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A second-order sufficient condition for a weak local minimum in an optimal control problem with an inequality control constraint*

by

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Abstract: This paper is devoted to a sufficient second-order condition for a weak local minimum in a simple optimal control problem with one control constraint $G(u) \le 0$, given by a C^2-function. A similar second-order condition was obtained earlier by the author for a strong minimum in a much more general problem. In the present paper, we would like to take a narrower perspective than before and thus provide shorter and simpler proofs. In addition, the paper uses the first and second order tangents to the set U, defined by the inequality $G(u) \le 0$. The main difficulty of the proof, clearly shown in the paper, refers to the set, where the gradient H_u of the Hamiltonian is small, but the condition of quadratic growth of the Hamiltonian is satisfied. The paper can be valuable for self-explanation and provides a basis for extensions.

Keywords: critical cone, quadratic form, first and second order tangents, second order optimality condition, weak local minimum, inequality control constraint, Pontryagin's maximum principle

1. Introduction

In this paper, we discuss sufficient second-order conditions for a weak local minimum in the following optimal control problem on the interval $[0, 1]$:

$$\min\ J(x,u) := F(x(0), x(1)), \tag{1}$$

$$\dot{x}(t) = f(x(t), u(t)) \quad \text{for a.a.} \quad t \in [0,1], \tag{2}$$

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$$G(u(t)) \le 0 \quad \text{for a.a.} \quad t \in [0,1], \tag{3}$$

where $F : \mathbb{R}^{2n} \to \mathbb{R}$, $f : \mathbb{R}^{n+m} \to \mathbb{R}^n$, and $G : \mathbb{R}^m \to \mathbb{R}$ are of class C^2, $u \in L^\infty$, $x \in W^{1,1}$.

There is an extensive literature on sufficient second-order conditions in optimal control, see, for example, Bonnans and Hermant (2009), Bonnans and Osmolovskii (2010, 2012), Bonnans and Shapiro (2000), Levitin, Milyutin and Osmolovskii (1978), Malanowski (1994, 2001), Maurer (1981), Maurer and Pickenhain (1981), Milyutin and Osmolovskii (1998), Osmolovskii (2011, 2012), Osmolovskii and Maurer (2012), Zeidan (1984) and further literature cited in these papers. We do not mention here the works related to second-order conditions for singular arcs.

The most general results on sufficient conditions of the second order in optimal control were published by the author in Osmolovskii (2011). The conditions, contained in Osmolovskii (2011) took into account possible jumps of the optimal control at a finite number of points. Their proofs are long and difficult. To make the proofs more accessible, the author published in Osmolovskii (2012) a simplified version of these results, in which the assumptions did not allow jumps of the optimal control. Namely, in addition to the C^2-smoothness of the data, the following assumption was introduced: a.e. in $[t_0, t_1]$

$$H(\hat{x}(t), u, \hat{p}(t)) - H(\hat{x}(t), \hat{u}(t), \hat{p}(t)) \ge c|u - \hat{u}(t)|^2 \quad \forall\, u \in U \tag{4}$$

with some $c > 0$. Here, $H = pf$ is the Hamiltonian of the problem, $(\hat{x}, \hat{u})$ is the admissible state-control pair, examined for optimality, $\hat{p}$ is the adjoint variable, $U \subset \mathbb{R}^m$ is the control constraint. The set U in Osmolovskii (2012) was defined as:

$$U = \{u \in \mathbb{R}^m : \quad G_i(u) \le 0, \quad i = 1, \dots, k\},$$

where $G_i : \mathbb{R}^m \to \mathbb{R}$ are C^2-mappings, such that at every point $u \in U$ the gradients $g_i'(u)$, $i \in I_G(u)$ are linearly independent, where $I_G(u) = \{i : G_i(u) \le 0\}$ is the set of active indices.

Condition (4) obviously strengthens Pontryagin's minimum principle (we prefer to use the notion of minimum principle rather than that of the maximum principle), and we call it the *quadratic growth condition for the Hamiltonian.* It was shown in Bonans and Osmolovskii (2012) that together with the assumption of C^2-smoothness of the data this condition guarantees the continuity of the control $\hat{u}$. Note that in Osmolovskii (2011) we used a much finer growth condition for H, allowing jumps of $\hat{u}$. In this paper, the condition (4) is assumed to hold only on a set of small measure, so the control $\hat{u}$ can only be measurable. This set of a small measure has the form

$$m(\varepsilon) := \{t \in [0,1] : \; 0 < |H_u(\hat{x}(t), \hat{u}(t), \hat{p}(t))| < \varepsilon\}, \tag{5}$$

where $\varepsilon > 0$ is arbitrarily small. The main difficulties of the proof are connected with this set. The proof uses the ideas of the paper Osmolovskii (2012), but compared to the proof contained in that paper, it is much simpler and clearer, mainly due to the simplicity of the problem, but also due to the use of some new tricks.

