) = \mathcal{D}(\mathcal{A}_0), \quad \mathcal{A}^*\psi = \frac{\partial \psi}{\partial a} - \mu\psi + A^*\psi + \beta(a)\psi(0).$$

Moreover, $\mathcal{B}^* \in \mathcal{L}(L^2(0, a_\dagger; \mathcal{D}(A^*)); \mathcal{U})$, defined by

$$\mathcal{B}^*\psi = \mathbb{1}_{(a_1, a_2)} B^*\psi,$$

where $B^* \in \mathcal{L}(\mathcal{D}(A^*), U)$ is the adjoint of the operator B.

We end this subsection with the following result, which will be required later on.

LEMMA 2.3 *Assume the hypothesis of Lemma 2.2. Then, there exists a constant* $C_\tau > 0$ *such that the solution* φ *to the system*

$$\dot{\varphi} = \mathcal{A}_0\varphi + f(t) \quad t \in [0,\tau], \qquad \varphi(0) = 0, \tag{32}$$

satisfies

$$\int_0^\tau \|\mathcal{B}^*\varphi\|_{\mathcal{U}}^2 \leqslant C_\tau \|f\|_{L^2(0,\tau;\mathcal{X})}, \tag{33}$$

for every $f \in L^2(0,\tau;\mathcal{X})$.

PROOF We first note that $\mathcal{B} \in \mathcal{L}(\mathcal{U}, \mathcal{X}_{-1})$, defined in (16), is also an admissible control operator for the semigroup $\mathbb{T}^0$, generated by $\mathcal{A}_0$. The result follows from Curtain and Weiss (1989, Theorem 5.1 and Remark 5.4). □

3. An observability inequality

3.1. The results

As mentioned above, the null-controllability of a pair $(\mathcal{A}, B)$ is equivalent to the final state observability of the pair $(\mathcal{A}^*, \mathcal{B}^*)$, see Tucsnak and Weiss (2009, Theorem 11.2.1). Recall that the final-state observability of $(\mathcal{A}^*, \mathcal{B}^*)$ is defined as

DEFINITION 3.1 *(Tucsnak and Weiss, 2009) The pair* $(\mathcal{A}^*, \mathcal{B}^*)$ *is final state observable in time* τ *if there exists a* $k_\tau > 0$ *such that*

$$\|\mathbb{T}_\tau^* q_0\|_{\mathcal{X}}^2 \leqslant k_\tau^2 \int_0^\tau \|\mathcal{B}^*\mathbb{T}_\tau^* q_0\|_{\mathcal{U}}^2, \qquad (q_0 \in \mathcal{D}(\mathcal{A}^*)).$$

For $\mathcal{A}$ defined in (2) and $q_0 \in \mathcal{X}$ we set

$$q(t) = \mathbb{T}_t^* q_0 \qquad (t \geqslant 0),$$

where $\mathbb{T}$ is the semigroup generated by $\mathcal{A}$. According to Proposition 2.4, q satisfies, for $t \geqslant 0, a \in (0, a_\dagger)$:

$$\begin{cases} \dfrac{\partial q}{\partial t} - \dfrac{\partial q}{\partial a} - A^* q - \beta(a) q(t,0) + \mu(a) q = 0, \\ q(t, a_\dagger) = 0, \\ q(0,a) = q_0(a). \end{cases} \tag{34}$$

In view of Tucsnak and Weiss (2009, Theorem 11.2.1), the statement in Theorem 1.1 is equivalent to the following theorem:

THEOREM 3.1 *Assume that β and μ satisfy the conditions (H1)-(H3). Moreover, suppose that the fertility rate β is such that*

$$\beta(a) = 0 \text{ for all } a \in (0, a_b), \tag{35}$$

for some $a_b \in (0, a_\dagger)$ and that $a_1 < a_b$. Let us assume that the pair (A^*, B^*) is final state observable in time $\tau > \tau_0$, with

$$0 \leqslant \tau_0 < \overline{\tau}, \quad \overline{\tau} = \min\{a_2 - a_1, a_b - a_1\}. \tag{36}$$

Then, the pair $(\mathcal{A}^*, \mathcal{B}^*)$ is final-state observable for every $\tau > a_1 + a_\dagger - a_2 + 2\tau_0$. In other words, for every $\tau > a_1 + a_\dagger - a_2 + \tau_0$ there exists $k_\tau > 0$ such that the solution q of (34) satisfies

$$\|q(\tau)\|^2_{\mathcal{X}} \leqslant k_\tau^2 \int_0^\tau \|\mathcal{B}^* q(t)\|^2_{\mathcal{U}} \, \mathrm{d}t, \qquad (q_0 \in \mathcal{D}(\mathcal{A}^*)). \tag{37}$$

REMARK 3.1 *Using the expression of $\mathcal{B}^*$ it is easy to see that the inequality (37) reads as*

$$\int_0^{a_\dagger} \|q(\tau, a)\|^2_X \, \mathrm{d}a \leqslant \kappa_\tau^2 \int_0^\tau \int_{a_1}^{a_2} \|B^* q(t, a)\|^2_U \, \mathrm{d}a\mathrm{d}t, \tag{38}$$

for any $q_0 \in \mathcal{D}(\mathcal{A}^*)$.

The main idea of the proof is to use final state observability of the pair (A^*, B^*) along the characteristic lines. We first have the following proposition, which is an easy consequence of the final state observability of the pair (A^*, B^*).

PROPOSITION 3.1 *Let us assume that the pair (A^*, B^*) is final state observable in any time $T > T_0$ with $T_0 \geqslant 0$. Let $\mathcal{C}(T)$ be the observability cost with $\mathcal{C}(T) \to \infty$ as $T \to T_0$. Let T_1, T_2 and T_3 be three real numbers such that*

$$0 \leqslant T_1 < T_2 \leqslant T_3 \quad \text{with} \quad T_2 - T_1 > T_0.$$

Then for every $w_0 \in \mathcal{D}(A^*)$, the solution w of the problem

$$\frac{dw}{dt} = A^* w \qquad t \in [T_1, T_3], \quad w(T_1) = w_0, \tag{39}$$

satisfies the estimate

$$\|w(T_3)\|^2_X \leqslant M e^{\omega(T_3 - T_2)} \mathcal{C}(T_2 - T_1) \int_{T_1}^{T_2} \|B^* w(s)\|^2_U \, \mathrm{d}s, \tag{40}$$

where M and ω are defined in (11).

PROOF By the semigroup property (11), it is easy to see that

$$\big\|w(T_3)\big\|^2_X \leqslant M e^{\omega(T_3 - T_2)} \big\|w(T_2)\big\|^2_X.$$

Now, applying the final state observability of (A^*, B^*) over the time interval $[T_1, T_2]$, we obtain

$$\left\|w(T_2)\right\|_X^2 \leqslant \mathcal{C}(T_2 - T_1)\left\|B^* w(s)\right\|_U^2 \,\mathrm{d}s.$$

Combining the above two estimates we conclude the proof of the proposition. □

The following three propositions are crucial in proving Theorem 3.1.

