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Analysis, adaptive control and synchronization of a novel 4-D hyperchaotic hyperjerk system and its SPICE implementation

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A hyperjerk system is a dynamical system, which is modelled by an *n*th order ordinary differential equation with $n \geqslant 4$ describing the time evolution of a single scalar variable. Equivalently, using a chain of integrators, a hyperjerk system can be modelled as a system of n first order ordinary differential equations with $n \geqslant 4$. In this research work, a 4-D novel hyperchaotic hyperjerk system has been proposed, and its qualitative properties have been detailed. The Lyapunov exponents of the novel hyperjerk system are obtained as $L_1 = 0.1448, L_2 = 0.0328, L_3 = 0$ and $L_4 = -1.1294$. The Kaplan-Yorke dimension of the novel hyperjerk system is obtained as $D_{KY} = 3.1573$. Next, an adaptive backstepping controller is designed to stabilize the novel hyperjerk chaotic system with three unknown parameters. Moreover, an adaptive backstepping controller is designed to achieve global hyperchaos synchronization of the identical novel hyperjerk systems with three unknown parameters. Finally, an electronic circuit realization of the novel jerk chaotic system using SPICE is presented in detail to confirm the feasibility of the theoretical hyperjerk model.

Key words: hyperchaos, hyperjerk system, adaptive control, backstepping control, synchronization.

1. Introduction

Chaos theory describes the qualitative study of unstable aperiodic behaviour in deterministic nonlinear dynamical systems. For the motion of a dynamical system to be chaotic, the system variables should contain nonlinear terms and it must satisfy three properties: boundedness, infinite recurrence and sensitive dependence on initial conditions [1]. The Lyapunov exponent of a dynamical system is a quantity that characterizes the rate of separation of infinitesimally close trajectories. The sensitive dependence on

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initial conditions of a dynamical system is characterized by the presence of a positive Lyapunov exponent. A positive Lyapunov exponent reflects a direction of *stretching* and *folding* and along with phase-space compactness indicates the presence of chaos in a dynamical system. An n-dimensional dynamical system has a spectrum of n Lyapunov exponents and the *maximal Lyapunov exponent* (MLE) of a chaotic system is defined as the largest positive Lyapunov exponent of the system.

The first famous chaotic system was accidentally discovered by Lorenz, when he was designing a 3-D model for atmospheric convection in 1963 [2]. Subsequently, Rössler discovered a 3-D chaotic system in 1976 [3], which is algebraically simpler than the Lorenz system. Indeed, Lorenz's system is a seven-term chaotic system with two quadratic nonlinearities, while Rössler's system is a seven-term chaotic system with just one quadratic nonlinearity. Some well-known paradigms of 3-D chaotic systems are Arneodo system [4], Sprott systems [5], Chen system [6], Lü-Chen system [7], Liu system [8], Cai system [9], T-system [10], etc. Many new chaotic systems have been also discovered like Li system [11], Sundarapandian systems [12, 13], Vaidyanathan systems [14, 15, 16, 17, 18, 19, 20, 21], Pehlivan system [22], Jafari system [23], Pham system [24], etc.

Chaos theory has applications in several fields of science and engineering such as oscillators [25], lasers [26], chemical reactions [27], biology [28], ecology [29], neural networks [30], robotics [31], fuzzy logic [32], electrical circuits [33], cryptosystems [34, 35], etc. Chaos communications is an application of chaos theory which is aimed to provide security in the transmission of information performed through telecommunications technologies [36, 37, 38]. For implementation of chaos communication systems, two chaotic oscillators are required as a transmitter (or master) and receiver (or slave). At the transmitter, a message is added onto a chaotic signal and then, the message is masked in the chaotic signal. When chaos synchronization is used, a basic scheme of a communications device is made by two identical chaotic systems, where one chaotic system is used as the transmitter, and the other chaotic system as the receiver. They are connected in a configuration where the transmitter drives the receiver in such a way that identical synchronization of chaos between the two oscillators is achieved. For the purpose of transmission of information, at the transmitter, a message is added as a small perturbation to the chaotic signal that drives the receiver. In this way, the message transmitted is masked by the chaotic signal. When the receiver synchronizes to the transmitter, the message is decoded by a subtraction between the signal sent by transmitter and its copy generated at the receiver by means of the synchronization of chaos mechanism.

A hyperchaotic system is generally defined as a chaotic system with at least two positive Lyapunov exponents [39]. Thus, the dynamics of a hyperchaotic system are expended in several different directions simultaneously. Thus, the hyperchaotic systems have more complex dynamical behaviour and hence they have miscellaneous applications in engineering such as secure communications [40], cryptosystems [41], encryption [42], electrical circuits [43], etc.

The minimum dimension for an autonomous, continuous-time, hyperchaotic system is four. Since the discovery of a first 4-D hyperchaotic system by Rössler in 1979 [44], many 4-D hyperchaotic systems have been found in the literature such as hyperchaotic Lorenz system [45], hyperchaotic Lü system [46], hyperchaotic Chen system [47], hyperchaotic Wang system [48], hyperchaotic Newton-Leipnik system [49], hyperchaotic Jia system [50], hyperchaotic Vaidyanathan systems [51, 52], etc. In mechanics, if the scalar $x(t)$ represents the *position* of a moving object at time t, then the first derivative, $\dot{x}(t)$, represents the *velocity*, the second derivative, $\ddot{x}(t)$, represents the *acceleration* and the third derivative, $\dddot{x}(t)$, represents the *jerk* or *jolt* [53]. In mechanics, a *jerk system* is described by an explicit third order ordinary differential equation describing the time evolution of a single scalar variable x according to the dynamics

$$\frac{d^3x}{dt^3} = f\left(\frac{d^2x}{dt^2}, \frac{dx}{dt}, x\right) \tag{1}$$

A particularly simple example of a jerk system is the famous Coullet system [54] given by

$$\frac{d^3x}{dt^3} + a\frac{d^2x}{dt^2} + \frac{dx}{dt} = g(x) \tag{2}$$

where $g(x)$ is a nonlinear function such as $g(x) = b(x^2 - 1)$, which exhibits chaos for $a = 0.6$ and $b = 0.58$. A generalization of the jerk dynamics is given by the dynamics

$$\frac{d^{(n)}x}{dt^n} = f\left(\frac{d^{(n-1)}x}{dt^{n-1}}, \ldots, \frac{dx}{dt}, x\right), \quad (n \geqslant 4) \tag{3}$$

An ordinary differential equation of the form (3) is called a *hyperjerk* system since it involves time derivatives of a jerk function [55].