The paper is organized as follows. In Section 2, the assumptions for the problem (1)-(3) are given, the first-order necessary optimality condition for a weak local minimum in this problem is recalled, the critical cone K and the quadratic form Ω are defined, a condition (using a second-order adjacent set to U), equivalent to the positive definiteness of Ω on K is proven, and finally, the main result of the paper is formulated: a second-order sufficient condition for the so-called quadratic growth of the cost, which implies a weak local minimum at the given point. The main result is stated in Theorem 2.1. Section 3 is entirely devoted to the proof of this theorem.

2. Main result

Let us formulate the assumptions for the problem (1)-(3).

Assumption 2.1 *We assume that $G'(u) \neq 0$ at all points $u \in \mathbb{R}^m$ such that* $G(u) = 0$ (regularity assumption).

In the sequel we use the notation

$$q = (x(0), x(1)) = (x_0, x_1), \quad w = (x, u), \quad \mathcal{W} = W^{1,1} \times L^\infty.$$

The norm of an element $w = (x, u) \in \mathcal{W}$ is defind as

$$\|w\| = \|x\|_{1,1} + \|u\|_\infty.$$

The local minimum in this norm is a *weak local minimum.*

We say that a pair $w = (x, u) \in \mathcal{W}$ is *admissible*, if equation (2) and inequality (3) hold. Let $\hat{w} = (\hat{x}, \hat{u})$ be an addmissible pair. Set $\hat{q} = (\hat{x}(0), \hat{x}(1))$.

Assumption 2.2 *The first order necessary optimality condition for a weak local minimum for the pair $\hat{w} = (\hat{x}, \hat{u})$ is fulfilled: there exist $\hat{p} \in W^{1,1}$ and $\hat{\lambda} \in L^\infty$, such that*

$$(-\hat{p}(0), \hat{p}(1)) = F'(\hat{q}), \tag{6}$$

$$-\dot{\hat{p}}(t) = \hat{p}(t)\, f_x(\hat{w}(t)) \quad \textit{for a.a.} \quad t \in [0, 1], \tag{7}$$

$$\hat{p}(t)\, f_u(\hat{w}(t)) + \hat{\lambda}(t) G'(\hat{u}(t)) = 0 \quad \textit{for a.a.} \quad t \in [0, 1], \tag{8}$$

$$\hat{\lambda}(t) \geq 0 \quad \textit{for a.a.} \quad t \in [0, 1], \tag{9}$$

$$\hat{\lambda}(t) G(\hat{u}(t)) = 0 \quad \textit{for a.a.} \quad t \in [0, 1]. \tag{10}$$

Note that for a given $\hat{w}$, the pair $(\hat{p}, \hat{\lambda})$ is uniquely determined by these conditions.

Now, we introduce the *Hamiltonian* and the *augmented Hamiltonian*

$$H(w,p) = p\,f(w), \quad \bar{H}(w,p,\lambda) = p\,f(w) + \lambda\,G(u).$$

Then, equations (7) and (8) take the forms

$$-\dot{\hat{p}}(t) = H_x(\hat{w}(t), \hat{p}(t)), \quad \bar{H}_u(\hat{w}(t), \hat{p}(t), \hat{\lambda}(t)) = 0.$$

Let us now formulate sufficient conditions of the second order for a weak local minimum. Define the *critical cone* K. Set

$$M_0 = \{t \in [0,1] :\ G(\hat{u}(t)) = 0\},$$

$$K = \Big\{ w \in \mathcal{W} :\ \dot{x}(t) = f'(\hat{w}(t))w(t),\ H_u(\hat{w}(t), \hat{p}(t))u(t) = 0 \\ \text{for a.a. } \ t \in [0,1]; G'(\hat{u}(t))u(t) \le 0 \quad \text{for a.a. } \ t \in M_0 \Big\}. \tag{11}$$

Note that the condition

$$G'(\hat{u}(t))u(t) \le 0 \ \text{ for a.a. } t \in M_0,$$

which appears in the definition of the critical cone, can be presented as

$$u(t) \in T_U^{\flat}(\hat{u}(t)) \quad \text{for a.a. } t \in [0,1],$$

where $T_U^{\flat}(\hat{u}(t))$ is the *first-order tangent* to the set

$$U = \{u \in \mathbb{R}^m :\ G(u) \le 0\}$$

at the point $\hat{u}(t)$, see, for instance, Aubin and Frankowska (1990).