PROPOSITION 3.2 *Let us assume the hypothesis of Theorem 3.1. Let*

$$\tau > \tau_0 + a_1.$$

Then, for every $q_0 \in \mathcal{D}(\mathcal{A}^*)$, the solution q of the system (34), verifies

$$\int_0^{a_1} \|q(\tau, a)\|_X^2 \mathrm{d}a \leqslant$$
$$M C_\mu e^{\omega a_1} \max\left\{\mathcal{C}(\tau - a_1), \mathcal{C}(a_2 - a_1)\right\} \int_0^\tau \int_{a_1}^{a_2} \|B^* q(t,a)\|_U^2 \,\mathrm{d}a\,\mathrm{d}t, \quad (41)$$

where $C_\mu = \mathrm{e}^{2\|\mu\|_{L^1[0,a_0]}}$.

PROOF Let us recall that $\overline{\tau}$ is defined by $\overline{\tau} = \min\{a_2 - a_1, a_b - a_1\}$. Thus, without loss of generality we can assume that $a_2 \leqslant a_b$. Since $\beta(a) = 0$ for all $a \in (0, a_2)$, q satisfies

$$\frac{\partial q}{\partial t} - \frac{\partial q}{\partial a} - A^* q + \mu(a) q = 0, \quad t \geqslant 0, a \in (0, a_2). \quad (42)$$

We set

$$\widetilde{q}(t,a) = q(t,a)\,\mathrm{e}^{-\displaystyle\int_0^a \mu(r)\,\mathrm{d}r}. \quad (43)$$

Then, $\widetilde{q}$ satisfies

$$\frac{\partial \widetilde{q}}{\partial t} - \frac{\partial \widetilde{q}}{\partial a} - A^* \widetilde{q} = 0, \quad t \geqslant 0, a \in (0, a_2). \quad (44)$$

Without loss of generality, let us assume that

$$\tau < a_2, \quad \tau > a_2 - a_1. \quad (45)$$

We set $b_0 = a_2 - \tau$ and we split the interval $(0, a_1)$ as follows

$$(0, a_1) = (0, b_0) \cup (b_0, a_1). \quad (46)$$

Let us remark that, the choices in (45) are made to cover all possible scenarios. Indeed, if $\tau < a_2 - a_1$, we can choose $b_0 = a_1$, or, if $\tau > a_2$, we choose $b_0 = 0$. We are going to use Proposition 3.1 along the characteristics. In the remaining part of the proof we give upper bounds for $\displaystyle\int_I \|\widetilde{q}(\tau,a)\|_X^2 \mathrm{d}a$ where I is successively each one of the intervals appearing in the decomposition (46).

Upper bound on $(0, b_0)$:

For a.e. $a \in (0, b_0)$, we first set

$$w(s) = \widetilde{q}(s, a+\tau-s) \qquad\qquad s \in (0,\tau).$$

Then, w satisfies

$$\frac{\partial w}{\partial s} - A^* w = 0, \quad s \in (0,\tau). \tag{47}$$

Applying Proposition 3.1, with $T_0 = \tau_0$, $T_1 = 0$, $T_2 = \tau + a - a_1$ and $T_3 = \tau$, we obtain

$$\|w(\tau)\|_X^2 \leqslant M e^{\omega(a_1-a)} \mathcal{C}(\tau+a-a_1) \int_0^{\tau+a-a_1} \|B^* w(s)\|_U^2 \,\mathrm{d}s.$$

In terms of $\widetilde{q}$, the above inequality writes

$$\begin{aligned}\|\widetilde{q}(\tau,a)\|_X^2 \leqslant M e^{\omega(a_1-a)} \mathcal{C}(\tau+a-a_1) \int_0^{\tau+a-a_1} \|B^*\widetilde{q}(s, a+\tau-s, x)\|_U^2 \,\mathrm{d}s = \\ = M e^{\omega(a_1-a)} \mathcal{C}(\tau+a-a_1) \int_{a_1}^{\tau+a} \|B^*\widetilde{q}(\tau+a-s, s)\|_U^2 \,\mathrm{d}s.\end{aligned}$$

Integrating with respect to a over $(0, b_0)$ we obtain

$$\begin{aligned}\int_0^{b_0} \|\widetilde{q}(\tau,a)\|_X^2 \,\mathrm{d}a &\leqslant M e^{\omega a_1} \mathcal{C}(\tau-a_1) \int_0^{b_0} \int_{a_1}^{\tau+a} \|B^*\widetilde{q}(\tau+a-s, s)\|_U^2 \,\mathrm{d}s\mathrm{d}a \\ &= M e^{\omega a_1} \mathcal{C}(\tau-a_1) \int_{a_1}^{a_2} \int_{s-\tau}^{b_0} \|B^*\widetilde{q}(\tau+a-s, s)\|_U^2 \,\mathrm{d}a\mathrm{d}s \\ &= M e^{\omega a_1} \mathcal{C}(\tau-a_1) \int_{a_1}^{a_2} \int_0^{a_2-s} \|B^*\widetilde{q}(r, s)\|_U^2 \,\mathrm{d}r\mathrm{d}s \\ &\leqslant M e^{\omega a_1} \mathcal{C}(\tau-a_1) \int_0^{\tau} \int_{a_1}^{a_2} \|B^*\widetilde{q}(t, a)\|_U^2 \,\mathrm{d}a\mathrm{d}t. \end{aligned} \tag{48}$$

Upper bound on (b_0, a_1):