In this paper, we propose a 4-D novel hyperchaotic hyperjerk system by adding a quadratic nonlinearity to the Chlouverakis-Sprott hyperjerk system [56]. First, we detail the fundamental qualitative properties of the novel hyperchaotic hyperjerk system. We show that the novel hyperjerk system is dissipative. Then we derive the Lyapunov exponents and Kaplan-Yorke dimension of the novel hyperchaotic hyperjerk system. The study of control of a chaotic system investigates methods for designing feedback control laws that globally or locally asymptotically stabilize or regulate the outputs of a chaotic system. Next, this paper derives an adaptive backstepping control law that stabilizes the novel hyperjerk system, when the system parameters are unknown. The backstepping control method is a recursive procedure that links the choice of a Lyapunov function with the design of a controller and guarantees global asymptotic stability of strict feedback systems [57, 58]. This paper also derives an adaptive backstepping control law that achieves global chaos synchronization of the identical 4-D novel hyperchaotic hyperjerk systems with unknown parameters. Chaos synchronization problem deals with the synchronization of a couple of systems called the master or drive system and the slave or response system. To solve this problem, control laws are designed so that the output of the slave system tracks the output of the master system asymptotically with time. Because of the butterfly effect, the synchronization of chaotic systems is a challenging problem in the chaos literature even when the initial conditions of the master and slave systems are nearly identical because of the exponential divergence of the outputs of the two systems in the absence of any control. All the main adaptive results in this paper are proved using Lyapunov stability theory. MATLAB simulations are depicted to illustrate the phase portraits of the novel hyperchaotic hyperjerk system with two positive Lyapunov exponents, adaptive stabilization and synchronization results for the novel 4-D hyperchaotic hyperjerk system. Finally, an electronic circuit realization of the novel hyperchaotic hyperjerk system using SPICE is presented to confirm the feasibility of the theoretical model.

2. A 4-D novel hyperchaotic hyperjerk system

In [56], Chlouverakis and Sprott discovered a simple hyperchaotic hyperjerk system given by the dynamics

$$\frac{d^4x}{dt^4} + \frac{d^3x}{dt^3}x^4 + A\frac{d^2x}{dt^2} + \frac{dx}{dt} + x = 0 \tag{4}$$

In system form, the differential equation (4) can be expressed as

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= -x_1 - x_2 - Ax_3 - x_1^4 x_4 \end{cases} \tag{5}$$

When $A = 3.6$, the hyperjerk system (5) exhibits *hyperchaos* with Lyapunov exponents $L_1 = 0.132, L_2 = 0.035, L_3 = 0$ and $L_4 = -1.25$.

The Kaplan-Yorke dimension [59, 60] of a chaotic system of order n is defined as

$$D_{KY} = j + \frac{L_1 + \cdots + L_j}{|L_{j+1}|} \tag{6}$$

where $L_1 \geqslant L_2 \geqslant \cdots \geqslant L_n$ are the Lyapunov exponents of the chaotic system and j is the largest integer for which $L_1 + L_2 + \ldots + L_j \geqslant 0$. (Kaplan-Yorke conjecture states that for typical chaotic systems, $D_{KY} \approx D_L$, the information dimension of the system.) Thus, the Kaplan-Yorke dimension of the hyperjerk system (5) is easily calculated a $D_{KY} = 3.13$.

In this work, we propose a novel hyperjerk system by adding a quadratic nonlinearity to the Chlouverakis-Sprott hyperjerk system (5) and with a different set of values for the system parameters. Our novel hyperjerk system is given in system form as

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= -x_1 - x_2 - bx_1^2 - ax_3 - cx_1^4 x_4 \end{cases} \tag{7}$$

where a, b and c are positive parameters. In this paper, we shall show that the system (7) is *hyperchaotic* when the parameters a, b and c take the values

$$a = 3.7, \quad b = 0.2, \quad c = 1.5 \tag{8}$$

For the parameter values in (8), the Lyapunov exponents of the novel hyperjerk system (7) are obtained as

$$L_1 = 0.1448, \;\; L_2 = 0.0328, \;\; L_3 = 0, \;\; L_4 = -1.1294 \tag{9}$$

From the LE spectrum given in (9), it is easily seen that the maximal Lyapunov exponent (MLE) of our novel hyperchaotic hyperjerk system (7) is $L_1 = 0.1448$, which is greater than the MLE of the Chlouverakis-Sprott hyperchaotic hyperjerk system (5). Also, the Kaplan-Yorke dimension of the novel hyperjerk system (7) is calculated as $D_{KY} = 3.1573$, which is greater than the Kaplan-Yorke dimension of the Chlouverakis-Sprott hyperjerk system (5). This shows that the novel hyperchaotic hyperjerk system (7) exhibits more complex behaviour than the Chlouverakis-Sprott hyperchaotic hyperjerk system (5).

For numerical simulations, we take the initial values of the novel hyperjerk system (7) as $x_1(0) = 0.1, x_2(0) = 0.1, x_3(0) = 0.1$ and $x_4(0) = 0.1$. Figs. 1-4 depict the 3-D projections of the 4-D novel hyperjerk system (7) on (x_1, x_2, x_3), (x_1, x_2, x_4), (x_1, x_3, x_4) and (x_2, x_3, x_4) spaces respectively.

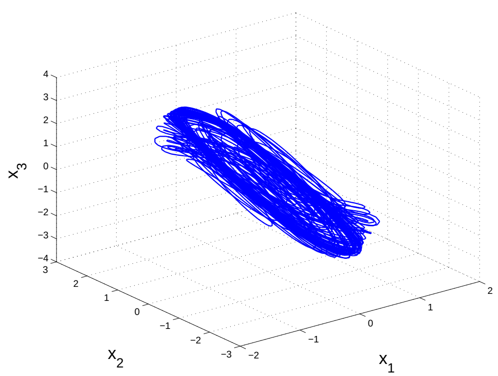

Figure 1: 3-D projection of the 4-D novel hyperjerk system on (x_1, x_2, x_3) space.

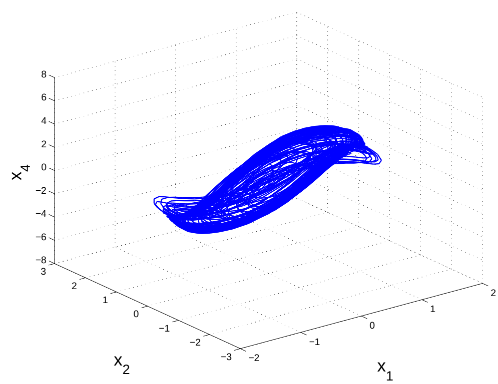

Figure 2: 3-D projection of the 4-D novel hyperjerk system on (x_1, x_2, x_4) space.