It is easy to see that K can be defined in the following equivalent way

$$K = \Big\{ w \in \mathcal{W} :\ F'(\hat{q})q \le 0, \quad \dot{x}(t) = f'(\hat{w}(t))w(t) \ \text{ a.e. on } [0,1], \\ G'(\hat{u}(t))u(t) \le 0 \ \text{ a.e. on } M_0 \Big\}$$

(this corresponds to the classical definition of a critical cone) and, moreover, $F'(\hat{q})q = 0$ for any element $w \in K$. But we will not use here these facts.

Let us show that the condition $K = \{0\}$ is not sufficient for local minimality of $\hat{w}$. We will show this for a problem of a different type.

EXAMPLE 2.1 *Let $m = 1$. Consider the problem*

$$\text{minimize} \quad J(u) := \int_0^1 tu - u^2 \, \mathrm{d}t, \quad u \geq 0.$$

Set $\hat{u} = 0$. Then, $\hat{\lambda}(t) = t$, $K = \{0\}$. But $\hat{u}$ is not a weak local minimizer, because there is a sequence

$$u_n(t) = \begin{cases} \frac{1}{\sqrt{n}} & 0 \leq t \leq \frac{1}{n} \\ 0, & \frac{1}{n} < t \leq 1, \end{cases}$$

such that $J(u_n) < 0$ for all $n = 1, 2, \ldots$.

ASSUMPTION 2.3 *There exist $C > 0$ and $\varepsilon > 0$ such that for a.a. $t \in m(\varepsilon)$ (see (5)) we have*

$$H(\hat{x}(t), u, \hat{p}(t)) - H(\hat{x}(t), \hat{u}(t), \hat{p}(t)) \geq C|u - \hat{u}(t)|^2$$
$$\text{whenever } |u - \hat{u}(t)| < \varepsilon, \ G(u) \leq 0. \tag{12}$$

Note that this assumption does not hold in Example 2.1.

Let us introduce the *quadratic form*:

$$\Omega(w) := \langle F''(\hat{q})q, q\rangle + \int_0^1 \langle \bar{H}_{ww}(\hat{w}(t), \hat{p}(t), \hat{\lambda}(t))w(t), w(t)\rangle \, \mathrm{d}t, \tag{13}$$

where $q = (x(0), x(1))$.

ASSUMPTION 2.4 *There exists $c_0 > 0$ such that*

$$\Omega(w) \geq c_0\big(|x(0)|^2 + \|u\|_2^2\big) \quad \forall w \in K. \tag{14}$$

PROPOSITION 2.1 *Assumption 2.4 is equivalent to the following one: there exists $c_0 > 0$ such that*

$$\Omega(w) \geq c_0\big(\|x\|_\infty^2 + \|u\|_2^2\big) \quad \forall w \in K. \tag{15}$$

PROOF Indeed, if $w \in K$, then

$$x(t) = x(0) + \int_0^t \big(f_x(\hat{w}(\tau))x(\tau) + f_u(\hat{w}(\tau))u(\tau)\big) \, \mathrm{d}\tau,$$

whence

$$\|x\|_{1,1} \leq c\big(|x(0)| + \|u\|_1\big)$$

with some $c > 0$. The required equivalence follows. □

Here is another equivalent form of this assumption, which will be used in further course of considerations.

Proposition 2.2 *Assumption 2.4 is equivalent to the following one: there exists $c_0 > 0$ such that*

$$\omega(w) + \int_0^1 H_u(\hat{w}(t), \hat{p}(t))v(t)\,\mathrm{d}t \geq c_0\big(\|x\|_\infty^2 + \|u\|_2^2\big) \tag{16}$$

for all $w = (x,u) \in K$ and for all $v \in L^\infty$ such that $v(t) \in T_U^{\flat(2)}(\hat{u}(t), u(t))$ a.e. on M_0, where

$$\omega(w) = \frac{1}{2}\langle F''(\hat{p})q, q\rangle + \frac{1}{2}\int_0^1 \langle H_{ww}(\hat{w}(t), \hat{p}(t))w(t), w(t)\rangle\,\mathrm{d}t,$$

and

$$T_U^{\flat(2)}(\hat{u}, u) = \big\{v \in \mathbb{R}^m : G'(\hat{u})v + \frac{1}{2}\langle G''(\hat{u})u, u\rangle \leq 0\big\}$$

is the second-order tangent to the set U for the pair $(\hat{u}, u) \in \mathbb{R}^{2m}$, see, for instance, Aubin and Frankowska (1990) and Cominetti (1990).