For a.e. $a \in (b_0, a_1)$, we define

$$w(s) = \widetilde{q}(s, a+\tau-s) \qquad\qquad s \in (\tau+a-a_2, \tau).$$

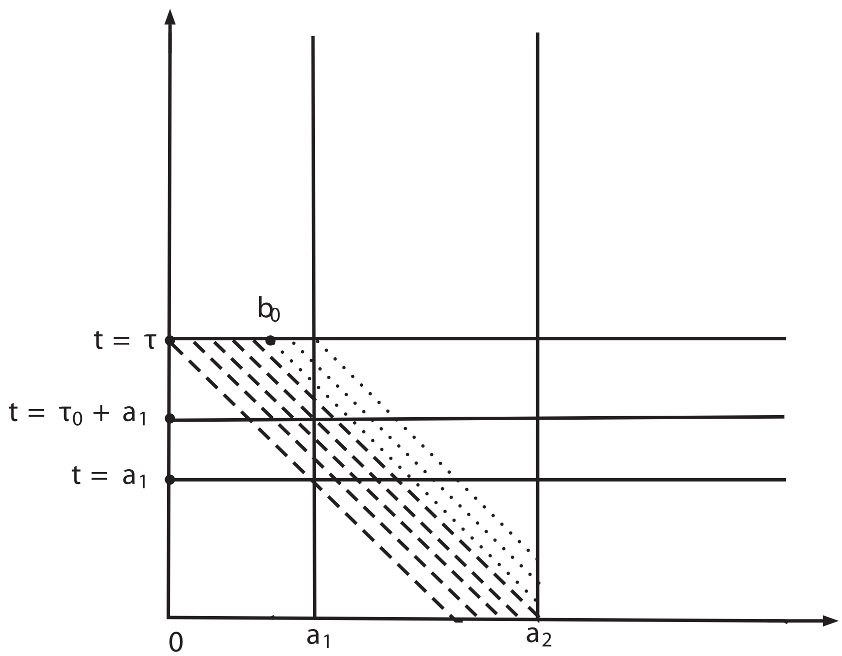

Figure 1. An illustration of the choice made in (45): **the dashed** (—–) region corresponds to the interval $(0, b_0)$. Since $\tau > a_1$, the trajectory $\gamma(s) := (\tau - s, a + s)$, $s \in [0, \tau]$ (or equivalently the backward characteristics starting from (τ, a)) enters the observation region $(a_1, a_2) \times (0, \tau)$ at $s = a_1 - a$. At $s = \tau$, $\gamma(s)$ hits the line $t = 0$ without leaving the observation region. The **dotted** (.....) region corresponds to the interval (b_0, a_1). In this case, the trajectory $\gamma(s)$ enters the observation domain at $s = a_1 - a$ and exits the observation region at $s = a_2 - a$. Since (A^*, B^*) is final state observable in time $\tau > \tau_0$, we need the length of the characteristics to be greater than τ_0 within the observation region. Thus, we need $\tau > \tau_0 + a_1$ in order to observe $\widetilde{q}$ at final time

Then, w satisfies

$$\frac{\partial w}{\partial s} - A^* w = 0, \qquad s \in (\tau + a - a_2, \tau). \tag{49}$$

By applying Proposition 3.1 with $T_0 = \tau_0$, $T_1 = \tau + a - a_2$, $T_2 = \tau + a - a_1$ and $T_3 = \tau$ we obtain that

$$\|w(\tau)\|_X^2 \leqslant M e^{\omega(a_1 - a)} \mathcal{C}(a_2 - a_1) \int_{\tau + a - a_2}^{\tau + a - a_1} \|B^* w(s)\|_U^2 \, \mathrm{d}s.$$

In terms of $\widetilde{q}$, the above inequality becomes

$$\begin{aligned}\|\widetilde{q}(\tau,a)\|_X^2 &\leqslant Me^{\omega(a_1-a)}\mathcal{C}(a_2-a_1)\int_{\tau+a-a_2}^{\tau+a-a_1}\|B^*\widetilde{q}(s,a+\tau-s)\|_U^2\,\mathrm{d}s\\ &= Me^{\omega(a_1-a)}\mathcal{C}(a_2-a_1)\int_{a_1}^{a_2}\|B^*\widetilde{q}(\tau+a-s,s)\|_U^2\,\mathrm{d}s.\end{aligned}$$

Integrating with respect to a over (b_0, a_1) yields

$$\begin{aligned}\int_{b_0}^{a_1}\|\widetilde{q}(\tau,a)\|_X^2\,\mathrm{d}a &\leqslant Me^{\omega(a_1-b_0)}\mathcal{C}(a_2-a_1)\int_{b_0}^{a_1}\int_{a_1}^{a_2}\|B^*\widetilde{q}(\tau+a-s,s)\|_U^2\,\mathrm{d}s\mathrm{d}a\\ &= Me^{\omega(a_1-b_0)}\mathcal{C}(a_2-a_1)\int_{a_1}^{a_2}\int_{b_0}^{a_1}\|B^*\widetilde{q}(\tau+a-s,s)\|_U^2\,\mathrm{d}a\mathrm{d}s\\ &= Me^{\omega(a_1-b_0)}\mathcal{C}(a_2-a_1)\int_{a_1}^{a_2}\int_{\tau+b_0-s}^{\tau+a_1-s}\|B^*\widetilde{q}(r,s)\|_U^2\,\mathrm{d}r\mathrm{d}s\\ &\leqslant Me^{\omega a_1}\mathcal{C}(a_2-a_1)\int_{a_1}^{a_2}\int_0^{\tau}\|B^*\widetilde{q}(r,s)\|_U^2\,\mathrm{d}r\mathrm{d}s\\ &= Me^{\omega a_1}\mathcal{C}(a_2-a_1)\int_0^{\tau}\int_{a_1}^{a_2}\|B^*\widetilde{q}(t,a)\|_U^2\,\mathrm{d}a\mathrm{d}t. \qquad (50)\end{aligned}$$

Therefore, by combining (48) and (50), we get

$$\begin{aligned}&\int_0^{a_1}\|\widetilde{q}(\tau,a)\|_X^2\,\mathrm{d}a\\ &\leqslant Me^{\omega a_1}\max\left\{\mathcal{C}(\tau-a_1),\mathcal{C}(a_2-a_1)\right\}\int_0^{\tau}\int_{a_1}^{a_2}\|B^*\widetilde{q}(t,a)\|_U^2\,\mathrm{d}a\,\mathrm{d}t. \qquad (51)\end{aligned}$$

Finally, using the above estimate and the definition of $\widetilde{q}$ in (43), we obtain (41). This completes the proof of the proposition. □

Next, we consider the system (34) with $\beta = 0$. More precisely, we consider the system

$$\begin{cases}\dfrac{\partial z}{\partial t}-\dfrac{\partial z}{\partial a}-A^*z+\mu(a)z=0, & (t,a)\in(0,\tau)\times(0,a_\dagger)\\ z(t,a_\dagger)=0, & t\in(0,\tau)\\ z(0,a)=z_0(a) & a\in(0,a_\dagger).\end{cases} \qquad (52)$$

Proposition 3.3 *Let us assume the hypothesis of Theorem 3.1. Let*

$$\tau > \tau_0 \text{ and } a_1 < a_0 < a_2 - \tau_0.$$

Then, for every $z_0 \in \mathcal{D}(\mathcal{A}^*)$, the solution z of the system (52) verifies

$$\begin{aligned}&\int_{a_1}^{a_0}\|z(\tau,a)\|_X^2\mathrm{d}a\\ &\leqslant MC_\mu e^{\omega a_1}\max\left\{\mathcal{C}(\tau),\mathcal{C}(a_2-a_0)\right\}\int_0^{\tau}\int_{a_1}^{a_2}\|B^*z(t,a)\|_U^2\,\mathrm{d}a\,\mathrm{d}t, \qquad (53)\end{aligned}$$

where $C_\mu = \mathrm{e}^{2\|\mu\|_{L^1[0,a_0]}}$.