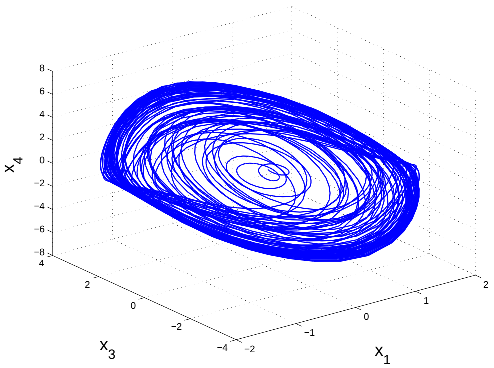

Figure 3: 3-D projection of the 4-D novel hyperjerk system on (x_1, x_3, x_4) space.

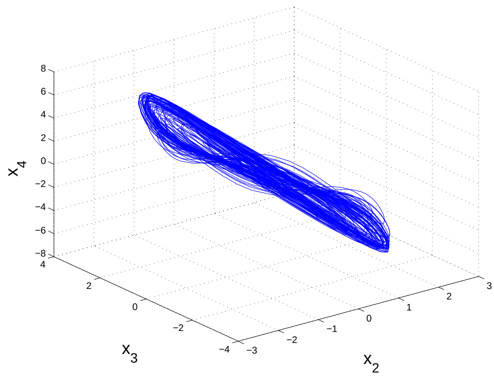

Figure 4: 3-D projection of the 4-D novel hyperjerk system on (x_2, x_3, x_4) space.

3. Analysis of the 4-D novel hyperjerk system

3.1. Equilibrium Points

The equilibrium points of the 4-D novel hyperjerk system (7) are obtained by solving the equations

$$\left.\begin{array}{rcl} f_1(x_1,x_2,x_3,x_4) &=& x_2 &=& 0 \\ f_2(x_1,x_2,x_3,x_4) &=& x_3 &=& 0 \\ f_3(x_1,x_2,x_3,x_4) &=& x_4 &=& 0 \\ f_4(x_1,x_2,x_3,x_4) &=& -x_1 - x_2 - bx_1^2 - ax_3 - cx_1^4 x_4 &=& 0 \end{array}\right\} \tag{10}$$

We take the parameter values as in the hyperchaotic case (8). Thus, the equilibrium points of the system (7) are characterized by the equations

$$x_1(1 + 0.2x_1) = 0, \;\; x_2 = 0, \;\; x_3 = 0, \;\; x_4 = 0 \tag{11}$$

Solving the system (11), we get the equilibrium points of the system (7) as

$$E_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } E_1 = \begin{bmatrix} -5 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{12}$$

To test the stability type of the equilibrium points E_0 and E_1, we calculate the Jacobian matrix of the novel hyperjerk system (7) at any point $\boldsymbol{x} \in \mathbb{R}^4$ as

$$J(\mathbb{R}x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1-0.4x_1-6x_1^3x_4 & -1 & -3.7 & -1.5x_1^4 \end{bmatrix} \tag{13}$$

We note that

$$J_0 \triangleq J(E_0) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -1 & -3.7 & 0 \end{bmatrix} \tag{14}$$

which has the eigenvalues

$$\lambda_{1,2} = 0.1550 \pm 1.8674i, \quad \lambda_{3,4} = -0.1550 \pm 0.5107i \tag{15}$$

This shows that the equilibrium point E_0 is a saddle-focus point. Next, we note that

$$J_1 \triangleq J(E_1) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & -1 & -3.7 & -937.5 \end{bmatrix} \tag{16}$$

which has the eigenvalues

$$\lambda_1 = -937.5, \ \lambda_2 = 0.1, \ \lambda_{3,4} = -0.0005 \pm 0.0009i \tag{17}$$

This shows that the equilibrium point E_1 is also a saddle-focus point. Hence, the novel hyperjerk system (7) has two equilibrium points E_0 and E_1 defined by (12), which are both saddle-foci. Hence, E_0 and E_1 are both unstable equilibrium points.

3.2. Lyapunov exponents and Kaplan-Yorke dimension

For the parameter values $a = 3.7, b = 0.2$ and $c = 1.5$, the Lyapunov exponents of the novel hyperjerk system (7) are numerically obtained using MATLAB as

$$L_1 = 0.1448, \ L_2 = 0.0328, \ L_3 = 0 \ \text{and} \ L_4 = -1.1294 \tag{18}$$

Since the LE spectrum in (18) has two positive Lyapunov exponents, the novel hyperjerk system (7) is hyperchaotic. Since $L_1 + L_2 + L_3 + L_4 = -0.9518 < 0$, it follows that the novel hyperjerk system (7) is dissipative. Also, the Kaplan-Yorke dimension of the novel hyperchaotic hyperjerk system (7)is obtained as

$$D_{KY} = 3 + \frac{L_1 + L_2 + L_3}{|L_4|} = 3.1573, \tag{19}$$

which is fractional.

4. Adaptive control of the 4-D novel hyperjerk system with unknown parameters

In this section, we use backstepping control method to derive an adaptive feedback control law for globally stabilizing the 4-D novel hyperjerk system with unknown parameters. Thus, we consider the 4-D novel jerk chaotic system given by

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= -x_1 - x_2 - bx_1^2 - ax_3 - cx_1^4 x_4 + u \end{cases} \tag{20}$$

where a, b and c are unknown constant parameters, and u is a backstepping control law to be determined using estimates $\hat{a}(t), \hat{b}(t)$ and $\hat{c}(t)$ for a, b and c, respectively. The parameter estimation errors are defined as:

$$\begin{cases} e_a(t) &= a - \hat{a}(t) \\ e_b(t) &= b - \hat{b}(t) \\ e_c(t) &= c - \hat{c}(t) \end{cases} \tag{21}$$

Differentiating (21) with respect to t, we obtain the following equations:

$$\begin{cases} \dot{e}_a(t) &= -\dot{\hat{a}}(t) \\ \dot{e}_b(t) &= -\dot{\hat{b}}(t) \\ \dot{e}_c(t) &= -\dot{\hat{c}}(t) \end{cases} \tag{22}$$

Next, we shall state and prove the main result of this section.