Proof Indeed, if $w = (x,u) \in K$, $v \in L^\infty$, $v(t) \in T_U^{\flat(2)}(\hat{u}(t), u(t))$ a.e. on M_0, then

$$H_u(\hat{w}(t), \hat{p}(t))v(t) = -\hat{\lambda}(t)G'(\hat{u}(t))v(t) \geq \frac{1}{2}\hat{\lambda}(t)\langle G''(\hat{u}(t))u(t), u(t)\rangle \quad \text{a.e. on} \quad [0,1],$$

and therefore

$$\omega(w) + \int_0^1 H_u(\hat{w}(t), \hat{p}(t))v(t)\,\mathrm{d}t \geq \Omega(w).$$

Hence, condition (15) implies condition (16).

Moreover, due to Assumption 2.1, for any $w = (x,u) \in K$ there exists $v \in L^\infty$ such that

$$G'(\hat{u}(t))v(t) + \frac{1}{2}\langle G''(\hat{u}(t))u(t), u(t)\rangle = 0 \quad \text{a.e. on} \quad M_0.$$

Hence $v(t) \in T_U^{\flat(2)}(\hat{u}(t), u(t))$ a.e. on M_0 and

$$H_u(\hat{w}(t), \hat{p}(t))v(t) = -\hat{\lambda}(t)G'(\hat{u}(t))v(t) = \frac{1}{2}\hat{\lambda}(t)\langle G''(\hat{u}(t))u(t), u(t)\rangle \quad \text{a.e. on} \quad [0,1].$$

Consequently,

$$\omega(w) + \int_0^1 H_u(\hat{w}(t), \hat{p}(t))v(t)\,\mathrm{d}t = \Omega(w).$$

Therefore, conditions (16) and (15) are equivalent. □

The following theorem holds.

Theorem 2.1 (sufficient second order condition) *Let Assumptions 2.1-2.4 be fulfilled. Then there exist* $\delta > 0$ *and* $c > 0$ *such that*

$$J(w) - J(\hat{w}) \geq c\big(\|x - \hat{x}\|_\infty^2 + \|u - \hat{u}\|_2^2\big) \tag{17}$$

for all admissible $w = (x, u) \in \mathcal{W}$ *such that* $\|w - \hat{w}\|_\infty < \delta$.

We conclude this section with a brief note on the numerical verification of the estimates (14) or (15) for the quadratic form Ω on the critical cone K. The "standard" method is to show that the associated matrix Riccati equation has a bounded solution; see, e.g., Malanowski (2001), Malanowski and Maurer (1996), Maurer and Pickenhain 1981), and the author's book with H. Maurer, i.e. Osmolovskii and Maurer (2012).

3. Proof of the main result

Here we give the proof of Theorem 2.1. In what follows, we omit the dependence on t for x, u, $\hat{x}$, $\hat{u}$, etc.

Step 1°

For $w = (x, u) \in \mathcal{W}$ we set

$$\Delta w = w - \hat{w}, \quad \gamma(\Delta w) = \|\Delta x\|_\infty^2 + \|\Delta u\|_2^2.$$

Assume that condition (17) does not hold. Then, there is a sequence of admissible points $w_n \neq \hat{w}$ such that $\|w_n - \hat{w}\|_\infty \to 0$ and

$$\Delta_n J := J(w_n) - J(\hat{w}) \leq o(\gamma_n), \tag{18}$$

where

$$\gamma_n = \gamma(\Delta w_n) > 0, \quad \Delta w_n = (\Delta x_n, \Delta u_n) = w_n - \hat{w}.$$

Set $\Delta_n f = f(w_n) - f(\hat{w})$. Since $\Delta \dot{x}_n = \Delta_n f$, we get

$$\Delta_n J = \Delta_n J + \int_0^1 \hat{p}(\Delta_n f - \Delta \dot{x}_n)\, \mathrm{d}t.$$

Further,

$$\int_0^1 \hat{p}\Delta\dot{x}_n\, \mathrm{d}t = \hat{p}\,\Delta x_n\,|_0^1 - \int_0^1 \dot{\hat{p}}\Delta x_n\, \mathrm{d}t = F'(\hat{p})\Delta q_n + \int_0^1 \hat{p} f_x(\hat{w})\Delta x_n\, \mathrm{d}t.$$

Therefore,

$$\begin{aligned}\Delta_n J &= \Delta_n F - F'(\hat{p})\Delta q_n + \int_0^1 \big(\hat{p}\Delta_n f - \hat{p} f_x(\hat{w})\Delta x_n\big)\, \mathrm{d}t \\ &= \Delta_n F - F'(\hat{p})\Delta q_n + \int_0^1 \big(\Delta_n H - H_x(\hat{w}, \hat{p})\Delta x_n\big)\, \mathrm{d}t,\end{aligned} \tag{19}$$

where $\Delta_n H = H(w_n, \hat{p}) - H(\hat{w}, \hat{p})$.

