On identities for derivative operators

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Abstract
Let X be a commutative algebra with unity e and let D be a derivative on X that means the Leibniz rule is satised: $D(f\ g) = D(f)\ g + f\,D(g)$. If $D^{(k)}$ is k-th iteration of D then we prove that the following identity holds for any positive integer k

$$\frac{1}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}f^j D^{(m)}(gf^{k-j}) = \Phi_{k,m}(g,f) = \begin{cases}0, & 0 \le m < k,\\ gD(f)^k, & k = m\end{cases}$$

As an application we prove a sharp version of Bernstein's inequality for trigonometric polynomials.

Key words: Derivative operators, polynomial inequalities

1. An identity.

Let X be a commutative algebra with unity e and let D be a derivative on X that means the Leibniz rule is satisfied: $D(f \cdot g) = D(f) \cdot g + f \cdot D(g)$. If $D^{(k)}$ is k-th iteration of D (with $D^{(0)}(f) = f$) then we define for $f, g \in X$

$$\Phi_{k,m}(g, f) := \frac{1}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}f^j D^{(m)}(gf^{k-j}).$$

It is easy to check the following properties of $\Phi_{k,m}$.

Proposition 1.1.

- $\Phi_{0,m}(g, f) = D^{(m)}(g)$ *for* $m \ge 0$
- *Key observation:*
$$\Phi_{k,0}(g, f) = 0, \ \ k \ge 1.$$
- *Basic recurrence:*
$$\Phi_{k,m}(g, f) = D(f)\Phi_{k-1,m-1}(g, f) + D(\Phi_{k,m-1}(g, f)).$$

Proof. Only the recurrence is not clear. To see it, let us calculate $D(\Phi_{k,m-1}(g, f))$

$$D(\Phi_{k,m-1}(g, f)) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}D(f^j D^{(m-1)}(gf^{k-j}))$$

$$= \frac{1}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}\left(jf^{j-1}D(f)D^{(m-1)}(gf^{k-j}) + f^j D^{(m)}(gf^{k-j})\right)$$

$$= \Phi_{k,m}(g, f) + D(f)\frac{1}{(k-1)!}\sum_{j=1}^{k}(-1)^j\binom{k}{j}\frac{j}{k}f^{j-1}D^{(m-1)}(gf^{k-j})$$

$$= \Phi_{k,m}(g, f) - D(f)\Phi_{k-1,m-1}(g, f),$$

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which is equivalent to the recurrence formula. □

Theorem 1.2. *Let $D : X \longrightarrow X$ be a derivation $(D(g \cdot f) = g \cdot D(f) + D(g) \cdot f)$ on a commutative algebra X, then for any positive integer k we have*

$$\Phi_{k,m}(g,f) = \begin{cases} 0, \ 0 \leq m < k, \\ g \cdot D(f)^k, \ k = m. \end{cases}$$

Proof. (Induction with respect to k).

1. $k = 1$ $m = 0$

$$(\mathcal{L}) = \ e \cdot g \cdot f - f \cdot g = 0 = \ (\mathcal{R}).$$

$m = 1$

$$(\mathcal{L}) = \ D(g \cdot f) - f \cdot D(g) = g \cdot D(f) = \ (\mathcal{R}).$$

Remark 1.3. Those particular cases show, that Leibniz condition is necessary to hold the theorem ($k = 1, m = 1$) and commutative assumption is equivalent to the case $k = 1, m = 0$.

2. $Thm.(k-1) \Rightarrow \ Thm.(k)$ for $k \geq 2$.

To do this, we shall prove by induction with respect to m, the formula for $0 \leq m \leq k$.

$m = 0$

$$(\mathcal{L}) = \ gf^k \sum_{j=0}^{k} (-1)^j \binom{k}{j} e = 0 = \ (\mathcal{R}).$$

$Thm.(k,m) \Rightarrow Thm.(k, m+1)$ with $0 \leq m \leq k-1$.
We have

$$(\mathcal{L}) = \ \Phi_{k,m+1}(g,f) = D(f)\Phi_{k-1,m}(g,f) + D(\Phi_{k,m}(g,f)) = 0 = (\mathcal{R})$$

for $0 \leq m \leq k-2$. If $m = k-1$ then

$$(\mathcal{L}) = \ \Phi_{k,k}(g,f) = D(f)\Phi_{k-1,k-1}(g,f) + D(\Phi_{k,k-1}(g,f))$$
$$= D(f)gD(f)^{k-1} + 0 = gD(f)^k = \ (\mathcal{R}).$$

The proof is finished.

□

Remark 1.4. Theorem 1.2 has been presented, without a proof, in [5]. A knowledge of this result was a motivation to find its generalization by U. Abel [1], where paper [5] and thus Theorem 1.2 was noticed.

If we take $g = e$ then we obtain the following identity

$$D(f)^k = \frac{1}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} f^j D^{(k)}(f^{k-j}). \tag{1.1}$$

It was found in the case of polynomials of one variable and usual derivative operator by Beata Milówka [9] and, in the general case, by P. Ozorka in his PhD thesis (cf. informations in [5],[1]. The proof presented now is much simpler than original proofs by B. Milówka and P. Ozorka.) Later it was found by M. Baran that those identities were known earlier (cf. [8]) but nobody has applied them to polynomial inequalities as it was made by B. Milówka and P. Ozorka. It seems that it is a future for further applications of this type identities.

2. An application

We shall prove, as an application of Theorem 1.2, that there exists a constant B such that for any trigonometric polynomial T of degree N we have Bernstein's type inequality $||T'|| \leq BN||T||$. This is a sharp with respect to the exponent of N in this bound but the exact inequality holds with $B = 1$. There is known that S.N. Bernstein has obtained his bound with $B = 2$. We show, that we can take $B = 2e$.