Theorem 7 *The 4-D novel hyperjerk system* (20)*, with unknown parameters a, b and c, is globally and exponentially stabilized by the adaptive feedback control law,*

$$u(t) = -4x_1 - 9x_2 - [9 - \hat{a}(t)]x_3 - 4x_4 + \hat{b}(t)x_1^2 + \hat{c}(t)x_1^4 x_4 - kz_4, \tag{23}$$

where $k > 0$ is a gain constant,

$$z_4 = 3x_1 + 5x_2 + 3x_3 + x_4 \tag{24}$$

and the update law for the parameter estimates $\hat{a}(t), \hat{b}(t), \hat{c}(t)$ is given by

$$\begin{cases} \dot{\hat{a}}(t) &= -x_3 z_4 \\ \dot{\hat{b}}(t) &= -x_1^2 z_4 \\ \dot{\hat{c}}(t) &= -x_1^4 x_4 z_4 \end{cases} \tag{25}$$

Proof We prove this result via backstepping control method and Lyapunov stability theory. First, we define a quadratic Lyapunov function

$$V_1(z_1) = \frac{1}{2} z_1^2 \tag{26}$$

where

$$z_1 = x_1 \tag{27}$$

Differentiating V_1 along the dynamics (20), we get

$$\dot{V}_1 = z_1 \dot{z}_1 = x_1 x_2 = -z_1^2 + z_1(x_1 + x_2) \tag{28}$$

Now, we define

$$z_2 = x_1 + x_2 \tag{29}$$

Using (29), we can simplify the equation (28) as

$$\dot{V}_1 = -z_1^2 + z_1 z_2 \tag{30}$$

Secondly, we define a quadratic Lyapunov function

$$V_2(z_1, z_2) = V_1(z_1) + \frac{1}{2} z_2^2 = \frac{1}{2}\left(z_1^2 + z_2^2\right) \tag{31}$$

Differentiating V_2 along the dynamics (20), we get

$$\dot{V}_2 = -z_1^2 - z_2^2 + z_2(2x_1 + 2x_2 + x_3) \tag{32}$$

Now, we define

$$z_3 = 2x_1 + 2x_2 + x_3 \tag{33}$$

Using (33), we can simplify the equation (32) as

$$\dot{V}_2 = -z_1^2 - z_2^2 + z_2 z_3 \tag{34}$$

Thirdly, we define a quadratic Lyapunov function

$$V_3(z_1, z_2, x_3) = V_2(z_1, z_2) + \frac{1}{2} z_3^2 = \frac{1}{2}\left(z_1^2 + z_2^2 + z_3^2\right) \tag{35}$$

Differentiating V_3 along the dynamics (20), we get

$$\dot{V}_3 = -z_1^2 - z_2^2 - z_3^2 + z_3(3x_1 + 5x_2 + 3x_3 + x_4) \tag{36}$$

Now, we define

$$z_4 = 3x_1 + 5x_2 + 3x_3 + x_4 \tag{37}$$

Using (37), we can simplify the equation (36) as

$$\dot{V}_2 = -z_1^2 - z_2^2 - z_3^2 + z_3 z_4 \tag{38}$$

Finally, we define a quadratic Lyapunov function

$$V(z_1, z_2, z_3, z_4, e_a, e_b, e_c) = V_3(z_1, z_2, z_3) + \frac{1}{2}z_4^2 + \frac{1}{2}e_a^2 + \frac{1}{2}e_b^2 + \frac{1}{2}e_c^2 \tag{39}$$

which is a positive definite function on $\mathbb{R}^7$. Differentiating V along the dynamics (20), we get

$$\dot{V} = -z_1^2 - z_2^2 - z_3^2 - z_4^2 + z_4(z_4 + z_3 + \dot{z}_4) - e_a\dot{\hat{a}} - e_b\dot{\hat{b}} - e_c\dot{\hat{c}} \tag{40}$$

Eq. (40) can be written compactly as

$$\dot{V} = -z_1^2 - z_2^2 - z_3^2 - z_4^2 + z_4 S - e_a\dot{\hat{a}} - e_b\dot{\hat{b}} - e_c\dot{\hat{c}} \tag{41}$$

where

$$S = z_4 + z_3 + \dot{z}_4 = z_4 + z_3 + 3\dot{x}_1 + 5\dot{x}_2 + 3\dot{x}_3 + \dot{x}_4 \tag{42}$$

A simple calculation gives

$$S = 4x_1 + 9x_2 + (9 - a)x_3 + 4x_4 - bx_1^2 - cx_1^4 x_4 + u \tag{43}$$

Substituting the adaptive control law (23) into (43), we obtain

$$S = -(a - \hat{a}(t))x_3 - (b - \hat{b}(t))x_1^2 - (c - \hat{c}(t))x_1^4 x_4 - kz_4 \tag{44}$$

Using the definitions (22), we can simplify (44) as

$$S = -e_a x_3 - e_b x_1^2 - e_c x_1^4 x_4 - kz_4 \tag{45}$$

Substituting the value of S from (45) into (41), we obtain

$$\dot{V} = -z_1^2 - z_2^2 - z_3^2 - (1+k)z_4^2 + e_a(-x_3 z_4 - \dot{\hat{a}}) + e_b(-x_1^2 z_4 - \dot{\hat{b}}) + e_c(-x_1^4 x_4 z_4 - \dot{\hat{c}}) \tag{46}$$

Substituting the update law (25) into (46), we get

$$\dot{V} = -z_1^2 - z_2^2 - z_3^2 - (1+k)z_4^2, \tag{47}$$

which is a negative semi-definite function on $\mathbb{R}^7$. From (47), it follows that the vector $\mathbb{R}z(t) = (z_1(t), z_2(t), z_3(t), z_4(t))$ and the parameter estimation error $(e_a(t), e_b(t), e_c(t))$ are globally bounded, i.e.

$$\begin{bmatrix} z_1(t) & z_2(t) & z_3(t) & z_4(t) & e_a(t) & e_b(t) & e_c(t) \end{bmatrix} \in \mathbf{L}_\infty \tag{48}$$

Also, it follows from (47) that

$$\dot{V} \leqslant -z_1^2 - z_2^2 - z_3^2 - z_4^2 = -\|\mathbf{z}\|^2 \tag{49}$$

That is,

$$\|\mathbf{z}\|^2 \leqslant -\dot{V} \tag{50}$$

Integrating the inequality (50) from 0 to t, we get

$$\int_0^t |\mathbb{R}z(\tau)|^2 d\tau \leqslant V(0) - V(t) \tag{51}$$

From (51), it follows that $\mathbb{R}z(t) \in \mathbf{L}_2$. From Eq. (20), it can be deduced that $\mathbb{R}\dot{z}(t) \in \mathbf{L}_\infty$. Thus, using Barbalat's lemma, we conclude that $\mathbb{R}z(t) \to \mathbb{R}0$ exponentially as $t \to \infty$ for all initial conditions $\mathbb{R}z(0) \in \mathbb{R}^4$. Hence, it is immediate that $\mathbb{R}x(t) \to \mathbb{R}0$ exponentially as $t \to \infty$ for all initial conditions $\mathbb{R}x(0) \in \mathbb{R}^4$. This completes the proof. □

For the numerical simulations, the classical fourth-order Runge-Kutta method with step size $h = 10^{-8}$ is used to solve the system of differential equations (20) and (25), when the adaptive control law (23) is applied. The parameter values of the novel hyperjerk system (20) are taken as in the hyperchaotic case, *viz.* $a = 3.7, b = 0.2, c = 1.5$, and the positive gain constant as $k = 8$. Furthermore, as initial conditions of the novel hyperjerk system (20), we take $x_1(0) = 5.1$, $x_2(0) = -8.5$, $x_3(0) = 4.9$ and $x_4(0) = -7.6$. Also, as initial conditions of the parameter estimates, we take $\hat{a}(0) = 8.3$, $\hat{b}(0) = 5.4$ and $\hat{c}(0) = 10.2$. In Fig. 5, the exponential convergence of the controlled states $x_1(t), x_2(t), x_3(t)$ is depicted, when the adaptive control law (23) and (25) are implemented.