Step 2°

We have

$$\Delta_n H := H(\hat{x}+\Delta x_n, \hat{u}+\Delta u_n, \hat{p}) - H(\hat{x}, \hat{u}, \hat{p})$$

$$= H(\hat{x}+\Delta x_n, \hat{u}+\Delta u_n, \hat{p}) - H(\hat{x}, \hat{u}+\Delta u_n, \hat{p}) + H(\hat{x}, \hat{u}+\Delta u_n, \hat{p}) - H(\hat{x}, \hat{u}, \hat{p})$$

$$= H_x(\hat{x}, \hat{u}+\Delta u_n, \hat{p})\Delta x_n + \Delta_{un} H + r_n,$$

where

$$\Delta_{un} H := H(\hat{x}, \hat{u}+\Delta u_n, \hat{p}) - H(\hat{x}, \hat{u}, \hat{p}), \quad \|r_n\|_\infty = O(\gamma_n).$$

Let $\varepsilon_n \to 0+$. Set

$$m(\varepsilon_n) = \{t \in [0,1] :\ 0 < |H_u(\hat{x}, \hat{u}, \hat{p})| < \varepsilon_n\}.$$

Clearly, $m(\varepsilon_n) \subset M_0$ and $\operatorname{meas} m(\varepsilon_n) \to 0$ as $n \to \infty$. Since $G(u_n) \le 0$ for all n, then, due to Assumption 2.3, we have $\Delta_{un} H \ge C|\Delta u_n|^2$ for all sufficiently large n. Therefore,

$$\int_{m(\varepsilon_n)} \Delta_{un} H \,\mathrm{d}t \ge C \int_{m(\varepsilon_n)} |\Delta u_n|^2 \,\mathrm{d}t.$$

Consequently,

$$\int_{m(\varepsilon_n)} \big(\Delta_n H - H_x(\hat{x}, \hat{u}, \hat{p})\Delta x_n\big)\,\mathrm{d}t$$

$$\ge \int_{m(\varepsilon_n)} \big(H_x(\hat{x}, \hat{u}+\Delta u_n, \hat{p}) - H_x(\hat{x}, \hat{u}, \hat{p})\big)\Delta x_n \,\mathrm{d}t + C\int_{m(\varepsilon_n)} |\Delta u_n|^2 \,\mathrm{d}t + o(\gamma_n).$$

Since

$$\int_{m(\varepsilon_n)} |\Delta u_n| \cdot |\Delta x_n| \,\mathrm{d}t \le \|\Delta x_n\|_\infty \sqrt{\operatorname{meas} m(\varepsilon_n)}\|\Delta u_n\|_2 = o(\gamma_n),$$

we get

$$\int_{m(\varepsilon_n)} \big(H_x(\hat{x}, \hat{u}+\Delta u_n, \hat{p}) - H_x(\hat{x}, \hat{u}, \hat{p})\big)\Delta x_n \,\mathrm{d}t = o(\gamma_n).$$

Therefore,

$$\int_{m(\varepsilon_n)} \big(\Delta_n H - H_x(\hat{x}, \hat{u}, \hat{p})\Delta x_n\big)\,\mathrm{d}t \ge C\int_{m(\varepsilon_n)} |\Delta u_n|^2 \,\mathrm{d}t + o(\gamma_n). \tag{20}$$

Step 3°

Conditions (18)-(20) imply

$$o(\gamma_n) \geq \Delta_n F - F'(\hat{p})\Delta q_n + \int_0^1 \big(\Delta_n H - H_x(\hat{w},\hat{p})\Delta x_n\big)\,\mathrm{d}t$$

$$\geq \frac{1}{2}\langle F''(\hat{p})\Delta q_n, \Delta q_n\rangle + o(|\Delta q_n|^2) + \int_{[0,1]\setminus m(\varepsilon_n)} \big(\Delta_n H - H_x(\hat{w},\hat{p})\Delta x_n\big)\,\mathrm{d}t$$

$$+C\int_{m(\varepsilon_n)} |\Delta u_n|^2\,\mathrm{d}t + o'(\gamma_n). \tag{21}$$