PROOF The proof is similar to that of Proposition 3.2. Let us briefly explain the main steps. We consider the case

$$\tau < a_2 - a_1.$$

We split the interval (a_1, a_0) as (see Fig. 2)

$$(a_1, a_0) = (a_1, a_3) \cup (a_3, a_0) \quad \text{where } a_3 = a_2 - \tau.$$

If $\tau \geqslant a_2 - a_1$, then we choose $a_3 = a_1$. Then, we estimate

$$\int_I \|z(\tau, a)\|_X^2 \mathrm{d}a$$

where I is successively each of the intervals appearing in the above decomposition. These estimates are similar to the ones presented in Proposition 3.2, thus are omitted here. □

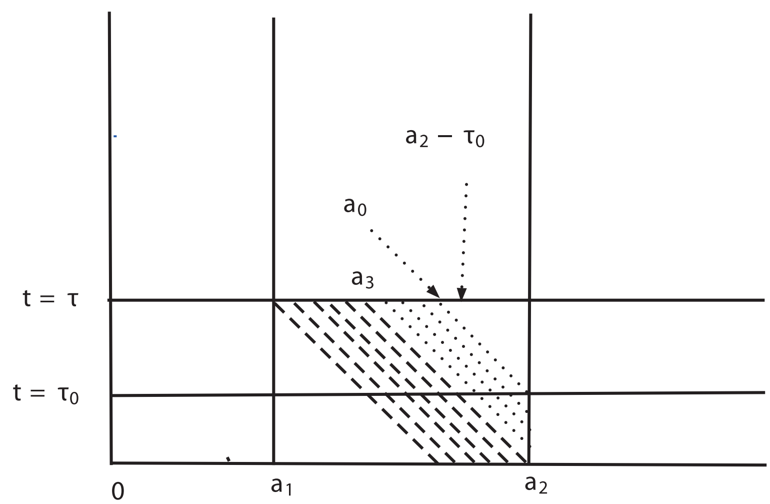

Figure 2. In this case, the trajectory $\gamma(s) = (\tau - s, a + s)$ starts inside the observation region. Thus, we just need $\tau > \tau_0$ in order to apply final state observability of the pair (A^*, B^*) along the characteristics

In the next proposition, we estimate $q(t, 0)$. More precisely, we prove the following:

PROPOSITION 3.4 *Let us assume the hypothesis of Theorem 3.1 and let $\tau > \tau_0 + a_1$ and $\eta \in (\tau_0 + a_1, \tau)$. Then, for every $q_0 \in \mathcal{D}(\mathcal{A}^*)$, the solution q of the system (34) satisfies*

$$\int_\eta^\tau \|q(t,0)\|_X^2 \, \mathrm{d}t \leqslant M e^{\omega a_1} \mathcal{C}(\eta - a_1) \int_0^\tau \int_{a_1}^{a_2} \|B^* q(t,a)\|_U^2 \, \mathrm{d}a\mathrm{d}t. \tag{54}$$

PROOF First of all, without loss of generality we can assume that $a_2 \leqslant a_b$ (otherwise we simply observe for small ages). Then, for all $t \geqslant 0$ and $a \in (0, a_2)$, q satisfies the system (42). Let $\widetilde{q}$ be defined as in (43). In particular, $\widetilde{q}$ satisfies (44). Here, we are also going to use Proposition 3.1 along the characteristics. Without loss of generality, let us assume that

$$a_2 \leqslant a_b \quad \text{and} \quad \eta < a_2 < \tau.$$

Case 1: For a.e. $t \in (a_2, \tau)$, we define

$$w(s,x) = \widetilde{q}(s, t-s), \quad s \in (t-a_2, t). \tag{55}$$

Then, w satisfies

$$\frac{\partial w}{\partial s} - A^* w = 0 \qquad s \in (t-a_2, t), \tag{56}$$

Using Proposition 3.1, with $t_0 = t - a_2$, $t_1 = t - a_1$ and $T = t$, we obtain

$$\|w(t)\|_X^2 \leqslant M e^{\omega a_1} \mathcal{C}(a_2 - a_1) \int_{t-a_2}^{t-a_1} \|B^* w(s,x)\|_U^2 \, \mathrm{d}s.$$

In terms of $\widetilde{q}$, the above inequality reads as

$$\begin{aligned}
&\|\widetilde{q}(t,0)\|_X^2 \leqslant \\
&M e^{\omega a_1} \mathcal{C}(a_2 - a_1) \int_{t-a_2}^{t-a_1} \|B^* \widetilde{q}(s, t-s)\|_U^2 \, \mathrm{d}s \\
&= M e^{\omega a_1} \mathcal{C}(a_2 - a_1) \int_{a_1}^{a_2} \|B^* \widetilde{q}(t-s, s)\|_U^2 \, \mathrm{d}s.
\end{aligned}$$

By integrating with respect to t over $[a_2, \tau]$, we obtain

$$\begin{aligned}
&\int_{a_2}^{\tau} \|\widetilde{q}(t,0)\|^2 \, \mathrm{d}t \leqslant M e^{\omega a_1} \mathcal{C}(a_2 - a_1) \int_{a_2}^{\tau} \int_{a_1}^{a_2} \|B^* \widetilde{q}(t-s, s)\|_U^2 \, \mathrm{d}s\mathrm{d}t \\
&= M e^{\omega a_1} \mathcal{C}(a_2 - a_1) \int_{a_1}^{a_2} \int_{a_2}^{\tau} \|B^* \widetilde{q}(t-s, s)\|_U^2 \, \mathrm{d}t\mathrm{d}s \\
&= M e^{\omega a_1} \mathcal{C}(a_2 - a_1) \int_{a_1}^{a_b} \int_{a_2-s}^{\tau-s} \|B^* \widetilde{q}(r, s)\|_U^2 \, \mathrm{d}r\mathrm{d}s \\
&\qquad \leqslant M e^{\omega a_1} \mathcal{C}(a_2 - a_1) \int_0^{\tau} \int_{a_1}^{a_2} \|B^* \widetilde{q}(t, a)\|_U^2 \, \mathrm{d}x\mathrm{d}a\mathrm{d}t.
\end{aligned} \tag{57}$$

Case 2: For a.e. $t \in (\eta, a_2)$, we define

$$w(s) = \widetilde{q}(s, t-s) \qquad\qquad s \in (0, t). \tag{58}$$

Then, w satisfies

$$\frac{\partial w}{\partial s} - A^* w = 0 \qquad s \in (0, t).$$

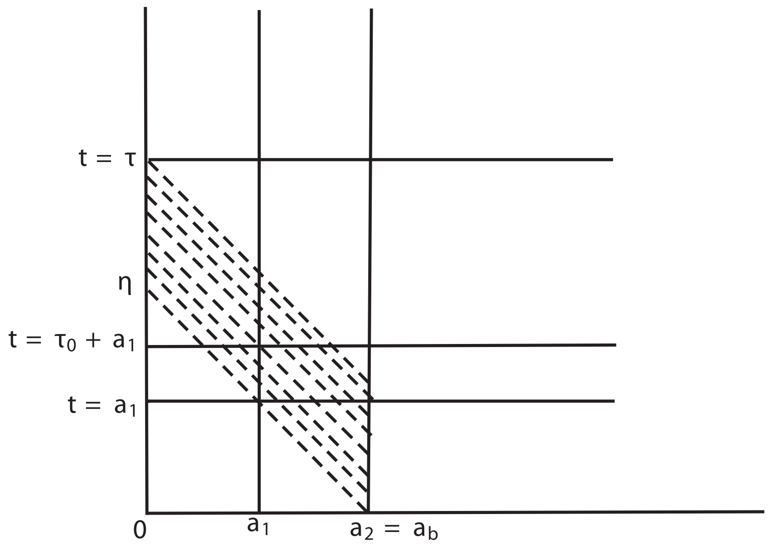

Figure 3. An illustration of the estimate of $\widetilde{q}(t,0)$. Here, we have chosen $a_2 = a_b$. Since $\tau > \tau_0 + a_1$, all the backward characteristics starting from $(t,0)$ enter the observation domain (the **dashed** region) and the length of the characteristics within the observation region is greater than τ_0