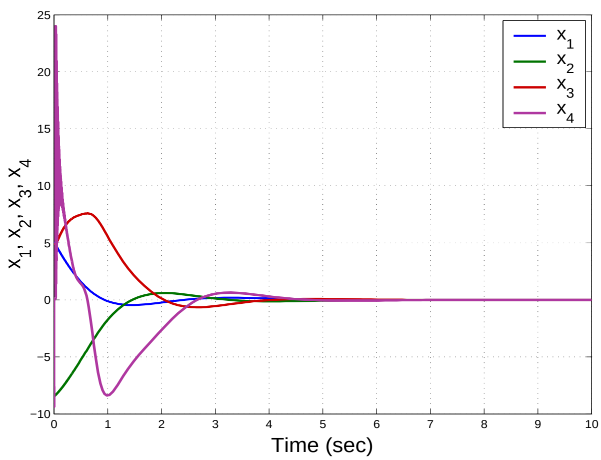

Figure 5: Time-history of the controlled states $x_1(t), x_2(t), x_3(t), x_4(t)$

5. Adaptive synchronization of the identical 4-D novel hyperjerk systems with unknown parameters

In this section, we use backstepping control method to derive an adaptive control law for globally and exponentially synchronizing the identical 4-D novel hyperjerk systems with unknown parameters.

As the master system, we consider the 4-D novel hyperjerk system given by

$$\left\{\begin{array}{lcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & x_3 \\ \dot{x}_3 & = & x_4 \\ \dot{x}_4 & = & -x_1 - x_2 - bx_1^2 - ax_3 - cx_1^4 x_4 \end{array}\right. \tag{52}$$

where x_1, x_2, x_3, x_4 are the states of the system, and a, b, c are unknown constant parameters. As the slave system, we consider the 4-D novel hyperjerk system given by

$$\left\{\begin{array}{lcl} \dot{y}_1 & = & y_2 \\ \dot{y}_2 & = & y_3 \\ \dot{y}_3 & = & y_4 \\ \dot{y}_4 & = & -y_1 - y_2 - by_1^2 - ay_3 - cy_1^4 y_4 + u \end{array}\right. \tag{53}$$

where y_1, y_2, y_3, y_4 are the states of the system, and u is a backstepping control to be determined using estimates $\hat{a}(t), \hat{b}(t)$ and $\hat{c}(t)$ for a, b and c, respectively.

We define the synchronization errors between the states of the master system (52) and the slave system (53) as

$$\left\{\begin{array}{lcl} e_1 & = & y_1 - x_1 \\ e_2 & = & y_2 - x_2 \\ e_3 & = & y_3 - x_3 \\ e_4 & = & y_4 - x_4 \end{array}\right. \tag{54}$$

Then the error dynamics is easily obtained as

$$\left\{\begin{array}{lcl} \dot{e}_1 & = & e_2 \\ \dot{e}_2 & = & e_3 \\ \dot{e}_3 & = & e_4 \\ \dot{e}_4 & = & -e_1 - e_2 - ae_3 - b(y_1^2 - x_1^2) - c(y_1^4 y_4 - x_1^4 x_4) + u \end{array}\right. \tag{55}$$

The parameter estimation errors are defined as:

$$\left\{\begin{array}{lcl} e_a(t) & = & a - \hat{a}(t) \\ e_b(t) & = & b - \hat{b}(t) \\ e_c(t) & = & c - \hat{c}(t) \end{array}\right. \tag{56}$$

Differentiating (56) with respect to t, we obtain the following equations:

$$\begin{cases} \dot{e}_a(t) &= -\dot{\hat{a}}(t) \\ \dot{e}_b(t) &= -\dot{\hat{b}}(t) \\ \dot{e}_c(t) &= -\dot{\hat{c}}(t) \end{cases} \tag{57}$$

Next, we shall state and prove the main result of this section.

Theorem 8 *The identical 4-D novel hyperjerk systems* (52) *and* (53) *with unknown parameters a, b and c are globally and exponentially synchronized by the adaptive control law*

$$u(t) = -4e_1 - 9e_2 - [9 - \hat{a}(t)]e_3 - 4e_4 + \hat{b}(t)\left(y_1^2 - x_1^2\right) + \hat{c}(t)\left(y_1^4 y_4 - x_1^4 x_4\right) - kz_4 \tag{58}$$

where $k > 0$ is a gain constant,

$$z_4 = 3e_1 + 5e_2 + 3e_3 + e_4, \tag{59}$$

and the update law for the parameter estimates $\hat{a}(t), \hat{b}(t), \hat{c}(t)$ is given by

$$\begin{cases} \dot{\hat{a}}(t) &= -e_3 z_4 \\ \dot{\hat{b}}(t) &= -\left(y_1^2 - x_1^2\right) z_4 \\ \dot{\hat{c}}(t) &= -\left(y_1^4 y_4 - x_1^4 x_4\right) z_4 \end{cases} \tag{60}$$

Proof We prove this result via backstepping control method and Lyapunov stability theory. First, we define a quadratic Lyapunov function

$$V_1(z_1) = \frac{1}{2} z_1^2 \tag{61}$$

where

$$z_1 = e_1 \tag{62}$$

Differentiating V_1 along the error dynamics (55), we get

$$\dot{V}_1 = z_1 \dot{z}_1 = e_1 e_2 = -z_1^2 + z_1(e_1 + e_2) \tag{63}$$

Now, we define

$$z_2 = e_1 + e_2 \tag{64}$$

Using (64), we can simplify the equation (63) as

$$\dot{V}_1 = -z_1^2 + z_1 z_2 \tag{65}$$

Secondly, we define a quadratic Lyapunov function

$$V_2(z_1, z_2) = V_1(z_1) + \frac{1}{2} z_2^2 = \frac{1}{2}\left(z_1^2 + z_2^2\right) \tag{66}$$

Differentiating V_2 along the error dynamics (55), we get

$$\dot{V}_2 = -z_1^2 - z_2^2 + z_2(2e_1 + 2e_2 + e_3) \tag{67}$$

Now, we define

$$z_3 = 2e_1 + 2e_2 + e_3 \tag{68}$$

Using (68), we can simplify the equation (67) as

$$\dot{V}_2 = -z_1^2 - z_2^2 + z_2 z_3 \tag{69}$$

Thirdly, we define a quadratic Lyapunov function

$$V_3(z_1, z_2, x_3) = V_2(z_1, z_2) + \frac{1}{2} z_3^2 = \frac{1}{2}\left(z_1^2 + z_2^2 + z_3^2\right) \tag{70}$$