We set

$$u_n' = \Delta u_n \chi_{m(\varepsilon_n)}, \quad \Delta u_n^0 = \Delta u_n - u_n', \quad \Delta w_n^0 = (\Delta x_n, \Delta u_n^0),$$

$$\gamma_n^0 = \gamma(\Delta w_n^0), \quad \gamma_n' = \int_0^1 |u_n'|\,\mathrm{d}t = \int_{m(\varepsilon_n)} |\Delta u_n|^2\,\mathrm{d}t.$$

Then

$$\gamma_n = \gamma_n^0 + \gamma_n'.$$

Further, set

$$\Delta_n^0 H := H(\hat{w} + \Delta w_n^0, \hat{p}) - H(\hat{w},\hat{p}).$$

Then

$$\int_{[0,1]\setminus m(\varepsilon_n)} \big(\Delta_n H - H_x(\hat{w},\hat{p})\Delta x_n\big)\,\mathrm{d}t = \int_{[0,1]\setminus m(\varepsilon_n)} \big(\Delta_n^0 H - H_x(\hat{w},\hat{p})\Delta x_n\big)\,\mathrm{d}t$$

$$= \int_0^1 \big(\Delta_n^0 H - H_x(\hat{w},\hat{p})\Delta x_n\big)\,\mathrm{d}t - \int_{m(\varepsilon_n)} \big(\Delta_n^0 H - H_x(\hat{w},\hat{p})\Delta x_n\big)\,\mathrm{d}t.$$

Obviously, we have

$$\int_{m(\varepsilon_n)} \big(\Delta_n^0 H - H_x(\hat{w},\hat{p})\Delta x_n\big)\,\mathrm{d}t$$

$$= \int_{m(\varepsilon_n)} \big(H(\hat{x} + \Delta x_n, \hat{u}, \hat{p}) - H(\hat{x},\hat{u},\hat{p}) - H_x(\hat{w},\hat{p})\Delta x_n\big)\,\mathrm{d}t = o(\gamma_n).$$

Thus, we get

$$\int_{[0,1]\setminus m(\varepsilon_n)} \big(\Delta_n H - H_x(\hat{w},\hat{p})\Delta x_n\big)\,\mathrm{d}t = \int_0^1 \big(\Delta_n^0 H - H_x(\hat{w},\hat{p})\Delta x_n\big)\,\mathrm{d}t + o(\gamma_n). \tag{22}$$

Now, note that $H_w(\hat{w},\hat{p})\Delta w_n^0 = H_x(\hat{w},\hat{p})\Delta x_n + H_u(\hat{w},\hat{p})\Delta u_n^0$. Therefore, relations (21) and (22) imply

$$o(\gamma_n) \geq \frac{1}{2}\langle F''(\hat{p})\Delta q_n, \Delta q_n\rangle + \int_0^1 \big(\Delta_n^0 H - H_x(\hat{w},\hat{p})\Delta x_n\big)\,\mathrm{d}t + C\gamma_n'$$

$$= \frac{1}{2}\langle F''(\hat{p})\Delta q_n, \Delta q_n\rangle + \int_0^1 \big(\Delta_n^0 H - H_w(\hat{w},\hat{p})\Delta w_n^0\big)\,\mathrm{d}t + \int_0^1 H_u(\hat{w},\hat{p})\Delta u_n^0\,\mathrm{d}t + C\gamma_n'.$$

Since

$$\Delta_n^0 H - H_w(\hat{w},\hat{p})\Delta w_n^0 = \frac{1}{2}\langle H_{ww}(\hat{w},\hat{p})\Delta w_n^0, \Delta w_n^0\rangle + o(|\Delta w_n^0|^2)$$

(here and below, all estimates are satisfied uniformly on $[0,1]$), we obtain from here that

$$o(\gamma_n) \geq$$

$$\frac{1}{2}\langle F''(\hat{p})\Delta q_n, \Delta q_n\rangle + \int_0^1 \langle H_{ww}(\hat{w},\hat{p})\Delta w_n^0, \Delta w_n^0\rangle\,\mathrm{d}t + \int_0^1 H_u(\hat{w},\hat{p})\Delta u_n^0\,\mathrm{d}t + C\gamma_n',$$

or, equivalently,

$$\omega(\Delta w_n^0) + \int_0^1 H_u(\hat{w},\hat{p})\Delta u_n^0\,\mathrm{d}t + C\gamma_n' \leq o(\gamma_n). \tag{23}$$

We will analyze this condition.