By applying Proposition 3.1, with $t_0 = 0$, $t_1 = t - a_1$ and $T = t$, we obtain

$$\|w(t)\|_X^2 \leqslant Me^{\omega a_1}\mathcal{C}(t-a_1)\int_0^{t-a_1} \|B^*w(s)\|_U^2 \, \mathrm{d}s.$$

This yields

$$\begin{aligned}
&\|\widetilde{q}(t,0)\|_X^2 \\
&\leqslant Me^{\omega a_1}\mathcal{C}(t-a_1)\int_0^{t-a_1} \|B^*\widetilde{q}(s,t-s)\|_U^2 \, \mathrm{d}s \\
&= Me^{\omega a_1}\mathcal{C}(t-a_1)\int_{a_1}^{t} \|B^*\widetilde{q}(t-s,s)\|_U^2 \, \mathrm{d}s.
\end{aligned}$$

Integration with respect to t over $[\eta, a_2]$ yields

$$\begin{aligned}
&\int_\eta^{a_2} \|\widetilde{q}(t,0)\|_X^2 \, \mathrm{d}t \\
&\leqslant Me^{\omega a_1}\mathcal{C}(\eta-a_1)\int_\eta^{a_2}\int_{a_1}^{t} \|B^*\widetilde{q}(t-s,s)\|_U^2 \, \mathrm{d}s\mathrm{d}t \\
&\leqslant Me^{\omega a_1}\mathcal{C}(\eta-a_1)\int_0^{a_2}\int_{a_1}^{t} \|B^*\widetilde{q}(t-s,s)\|_U^2 \, \mathrm{d}s\mathrm{d}t
\end{aligned}$$

$$= Me^{\omega a_1}\mathcal{C}(\eta - a_1)\int_{a_1}^{a_2}\int_{s}^{a_2}\|B^*q(t-s,s)\|_U^2\ \mathrm{d}t\mathrm{d}s$$
$$= Me^{\omega a_1}\mathcal{C}(\eta - a_1)\int_{a_1}^{a_2}\int_{0}^{a_2-s}\|B^*q(r,s)\|_U^2\ \mathrm{d}r\mathrm{d}s$$
$$\leqslant Me^{\omega a_1}\mathcal{C}(\eta - a_1)\int_{0}^{\tau}\int_{a_1}^{a_2}\|B^*\widetilde{q}(t,a)\|_U^2\ \mathrm{d}a\mathrm{d}t. \quad (59)$$

By combining (57) and (59), we obtain

$$\int_{\eta}^{\tau}\|\widetilde{q}(t,0)\|_X^2\ \mathrm{d}t \leqslant Me^{\omega a_1}\mathcal{C}(\eta - a_1)\int_{0}^{\tau}\int_{a_1}^{a_2}\|B^*\widetilde{q}(t,a)\|_U^2\ \mathrm{d}a\mathrm{d}t.$$

Note that, from the definition of $\widetilde{q}$ in (43), we have $\widetilde{q}(t,0) = q(t,0)$. Thus, from the above estimate, we clearly obtain (54). □

3.2. Proof of the main result

We are now in a position to prove Theorem 3.1, thus, consequently, our main result in Theorem 1.1.

Proof of Theorem 3.1 The constant C_τ, appearing in this proof, depends only on $\tau, a_\dagger, \mu, \beta, A$ and B. Let us set

$$\delta = \tau - (a_1 + a_\dagger - a_2 + 2\tau_0) \text{ and } \eta = a_1 + \tau_0 + \frac{\delta}{2}.$$

Without loss of generality we can assume that τ is such that $a_1 < a_2 - \tau_0 - \delta/2$. (see Fig. 4). By Proposition 3.2, we already have that

$$\int_{0}^{a_1}\|q(\tau,a)\|_X^2\ \mathrm{d}a \leqslant C_\mu e^{\omega a_1}\mathcal{C}(\tau_0 + \delta/2)\int_{0}^{\tau}\int_{a_1}^{a_2}\|B^*q(t,a)\|_U^2\ \mathrm{d}a\mathrm{d}t. \quad (60)$$

Thus, the rest of the proof is devoted to the estimate of

$$\int_{a_1}^{a_\dagger}\|q(\tau,a)\|_X^2\ \mathrm{d}a.$$

With this in mind, let us define

$$q_\eta(a) := q(\eta,a), a \in (0,a_\dagger) \text{ and } V(t,a) := \beta(a)q(t,0), t \in (\eta,\tau), a \in (0,a_\dagger). \quad (61)$$

We write

$$q(t,a) = q_1(t,a) + q_2(t,a), \qquad t \in (\eta,\tau), a \in (0,a_\dagger), \quad (62)$$

where q_1 solves

$$\begin{cases} \dfrac{\partial q_1}{\partial t} - \dfrac{\partial q_1}{\partial a} - A^*q_1 + \mu(a)q_1 = 0, & t \in (\eta,\tau), a \in (0,a_\dagger), \\ q_1(t,a_\dagger) = 0, & t \in (\eta,\tau), \\ q_1(\eta,a) = q_\eta(a), & a \in (0,a_\dagger), \end{cases} \quad (63)$$

and q_2 solves

$$\begin{cases} \dfrac{\partial q_2}{\partial t} - \dfrac{\partial q_2}{\partial a} - A^* q_2 + \mu(a) q_2 = V(t,a), & t \in (\eta,\tau), a \in (0,a_\dagger), \\ q_2(t,a_\dagger) = 0, & t \in (\eta,\tau), \\ q_2(\eta,a) = 0, & a \in (0,a_\dagger). \end{cases} \tag{64}$$

Using Duhamel's formula, we can write q_2 as

$$q_2(t,a) = \int_\eta^t \mathbb{T}^0_{t-s} V(s,\cdot)\ \mathrm{d}s, \tag{65}$$

where $\mathbb{T}^0$ is the C^0 semigroup defined in (31). Using (30) and Proposition 3.4 we get

$$\begin{aligned} &\int_{a_1}^{a_\dagger} \|q_2(\tau,a)\|_X^2\ \mathrm{d}a \\ &\leqslant C_\tau \int_\eta^\tau \|q(t,0)\|_X^2\ \mathrm{d}t \leqslant C_\tau \mathcal{C}(\tau_0+\delta/2) \int_0^\tau \int_{a_1}^{a_2} \|B^* q(t,a)\|_U^2\ \mathrm{d}a\mathrm{d}t. \end{aligned} \tag{66}$$

On the other hand, we write

$$\int_{a_1}^{a_\dagger} \|q_1(\tau,a)\|_X^2\ \mathrm{d}a = \int_{a_1}^{a_2-\tau_0-\delta/2} \|q_1(\tau,a)\|_X^2\ \mathrm{d}a + \int_{a_2-\tau_0-\delta/2}^{a_\dagger} \|q_1(\tau,a)\|^2\ \mathrm{d}a. \tag{67}$$