Differentiating V_3 along the error dynamics (55), we get

$$\dot{V}_3 = -z_1^2 - z_2^2 - z_3^2 + z_3(3e_1 + 5e_2 + 3e_3 + e_4) \tag{71}$$

Now, we define

$$z_4 = 3e_1 + 5e_2 + 3e_3 + e_4 \tag{72}$$

Using (72), we can simplify the equation (71) as

$$\dot{V}_2 = -z_1^2 - z_2^2 - z_3^2 + z_3 z_4 \tag{73}$$

Finally, we define a quadratic Lyapunov function

$$V(z_1, z_2, z_3, z_4, e_a, e_b, e_c) = V_3(z_1, z_2, z_3) + \frac{1}{2} z_4^2 + \frac{1}{2} e_a^2 + \frac{1}{2} e_b^2 + \frac{1}{2} e_c^2 \tag{74}$$

which is a positive definite function on $\mathbb{R}^7$. Differentiating V along the error dynamics (55), we get

$$\dot{V} = -z_1^2 - z_2^2 - z_3^2 - z_4^2 + z_4(z_4 + z_3 + \dot{z}_4) - e_a \dot{\hat{a}} - e_b \dot{\hat{b}} - e_c \dot{\hat{c}} \tag{75}$$

Eq. (75) can be written compactly as

$$\dot{V} = -z_1^2 - z_2^2 - z_3^2 - z_4^2 + z_4 S - e_a \dot{\hat{a}} - e_b \dot{\hat{b}} - e_c \dot{\hat{c}} \tag{76}$$

where

$$S = z_4 + z_3 + \dot{z}_4 = z_4 + z_3 + 3\dot{e}_1 + 5\dot{e}_2 + 3\dot{e}_3 + \dot{e}_4 \tag{77}$$

A simple calculation gives

$$S = 4e_1 + 9e_2 + (9 - a)e_3 + 4e_4 - b\left(y_1^2 - x_1^2\right) - c\left(y_1^4 y_4 - x_1^4 x_4\right) + u \tag{78}$$

Substituting the adaptive control law (58) into (78), we obtain

$$S = -\left[a - \hat{a}(t)\right] e_3 - \left[b - \hat{b}(t)\right] \left(y_1^2 - x_1^2\right) - \left[c - \hat{c}(t)\right] \left(y_1^4 y_4 - x_1^4 x_4\right] - k z_4 \tag{79}$$

Using the definitions (57), we can simplify (79) as

$$S = -e_a e_3 - e_b \left(y_1^2 - x_1^2\right) - e_c \left(y_1^4 y_4 - x_1^4 x_4\right) - k z_4 \tag{80}$$

Substituting the value of S from (80) into (76), we obtain

$$\left\{ \begin{array}{rcl} \dot{V} & = & -z_1^2 - z_2^2 - z_3^2 - (1+k) z_4^2 + e_a(-e_3 z_4 - \dot{\hat{a}}) \\ & & + e_b \left[-\left(y_1^2 - x_1^2\right) z_4 - \dot{\hat{b}} \right] + e_c \left[-\left(y_1^4 y_4 - x_1^4 x_4\right) z_4 - \dot{\hat{c}} \right] \end{array} \right. \tag{81}$$

Substituting the update law (60) into (81), we get

$$\dot{V} = -z_1^2 - z_2^2 - z_3^2 - (1+k) z_4^2, \tag{82}$$

which is a negative semi-definite function on $\mathbb{R}^7$. From (82), it follows that the vector $\mathbb{R}z(t) = (z_1(t), z_2(t), z_3(t), z_4(t))$ and the parameter estimation error $(e_a(t), e_b(t), e_c(t))$ are globally bounded, i.e.

$$\left[\begin{array}{ccccccc} z_1(t) & z_2(t) & z_3(t) & z_4(t) & e_a(t) & e_b(t) & e_c(t) \end{array} \right] \in \mathbf{L}_\infty \tag{83}$$

Also, it follows from (82) that

$$\dot{V} \leqslant -z_1^2 - z_2^2 - z_3^2 - z_4^2 = -\|\mathbf{z}\|^2 \tag{84}$$

That is,

$$\|\mathbf{z}\|^2 \leqslant -\dot{V} \tag{85}$$

Integrating the inequality (85) from 0 to t, we get

$$\int_0^t |\mathbb{R}z(\tau)|^2 \, d\tau \leqslant V(0) - V(t) \tag{86}$$

From (86), it follows that $\mathbb{R}z(t) \in \mathbf{L}_2$. From Eq. (55), it can be deduced that $\mathbb{R}\dot{z}(t) \in \mathbf{L}_\infty$. Thus, using Barbalat's lemma, we conclude that $\mathbb{R}z(t) \to \mathbb{R}0$ exponentially as $t \to \infty$ for all initial conditions $\mathbb{R}z(0) \in \mathbb{R}^4$. Hence, it is immediate that $\mathbb{R}e(t) \to \mathbb{R}0$ exponentially as $t \to \infty$ for all initial conditions $\mathbb{R}e(0) \in \mathbb{R}^4$. This completes the proof. □

For the numerical simulations, the classical fourth-order Runge-Kutta method with step size $h = 10^{-8}$ is used to solve the system of differential equations (52) and (53). The parameter values of the novel hyperjerk systems are taken as in the hyperchaotic case, *viz.* $a = 3.7, b = 0.2, c = 1.5$ and the positive gain constant as $k = 8$. Also, as initial

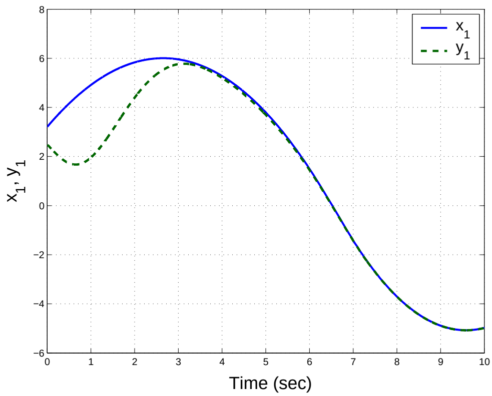

Figure 6: Synchronization of the states $x_1(t)$ and $y_1(t)$.

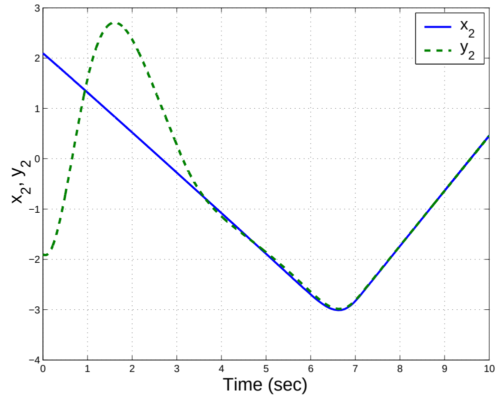

Figure 7: Synchronization of the states $x_2(t)$ and $y_2(t)$.