Step 4°

Since $\omega(\Delta w_n^0) \leq O(\gamma_n^0) \leq O(\gamma_n)$, relation (23) implies

$$\int_0^1 H_u(\hat{w},\hat{p})\Delta u_n^0\,\mathrm{d}t \leq O(\gamma_n). \tag{24}$$

Further, condition $G(\hat{u} + \Delta u_n^0) \leq 0$ yields $\Delta_{un}^0 H \geq C|\Delta u_n|^2$, and then

$$H_u(\hat{w},\hat{p})\Delta u_n^0 \geq O(|\Delta u_n^0|^2) \quad \text{a.e. on} \quad M_0.$$

It follows that

$$(H_u(\hat{w},\hat{p})\Delta u_n^0)^- \leq O(|\Delta u_n^0|^2) \quad \text{a.e. on} \quad M_0, \tag{25}$$

where $a^+ = \max\{a,0\}$, $a^- = \max\{-a,0\}$, $a = a^+ - a^-$ for $a \in \mathbb{R}$.

We analyse conditions (24) and (25). Let us represent condition (24) in the form

$$\int_0^1 (H_u(\hat{w},\hat{p})\Delta u_n^0)^+\,\mathrm{d}t - \int_0^1 (H_u(\hat{w},\hat{p})\Delta u_n^0)^-\,\mathrm{d}t \leq O(\gamma_n).$$

Since, in view of (25),

$$\int_0^1 (H_u(\hat{w},\hat{p})\Delta u_n^0)^- \,\mathrm{d}t \le O(\gamma_n),$$

we obtain

$$\int_0^1 (H_u(\hat{w},\hat{p})\Delta u_n^0)^+ \,\mathrm{d}t \le O(\gamma_n).$$

Consequently,

$$\int_0^1 |H_u(\hat{w},\hat{p})\Delta u_n^0| \,\mathrm{d}t \le O(\gamma_n). \tag{26}$$

Step 5°

Condition $G(\hat{u}+\Delta u_n^0) \le 0$ implies

$$G'(\hat{u})\Delta u_n^0 + \frac{1}{2}\langle G''(\hat{u})\Delta u_n^0, \Delta u_n^0\rangle \le o(|\Delta u_n^0|^2) \quad \text{a.e. on} \quad M_0. \tag{27}$$

By multiplying this inequality by $\hat{\lambda} \ge 0$ and by taking into account that $\hat{\lambda}G'(\hat{u}) = -H_u(\hat{w},\hat{p})$, we get

$$-H_u(\hat{w},\hat{p})\Delta u_n^0 + \frac{1}{2}\hat{\lambda}\langle G''(\hat{u})\Delta u_n^0, \Delta u_n^0\rangle \le o(|\Delta u_n^0|^2) \quad \text{a.e. on} \quad M_0, \tag{28}$$

whence

$$-\int_0^1 H_u(\hat{w},\hat{p})\Delta u_n^0 \,\mathrm{d}t + \int_0^1 \frac{1}{2}\hat{\lambda}\langle G''(\hat{u})\Delta u_n^0, \Delta u_n^0\rangle \,\mathrm{d}t \le o(\gamma_n).$$

Upon adding this inequality to (23) and using that $\bar{H}(w,p,\lambda) = p\,f(w) + \lambda\,G(u)$, we obtain

$$\Omega(\Delta w_n^0) + C\gamma_n' \le o(\gamma_n). \tag{29}$$

We consider two possible cases:

$$(i) \quad \liminf \frac{\gamma_n^0}{\gamma_n} = 0, \qquad (ii) \quad \liminf \frac{\gamma_n^0}{\gamma_n} > 0,$$

where $\gamma_n > 0$ for all n.

Case (i).

Step 6°

In this case, there is a subsequence such that $\gamma_n^0/\gamma_n \to 0$ on this subsequence. Assume that this condition holds for the sequence itself. Then, $\gamma_n^0 = o(\gamma_n)$. Since, obviously, $|\Omega(\Delta w_n^0)| \le O(\gamma_n^0)$, condition (29) yields

$$C\gamma_n' \le o(\gamma_n) + O(\gamma_n^0) = o_1(\gamma_n),$$

i.e., $\gamma_n' = o(\gamma_n)$. The latter contradicts the conditions $\gamma_n^0 = o(\gamma_n)$ and $\gamma_n^0 + \gamma_n' = \gamma_n > 0$.

Case (ii).