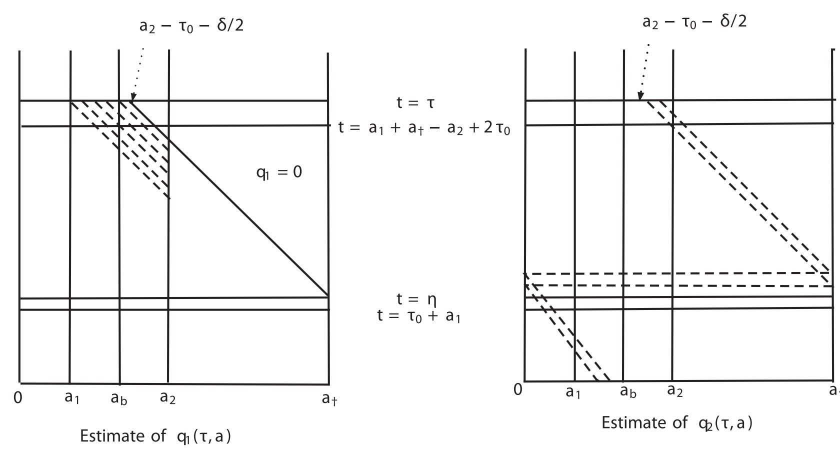

Figure 4.

From the semigroup representation of $\mathbb{T}^0_t$ in (31), we have

$$q_1(t,a) = 0 \text{ for } t-\eta \geqslant a_\dagger - a. \tag{68}$$

In particular,

$$q_1(\tau, a) = 0 \text{ for } a \in [a_2 - \tau_0 - \delta/2, a_\dagger].$$

Therefore,

$$\int_{a_1}^{a_\dagger} \|q_1(\tau, a)\|_X^2 \, \mathrm{d}a = \int_{a_1}^{a_2-\tau_0-\delta/2} \|q_1(\tau, a)\|_X^2 \, \mathrm{d}a. \tag{69}$$

Since $\tau - \eta > \tau_0$, by applying Proposition 3.3 to q_1 with $a_0 = a_2 - \tau_0 - \delta/2$, we obtain

$$\int_{a_1}^{a_2-\tau_0-\delta/2} \|q_1(\tau, a)\|_X^2 \, \mathrm{d}a \leqslant C_\tau \mathcal{C}(\tau_0 + \delta/2) \int_\eta^\tau \int_{a_1}^{a_2} \|B^* q_1(t, a)\|_U^2 \, \mathrm{d}a\mathrm{d}t. \tag{70}$$

Using Lemma 2.3 and Proposition 3.4 we deduce that

$$\begin{aligned}
&\int_\eta^\tau \int_{a_1}^{a_2} \|B^* q_1(t, a)\|_U^2 \, \mathrm{d}a\mathrm{d}t \\
&\leqslant 2\left(\int_\eta^\tau \int_{a_1}^{a_2} \|B^* q(t, a)\|_U^2 \, \mathrm{d}a\mathrm{d}t + \int_\eta^\tau \int_{a_1}^{a_2} \|B^* q_2(t, a)\|_U^2 \, \mathrm{d}a\mathrm{d}t\right) \\
&\leqslant C_\tau \left(\int_\eta^\tau \int_{a_1}^{a_2} \|B^* q(t, a)\|_U^2 \, \mathrm{d}a\mathrm{d}t + \int_\eta^\tau \|q(t, 0)\|_X^2 \, \mathrm{d}t\right) \\
&\leqslant C_\tau \left(1 + \mathcal{C}(\tau_0 + \delta/2)\right) \int_\eta^\tau \int_{a_1}^{a_2} \|B^* q(t, a)\|_U^2 \, \mathrm{d}a\mathrm{d}t.
\end{aligned}$$

Combining the above estimate together with (69) and (70), we obtain

$$\int_{a_1}^{a_\dagger} \|q_1(\tau, a)\|_X^2 \, \mathrm{d}a \leqslant C_\tau \left(1 + \mathcal{C}(\tau_0 + \delta/2)\right)^2 \int_0^\tau \int_{a_1}^{a_2} \|B^* q(t, a)\|_U^2 \, \mathrm{d}a\mathrm{d}t. \tag{71}$$

The above estimate, together with (62) and (3.2), yields

$$\begin{aligned}
&\int_{a_1}^{a_\dagger} \|q(\tau, a)\|_X^2 \, \mathrm{d}a \\
&\leqslant C_\tau \left((1 + \mathcal{C}(\tau_0 + \delta/2))^2 + \mathcal{C}(\tau_0 + \delta/2)\right) \int_0^\tau \int_{a_1}^{a_2} \|B^* q(t, a)\|_U^2 \, \mathrm{d}a\mathrm{d}t.
\end{aligned} \tag{72}$$

Finally, combining the above estimate with (60) we obtain (37), with

$$\kappa_\tau^2 = C_\tau \left[2 + \mathcal{C}\left(\frac{\tau - (a_1 + a_\dagger - a_2)}{2}\right)\right]^2. \tag{73}$$

This completes the proof of the theorem. □

4. Applications

The aim of this section is to apply the controllability result obtained in Theorem 1.1 for different classes of operators A and B.

4.1. Finite dimensional diffusion

Let us take $X = \mathbb{R}^n$ and $U = \mathbb{R}^m$ with $m \leqslant n$. Let A be a real $n \times n$ matrix and B be a real $n \times m$ matrix. Let us assume that

$$\operatorname{rank}[B, AB, \dots A^{n-1}B] = n. \tag{74}$$

In particular, we assume that the pair (A, B) is null-controllable for arbitrary time (i.e. $\tau_0 = 0$). Then, by Theorem 1.1, the system (6) is null controllable in time $\tau > a_1 + a_\dagger - a_2$.

A Special Case: Let us choose:

$$n = m = 1, \quad A = 0 \text{ and } B = 1,$$

i.e., we consider the classical diffusion free Lotka-McKendrick system. This system has already been studied in Barbu, Ianelli and Martcheva (2001); Hegoburu, Magal and Tucsnak (2018); Maity (2019), as well as Hegoburu and Aniţa (2019). By applying Theorem 1.1 to this particular case, we recover the result obtained in Hegoburu and Aniţa (2019, Theorem 1.1) (see also Hegoburu, Magal and Tucsnak, 2018; Maity, 2019).

4.2. Transport equation with age structure

Let $\Omega = (0, L)$. We consider the following control problem

$$\begin{cases} \dfrac{\partial p}{\partial t} + \dfrac{\partial p}{\partial a} + \dfrac{\partial}{\partial x}(v(x)p) + \mu(a)p = 0, & (t, a, x) \in (0, \tau) \times (0, a_\dagger) \times \Omega, \\ p(t, a, 0) = \mathbb{1}_{(a_1, a_2)} u(t, a), & (t, a) \in (0, \tau) \times (0, a_\dagger), \\ p(t, 0, x) = \displaystyle\int_0^{a_\dagger} \beta(a) p(t, a, x) \, \mathrm{d}a, & (t, x) \in (0, \tau) \times \Omega, \\ p(0, a, x) = p_0(a, x) & (a, x) \in \times (0, a_\dagger) \times \Omega, \end{cases} \tag{75}$$

where $v \in C^1[0, L]$ and $v(x) \geqslant \bar{v} > 0$. We take $X = L^2(\Omega)$ and $U = \mathbb{R}$. The operator A is defined by

$$\mathcal{D}(A) = \left\{ \varphi \in H^1(0, L) \mid \varphi(0) = 0 \right\}, \quad A\varphi = -\frac{\partial}{\partial x}(v\varphi).$$

The control operator B is defined by

$$Bu = u\delta_0,$$

where δ_0 is the Dirac mass at 0. It is well known that the pair (A, B) is null controllable in time $\tau > \dfrac{L}{\bar{v}}$. Therefore, in order to apply Theorem 1.1, we choose L or v such that

$$\frac{L}{\bar{v}} < \min\{a_2 - a_1, a_b - a_1\}. \tag{76}$$

Thus, the system (75) is null controllable in time $\tau > a_\dagger + a_1 - a_2 + \dfrac{2L}{\bar{v}}$.