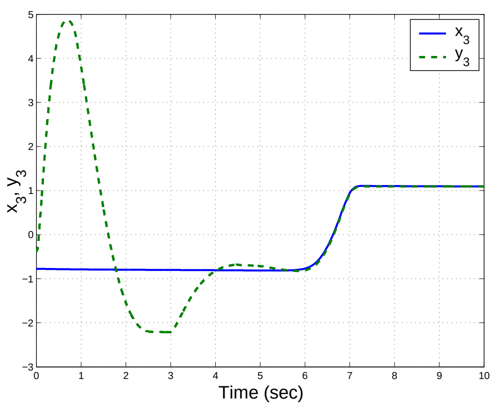

Figure 8: Synchronization of the states $x_3(t)$ and $y_3(t)$.

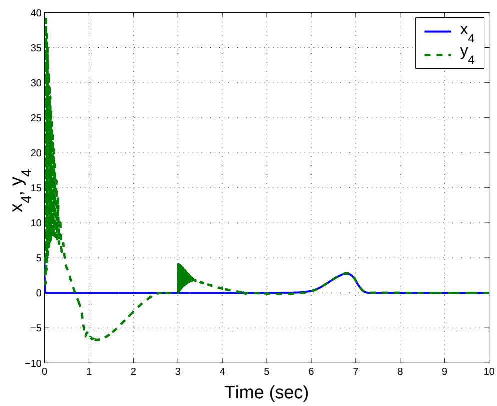

Figure 9: Synchronization of the states $x_4(t)$ and $y_4(t)$.

conditions of the master system (52), we take $x_1(0) = 3.2$, $x_2(0) = 2.1$, $x_3(0) = -0.8$ and $x_4(0) = 4.3$. As initial conditions of the slave system (53), we take $y_1(0) = 2.5$, $y_2(0) = -1.9$, $y_3(0) = -0.4$ and $y_4(0) = 1.7$. Furthermore, as initial conditions of the parameter estimates $\hat{a}(t), \hat{b}(t)$ and $\hat{c}(t)$, we take $\hat{a}(0) = 6.4$, $\hat{b}(0) = 3.9$ and $\hat{c}(0) = 5.1$.

In Figs. 6-9, the complete synchronization of the identical 4-D novel hyperchaotic hyperjerk systems (52) and (53) is shown, when the adaptive control law and the parameter update law are implemented. Also, in Fig. 10, the time-history of the complete synchronization errors $e_1(t), e_2(t), e_3(t), e_4(t)$, is shown.

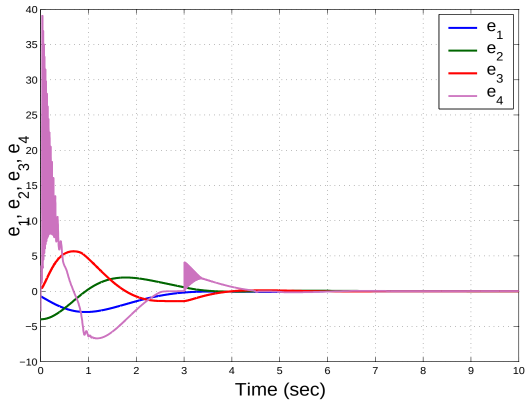

Figure 10: Time-history of the synchronization errors e_1, e_2, e_3, e_4.

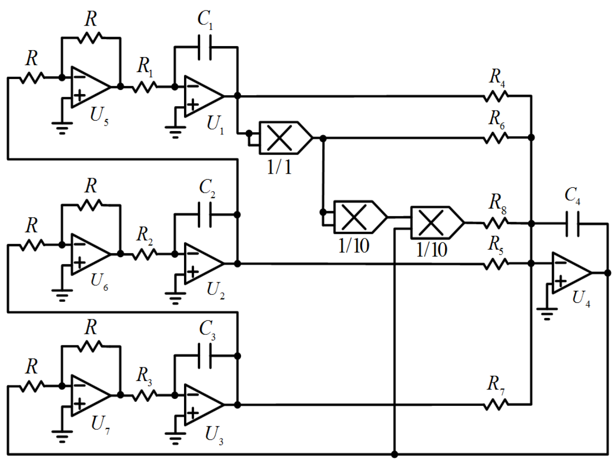

Figure 11: Circuit diagram for realizing the novel hyperjerk system (7).

6. SPICE implementation of the novel hyperjerk system

In this section, an electronic circuit modelling the novel hyperjerk system (7) is introduced. The circuit in Fig. 11 is designed by using operational amplifiers [12, 22] where the state variables x_1, x_2, x_3, and x_4 of system (7) are associated with the voltages across the capacitors C_1, C_2, C_3, and C_4, respectively. It is noting that it does not require the proportional compression transformation of state variables because chaotic attractors are in the dynamical range of operational amplifiers (see Figs. 1, 2). By applying Kirchhoff's laws to the designed electronic circuit, its nonlinear equations are derived in the following form

$$\begin{cases} \frac{dv_{C_1}}{dt} = \frac{1}{R_1C_1}v_{C_2} \\ \frac{dv_{C_2}}{dt} = \frac{1}{R_2C_2}v_{C_3} \\ \frac{dv_{C_3}}{dt} = \frac{1}{R_3C_3}v_{C_4} \\ \frac{dv_{C_4}}{dt} = -\frac{1}{R_4C_4}v_{C_1} - \frac{1}{R_5C_4}v_{C_2} - \frac{1}{R_6C_4}v_{C_1}^2 - \frac{1}{R_7C_4}v_{C_3} - \frac{1}{100R_8C_4}v_{C_1}^4 v_{C_4} \end{cases} \tag{87}$$

where v_{C_1}, v_{C_2}, v_{C_3}, and v_{C_4} are the voltages across the capacitors C_1, C_2, C_3, and C_4, respectively. Equations (87) match Eqs. (7) when the circuit components are selected as follows: $R_1 = R_2 = R_3 = R_4 = R_5 = R = 100k\Omega$, $R_6 = 500k\Omega$, $R_7 = 27.027k\Omega$, $R_8 = 0.667k\Omega$, and $C_1 = C_2 = C_3 = C_4 = 1nF$. The power supplies of all active devices are $\pm 15V_{DC}$ and the operational amplifiers TL084 are used.

The proposed circuit is implemented by using the electronic simulation package Cadence OrCAD. The obtained results are reported in Figs. 12 – 17 which display the (v_{C_1}, v_{C_2}), (v_{C_1}, v_{C_3}), (v_{C_2}, v_{C_3}), (v_{C_1}, v_{C_4}), (v_{C_2}, v_{C_4}), and (v_{C_3}, v_{C_4}) phase portraits respectively.