A trigonometric polynomial T of degree N has a form

$$T(t) = \sum_{j=0}^{N}(a_j \cos(jt) + b_j \sin(jt)),\ ||T|| = \max_{|t|\leq\pi} |T(t)|.$$

Everybody knows that $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} T(x)dx,\ a_j = \frac{1}{\pi}\int_{-\pi}^{\pi} T(x)\cos(jx)dx,\ b_j = \frac{1}{\pi}\int_{-\pi}^{\pi} T(x)\sin(jx)dx,\ j \geq 1$. Moreover $\frac{1}{\pi}\int_{-\pi}^{\pi} |T(x)|^2dx = 2|a_0|^2 + \sum_{j=1}^{N}(|a_j|^2 + |b_j|^2)$. It is clear that

$$||T^{(k)}||_2^2 = \frac{1}{\pi}\int_{-\pi}^{\pi} |T^{(k)}(x)|^2dx = \sum_{j=1}^{N} j^{2k}(|a_j|^2 + |b_j|^2)$$

$$\leq N^{2k}\sum_{j=1}^{N}(|a_j|^2 + |b_j|^2) = N^{2k}||T||_2^2,$$

$$||T|| \leq |a_0| + \sum_{j=1}^{N}(|a_j| + |b_j|) \leq (2N+1)^{1/2}\left(2|a_0|^2 + \sum_{j=1}^{N}(|a_j|^2 + |b_j|^2)\right)^{1/2}$$

$$= (2N+1)^{1/2}||T||_2.$$

Now, applying Theorem 1.2, we get

$$||T'||^k \leq \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}||T||^j||(T^{k-j})^{(k)}|| \leq \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}||T||^j(2N(k-j)+1)^{1/2}||(T^{k-j})^{(k)}||_2$$

$$\leq \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}||T||^j(2N(k-j)+1)^{1/2}(N(k-j))^k||(T^{k-j})||_2$$

$$\leq \sqrt{2}\frac{1}{k!}\sum_{j=0}^{k-1}\binom{k}{j}||T||^{j}(2N(k-j)+1)^{1/2}(N(k-j))^{k}||T||^{k-j}$$

$$= 2N^{k+1/2}\frac{1}{k!}\sum_{j=0}^{k-1}\binom{k}{j}((k-j)+1/2N)^{1/2}(k-j)^{k}||T||^{k}$$

$$\leq N^{k+1/2}\frac{1}{k!}2^{k+1}k^{k}(k+1/2)^{1/2}||T||^{k},$$

which gives $||T'|| \leq 2\left(2(k+1/2)^{1/2}k^{k}/k!\right)^{1/k} N^{1+1/2k}||T||$ and letting $k \to \infty$ we get inequality

(2.1) $$||T'|| \leq 2eN||T||.$$

As an application, applying a method from [2] (cf. also [3]), we get three bounds for algebraic polynomials:

(2.2) $$|P'(t)| \leq 2e(\deg P)(1-t^2)^{-1/2}||P||_{[-1,1]},\ t \in (-1,1),$$

(2.3) $$|P(t)| \leq 2e(\deg P+1)||P(t)\sqrt{1-t^2}||_{[-1,1]},$$

(2.4) $$||P'|_{[-1,1]} \leq 4e^2(\deg P)^2||P||_{[-1,1]}$$

The last one is Markov's inequality with sharp Markov's exponent 2 (cf. [4]). In the exact inequality exponent $4e^2$ is replaced by 1. Markov's inequality with constant e^2 was also showed in [6] by a similar method, where Milówka version of (1.1) was used.

References

[1] U. Abel, *An identity for formal derivatives in a commutative algebra*, Ann. Polon. Math. 119 (2017), no. 3, 195-202.

[2] M. Baran, *New approch to Markov inequality in L^p norms*, Approximation Theory: in Memory of A. K. Varma (N. K. Govil and alt., ed.), Marcel Dekker, New York (1998), 75-85.

[3] M. Baran, *Polynomial inequalities in Banach spaces*, Constructive approximation of functions, 2342, Banach Center Publ., 107, Polish Acad. Sci. Inst. Math., Warsaw, (2015).

[4] M. Baran, L. Białas-Cież, B. Milówka, *On the best exponent in Markov inequality*, Potential Analysis, 38 (2) (2013), 635–651.

[5] M. Baran, A. Kowalska, B. Milówka, P. Ozorka, *Identities for a derivation operator and their applications*, Dolomites Res. Notes Approx. 8 (2015), Special Issue, 102-110.

[6] M. Baran, B. Milówka, P. Ozorka, *Markov's property for k-th derivative*, Annales Polonici Mathematici, 106 (2012), 31–40.

[7] P. Borwein, T. Erdélyi, *Polynomials and Polynomial Inequalities*, Springer, Berlin, 1995, Graduate Texts in Mathematics 161.

[8] M. Klimek, *Remarks on derivations on normed algebras*, Zeszyty Nauk. Uniw. Jagiello. Prace Mat. No. 20 (1979), 161-166.

[9] B. Milówka, *Markov's inequality and a generalized Plesniak condition*, East J. Approx. 11 (2005), 291–300.

[10] Q. Rahman, G. Schmeisser, *Analytic theory of polynomials*, Clarendon Press, (2002).

[11] W. Rudin, *Functional Analysis*, McGraw-Hill Book

[12] E. M. Stein, *Interpolation in polynomial classes and Markoff's inequality*, Duke Math. J. 24 (1957), 467-476.