4.3. Population dynamics models with spatial diffusion

Let Ω be a smooth bounded domain in $\mathbb{R}^3$. Let us set $X = L^2(\Omega)$. We consider the Lotka-McKendrick system with spatial diffusion. For $(t, a, x) \in (0, \tau) \times (0, a_\dagger) \times \Omega$, let $p(t, a, x)$ be the distribution density of individuals with respect to age $a \geqslant 0$ and spatial position $x \in \Omega$ at some time $t \geqslant 0$. The control problem we consider is:

$$\begin{cases} \dfrac{\partial p}{\partial t} + \dfrac{\partial p}{\partial a} - \Delta p + \mu(a)p = d_1 \mathbb{1}_{(a_1,a_2)} \mathbb{1}_{\mathcal{O}} u_1, & (t,a,x) \in (0,\tau) \times (0,a_\dagger) \times \Omega \\ \dfrac{\partial p}{\partial n} = d_2 \mathbb{1}_{(a_1,a_2)} \mathbb{1}_{\Gamma} u_2, & (t,a,x) \in (0,\tau) \times (0,a_\dagger) \times \partial\Omega \\ p(t,0,x) = \displaystyle\int_0^{a_\dagger} \beta(a) p(t,a,x) \, \mathrm{d}a, & (t,x) \in (0,\tau) \times \Omega, \\ p(0,a,x) = p_0(a,x) & (a,x) \in (0,a_\dagger) \times \Omega, \end{cases} \tag{77}$$

where $\mathcal{O} \subset \Omega$ and $\Gamma \subset \partial\Omega$.

4.3.1. Interior control

We consider the case of $d_2 = 0$. In this case, we have

$$A = \Delta, \quad \mathcal{D}(A) = \left\{ \varphi \in H^2(\Omega) \mid \frac{\partial \varphi}{\partial n} = 0 \right\}, \tag{78}$$

and

$$B = \mathbb{1}_{\mathcal{O}}. \tag{79}$$

It is well known that the pair (A, B) is null controllable in arbitrary time, where A and B are defined as in (78) and (79), respectively (see, for instance, Fursikov and Imanuvilov, 1996). Therefore, by Theorem 1.1 the system (77) is null controllable in time $\tau > a_1 + a_\dagger - a_2$ by interior controls $u_1 \in L^2((0, \tau) \times (0, a_\dagger) \times \Omega)$. This result was already obtained in Maity, Tucsnak and Zuazua (2019).

4.3.2. Boundary control with respect to the spatial variable

We consider the case of $d_1 = 0$. In this case

$$B^* w = \mathbb{1}_\Gamma w, \quad w \in \mathcal{D}(A).$$

It is well known that (A^*, B^*) is the final state observable for any time, Seidman (1976). Thus, by applying Theorem 1.1, with $\tau_0 = 0$, we get that the system (77) is null controllable in time $\tau > a_1 + a_\dagger - a_2$ by controls $u_2 \in L^2((0, \tau) \times (0, a_\dagger) \times \Gamma)$.

4.4. Population dynamics models with degenerate diffusion

Let $\Omega = (0,1)$ and $\mathcal{O} = (\ell_1, \ell_2) \subset \Omega$. We consider the following age structured model with degenerate diffusion:

$$\begin{cases} \dfrac{\partial p}{\partial t} + \dfrac{\partial p}{\partial a} - k(x)\dfrac{\partial^2 p}{\partial x^2} + \mu(a)p = \mathbb{1}_{(a_1,a_2)}\mathbb{1}_{\mathcal{O}}u, & (t,a,x) \in (0,\tau) \times (0,a_\dagger) \times \Omega \\ p(t,a,0) = p(t,a,1) = 0, & (t,a) \in (0,\tau) \times (0,a_\dagger), \\ p(t,0,x) = \displaystyle\int_0^{a_\dagger} \beta(a)p(t,a,x)\,\mathrm{d}a, & (t,x) \in (0,\tau) \times \Omega, \\ p(0,a,x) = p_0(a,x) & (a,x) \in (0,a_\dagger) \times \Omega, \end{cases} \tag{80}$$

where k is a non-negative continuous function in $[0,1]$ and degenerate at the boundary, i.e.

$$k(0) = k(1) = 0. \tag{81}$$

Let us set the state and the control space as follows:

$$X = L^2_{1/k}(0,1) = \left\{ \varphi \in L^2(0,1) \mid \int_0^1 \frac{\varphi^2}{k}\,\mathrm{d}x < \infty \right\} \text{ and } U = L^2(0,1). \tag{82}$$

We consider the unbounded operator A on X defined by

$$\mathcal{D}(A) = \left\{ \varphi \in L^2_{1/k}(0,1) \cap H^1_0(0,1) \mid k\partial_{xx}\varphi \in L^2_{1/k}(0,1) \right\} \text{ and } A\varphi = k\partial_{xx}\varphi.$$

The operator B is defined by $B = \mathbb{1}_{\mathcal{O}}$. By Cannarsa, Fragnelli and Rocchetti (2008, Theorem 2.3), the operator A generates a C^0-semigroup on X. We now make several assumptions on the degenerate coefficient k so that the pair (A,B) is null controllable. Following Cannarsa, Fragnelli and Rocchetti (2008), we make the following assumptions on k: The function $k \in C^0[0,1] \cap C^3(0,1)$ is such that it satisfies (81) and $k > 0$ in $(0,1)$. Moreover, there exist $\varepsilon \in (0,1)$ such that

1) The function $\dfrac{x\partial_x k}{k} \in L^\infty(0,\varepsilon)$ and there exists $M_1 \in (0,2)$ and $C_1 > 0$ such that $\dfrac{x\partial_x k}{k} \leqslant M_1$ and $\left|\partial_{xx}\left(\dfrac{x\partial_x k}{k}\right)\right| \leqslant C_1 \dfrac{1}{k(x)}$ for all $x \in (0,\varepsilon)$;
2) The function $\dfrac{(x-1)\partial_x k}{k} \in L^\infty(1-\varepsilon,1)$ and there exist $M_2 \in (0,2)$ and $C_2 > 0$ such that $\dfrac{(x-1)\partial_x k}{k} \leqslant M_2$ and $\left|\partial_{xx}\left(\dfrac{(x-1)\partial_x k}{k}\right)\right| \leqslant C_2 \dfrac{1}{k(x)}$ for all $x \in (1-\varepsilon,1)$.