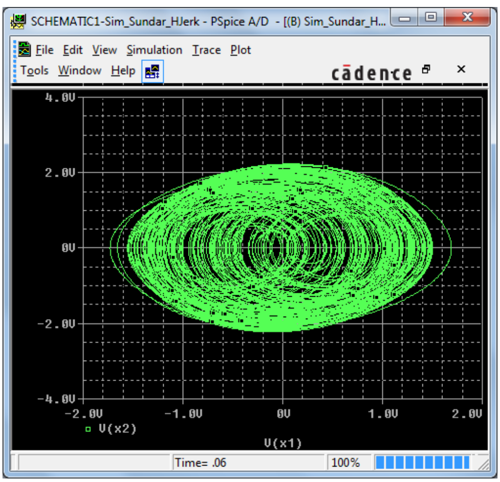

Figure 12: 2-D projection of the designed electronic circuit in (v_{C_1}, v_{C_2})–plane obtained from Cadence OrCAD.

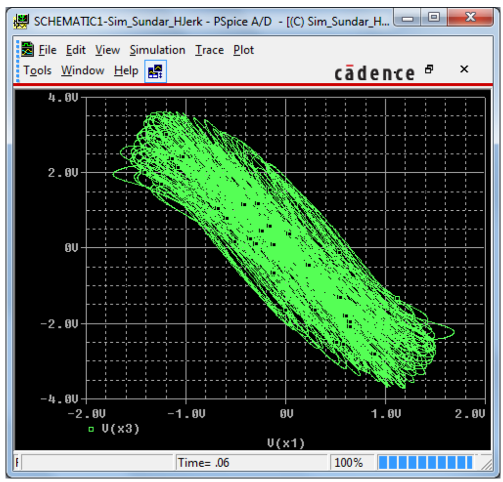

Figure 13: 2-D projection of the designed electronic circuit in (v_{C_1}, v_{C_3})–plane obtained from Cadence OrCAD.

7. Conclusion

In this paper, a new hyperjerk system, which is rarely reported in the literature, is proposed. It is worth noting that hyperchaos can be obversed in this hyperjerk system. Dynamics of the novel hyperjerk system are analysed through equilibrium points, 3-D

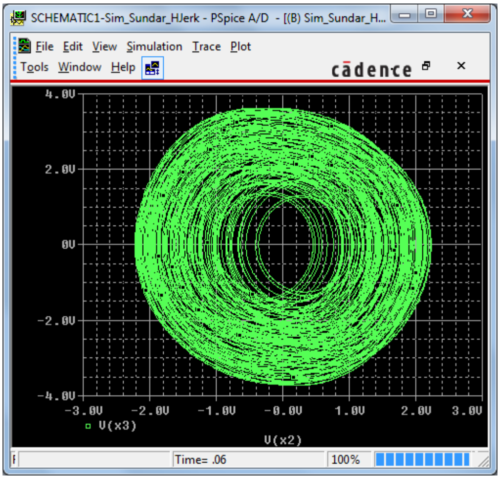

Figure 14: 2-D projection of the designed electronic circuit in (v_{C_2}, v_{C_3})–plane obtained from Cadence OrCAD.

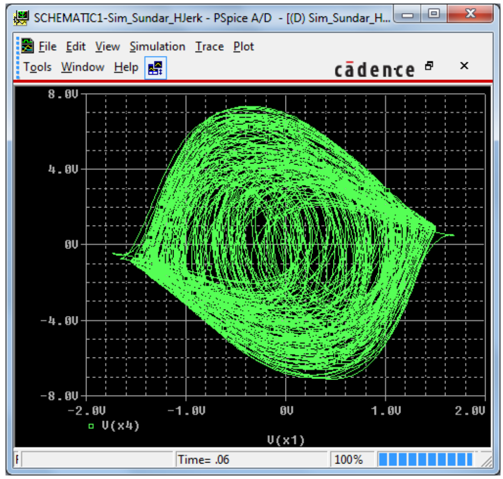

Figure 15: 2-D projection of the designed electronic circuit in (v_{C_1}, v_{C_4})–plane obtained from Cadence OrCAD.

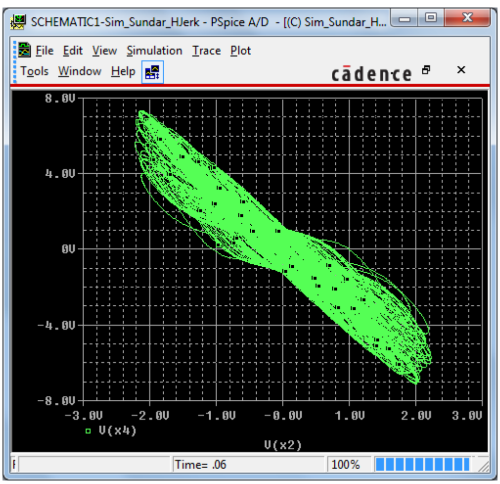

Figure 16: 2-D projection of the designed electronic circuit in (v_{C_2}, v_{C_4})–plane obtained from Cadence OrCAD.

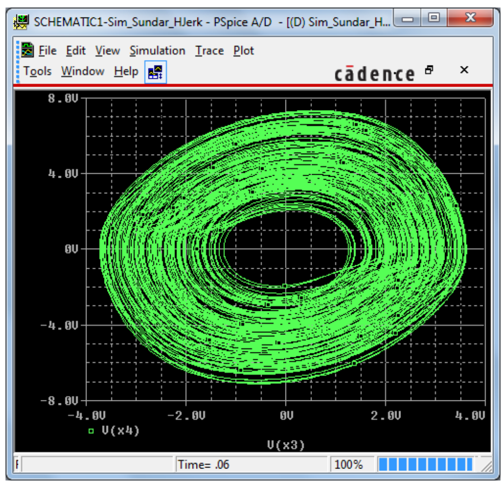

Figure 17: 2-D projection of the designed electronic circuit in (v_{C_1}, v_{C_4})–plane obtained from Cadence OrCAD.

projections, Lyapunov exponents and Kaplan-Yorke dimension. In addition, an adaptive backstepping controller is introduced to stabilize such hyperjerk system and achieve global hyperchaos synchronization. Moreover the feasibility of theoretical hyperjerk model is also confirmed by its electronic circuital implementation.

Hyperjerk systems are simple and elegant because they describe the time evolution of a single scalar variable. Additionally, mechanical systems can be presented conveniently in hyperjerk forms. Chaotic and hyperchaotic behaviors generating from such hyperjerk systems can be used in chaos-based applications. Hence, investigations of such proposed hyperjerk system and its applications should be further done in future works.

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