Under the above assumptions, by Cannarsa, Fragnelli and Rocchetti (2008, Theorem 4.5) the pair (A,B) is null controllable in any time. Therefore, by Theorem 1.1, the system (80) is null controllable in time $\tau > a_\dagger + a_1 - a_2$.

Remark 4.1 *Let us make the following remarks:*

- *Recently, similar controllability result for the system* (80) *was proven in Fragnelli (2018). Our result can be seen as an improvement of the above mentioned result, as we are able to tackle the case of a control, which is active for small ages and we show that our global controllability result applies to individuals of all ages, without the need to exclude ages in a neighbourhood of zero.*
- *Our method also applies to the case, in which the spatial variable is multidimensional. Of course, we need to make suitable assumptions on degeneracy. For instance, we can consider the case studied by Cannarsa, Martinez and Vancostenoble (2009, 2016). More precisely, let* Ω *be a smooth bounded domain in* $\mathbb{R}^2$*. The operator* A *is defined by*

  $$A\varphi = \operatorname{div}\left(M(x)\nabla\varphi\right),$$

  with appropriate boundary conditions. The control operator B *is defined by* $B = \mathbb{1}_{\mathcal{O}}$*, where* $\mathcal{O} \subset \Omega$*. Under suitable assumptions on the degenerate matrix* $M(x)$*, the pair* (A, B) *is null controllable in arbitrary time (see, for instance, Cannarsa, Martinez and Vancostonoble, 2009). Thus, the corresponding age structured model is also null controllable in time* $\tau > a_1 + a_\dagger - a_2$*.*

4.5. Fractional diffusion equation with age structure

Let $X = L^2(\Omega)$ and let $A := (-\Delta_D)^\alpha$ or $A := (-\Delta_N)^\alpha$, where $-\Delta_D$ and $-\Delta_N$ are the Dirichlet and the Neumann Laplacian in Ω and $\alpha > 1/2$. Let B be defined by (79). Then, (A, B) is null controllable in any time (see, for instance, Micu and Zuazua, 2006; Miller, 2006; Tenenbaum and Tucsnak, 2011). Therefore, the conclusion of Theorem 1.1 also holds with the above choice of (A, B).

4.6. Schrödinger equation with age structure

Let Ω be a square in $\mathbb{R}^2$, and we consider the Schrödinger operator as diffusion operator. More precisely, we take $X = L^2(\Omega)$

$$A = -i\Delta, \quad D(A) = H^2(\Omega) \cap H_0^1(\Omega).$$

Let B be defined by (79). Then, the pair (A, B) is null controllable in any time (see Jaffard, 1988). Thus, the conclusion of Theorem 1.1 holds with $\tau_0 = 0$.

Alternatively, we can take Ω to be a unit disc in $\mathbb{R}^2$ and $\mathcal{O} \subset \overline{\Omega}$ to be an open set, such that $\mathcal{O} \cap \partial\Omega \neq \emptyset$. The operators A and B are defined as above. The pair (A, B) is null controllable in any time, which was proven in Anantharaman, Léautand and Maciá (2016, Theorem 1.2). Therefore, Theorem 1.1 also holds in this setup.

5. Controllability with regular controls

In Theorem 1.1, we have shown that the age structured system (6) is null controllable by controls $u \in L^2((0,\tau) \times (0,a_\dagger);U)$. However, in many practical applications, we may need to choose controls in more regular spaces. For instance, while proving positivity of the controlled trajectory of the system (77) one needs to choose control $u_1 \in L^\infty((0,\tau) \times (0,a_\dagger) \times \Omega)$ (see Maity, Tucsnak and Zuazua, 2019, Theorem 4.6). The aim of this section is to show that null controllability by "smooth" controls of the pair (A,B) is also inherited by the pair $(\mathcal{A},\mathcal{B})$.

To this aim, let us fix $s \in \mathbb{N} \cup \{0\}$ and a Hilbert space V, so that $V \hookrightarrow U$. Following Pighin and Zuazua (2018) we introduce the notion of smooth controllability.

Definition 5.1 *We say that a pair (A,B) is smoothly null controllable in time τ, if for every $z_0 \in \mathcal{D}(A^s)$ there exists a control $u \in L^\infty(0,\tau,V)$ such that the solution of the system*

$$\dot{z}(t) = Az(t) + Bu(t) \quad t \in [0,\tau], \qquad z(0) = z_0$$

satisfies $z(\tau) = 0$.

The smooth controllability property of the system (6) can be stated as follows:

Theorem 5.1 *Let us assume the hypothesis of Theorem 1.1. Let us also assume that the pair (A,B) is smoothly null controllable in any time $\tau > \tau_0$, with*

$$0 \leqslant \tau_0 < \overline{\tau}, \quad \overline{\tau} = \min\{a_2 - a_1, a_b - a_1\}. \tag{83}$$

Then, for every $\tau > a_1 + a_\dagger - a_2 + 2\tau_0$ and for every $p_0 \in L^\infty(0,a_\dagger;\mathcal{D}(A^s))$ there exists a control $v \in L^\infty((0,\tau) \times (0,a_\dagger) \times V)$ such that the solution p of (6) satisfies

$$p(\tau,a) = 0 \text{ for all } a \in (0,a_\dagger). \tag{84}$$

The proof the above theorem is a consequence of a suitable observability inequality. Let us briefly describe the main steps. The principal idea is the same, i.e., to use observability property of the pair (A,B) along the characteristics. The smooth controllability in time τ of the pair (A,B) is equivalent to the following final state observability inequality (see, for instance, Pighin and Zuazua, 2018, Section 2): there exists a constant $k_\tau > 0$ such that for any $z_0 \in \mathcal{D}(A^*)$

$$\|\mathbb{S}_\tau^* z_0\|_{\mathcal{D}(A^s)^*} \leqslant k_\tau \int_0^\tau \|i^* B^* \mathbb{S}_t^* z_0\|_{V^*} dt, \tag{85}$$

where $\mathcal{D}(A^s)^*$ and V^* are the duals of $\mathcal{D}(A^s)^*$ and V respectively, with respect to the pivot spaces X and U and $i : V \to U$ is the inclusion map. By applying the above observability property of the pair (A, B) along the characteristics one can prove that: for every $\tau > a_1 + a_\dagger - a_2 + 2\tau_0$ and $q_0 \in \mathcal{D}(\mathcal{A}^*)$, the solution q of (34) satisfies

$$\int_0^{a_\dagger} \|q(\tau, a)\|_{\mathcal{D}(A^s)^*} \, \mathrm{d}a \leqslant \kappa_\tau^2 \int_0^\tau \int_{a_1}^{a_2} \|i^* B^* q(t, a)\|_{V^*} \, \mathrm{d}a\mathrm{d}t. \tag{86}$$

Next, using the classical duality argument (see, for instance, Maity, Tucsnak and Zuazua, 2019, Theorem 4.6, or Micu, Roventa and Tucsnak, 2012, Proposition 2.5) we can easily prove Theorem 5.1.

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