Step 7°

This is the main case, where we have $\gamma_n = O(\gamma_n^0)$. Let us represent (23) in the form

$$\frac{\gamma_n^0}{\gamma_n} \cdot \frac{\omega(\Delta w_n^0) + \int_0^1 H_u(\hat{w},\hat{p})\Delta u_n^0 \,\mathrm{d}t}{\gamma_n^0} + \frac{\gamma_n'}{\gamma_n} \cdot C \le o(1).$$

It follows that

$$\min\left\{\frac{\omega(\Delta w_n^0) + \int_0^1 H_u(\hat{w},\hat{p})\Delta u_n^0 \,\mathrm{d}t}{\gamma_n^0},\ C\right\} \le o(1).$$

Since $C > 0$, we get

$$\frac{\omega(\Delta w_n^0) + \int_0^1 H_u(\hat{w},\hat{p})\Delta u_n^0 \,\mathrm{d}t}{\gamma_n^0} \le o(1),$$

or, equivalently,

$$\omega(\Delta w_n^0) + \int_0^1 H_u(\hat{w},\hat{p})\Delta u_n^0 \,\mathrm{d}t \le o(\gamma_n^0). \tag{30}$$

Next, we analyze this condition using Assumption 2.4 in the form (16). In general, Δw_n^0 does not belong to the critical cone K, defined by (2). We find a sequence $\delta w_n \in K$, which is "close" in some sense to the sequence Δw_n^0, and then use condition (30).

Step 8°

Set

$$M_+(H_u) := \{t \in [0,1] :\ |H_u(\hat{x},\hat{u},\hat{p})| > 0\},$$
$$M_+(H_u,\varepsilon_n) := \{t \in [0,1] :\ |H_u(\hat{x},\hat{u},\hat{p})| \ge \varepsilon_n\},$$
$$M_0(H_u) := \{t \in M_0 :\ H_u(\hat{x},\hat{u},\hat{p}) = 0\}.$$

Then

$$M_0 = M_0(H_u) \cup M_+(H_u) = M_0(H_u) \cup m(\varepsilon_n) \cup M_+(H_u, \varepsilon_n). \tag{31}$$

In view of condition (27), there exists $\tilde{u}_{1n}$ such that

$$\tilde{u}_{1n}\chi_{M_0(H_u)} = \tilde{u}_{1n}, \quad G'(\hat{u})\big(\Delta u_n^0 + \tilde{u}_{1n}\big)\chi_{M_0(H_u)} \le 0, \tag{32}$$

$$|\tilde{u}_{1n}| \le O(|\Delta u_n^0|^2) \tag{33}$$

(hereinafter χ_M stands for the characteristic function of the set M), and therefore,

$$\|\tilde{u}_{1n}\|_1 \le O(\gamma_n), \quad \|\tilde{u}_{1n}\|_\infty \le O(\|\Delta u_n\|_\infty^2) = o(1). \tag{34}$$

Further, we set

$$H_u^0(\hat{w}, \hat{p}) = \frac{H_u(\hat{w}, \hat{p})}{|H_u(\hat{w}, \hat{p})|}, \quad t \in M_+(H_u).$$

There exists $\tilde{u}_{2n}$ such that

$$\tilde{u}_{2n}\chi_{M_+(H_u,\varepsilon_n)} = \tilde{u}_{2n}, \quad H_u(\hat{w}, \hat{p})\big(\Delta u_n^0 + \tilde{u}_{2n}\big)\chi_{M_+(H_u,\varepsilon_n)} = 0, \tag{35}$$

$$|\tilde{u}_{2n}| \le O(|H_u^0(\hat{w}, \hat{p})\Delta u_n^0)|)\chi_{M_+(H_u,\varepsilon_n)} \le \frac{1}{\varepsilon_n} O(|H_u(\hat{w}, \hat{p})\Delta u_n^0)|)\chi_{M_+(H_u,\varepsilon_n)}. \tag{36}$$

Consequently,

$$\|\tilde{u}_{2n}\|_\infty \le O(\|\Delta u_n\|_\infty) = o(1).$$

Taking into account the estimate (26), we obtain

$$\|\tilde{u}_{2n}\|_1 \le \frac{1}{\varepsilon_n} O(\gamma_n). \tag{37}$$

Choose $\varepsilon_n > 0$ such that

$$\frac{\|\Delta w_n\|_\infty}{\varepsilon_n} \to 0. \tag{38}$$

Then

$$\frac{1}{\varepsilon_n} O(\gamma_n) = o(\sqrt{\gamma_n}).$$

Consequently,

$$\|\tilde{u}_{2n}\|_1 = o(\sqrt{\gamma_n}). \tag{39}$$

Set $\tilde{u}_n = \tilde{u}_{1n} + \tilde{u}_{2n}$. Then, $\|\tilde{u}_n\|_\infty \le O(\|\Delta u_n\|_\infty) = o(1)$ and

$$\|\tilde{u}_n\|_1 = o(\sqrt{\gamma_n}), \quad \|\tilde{u}_n\|_2^2 \le \|\tilde{u}_n\|_\infty \|\tilde{u}_n\|_1 \le \frac{\|\tilde{u}_n\|_\infty}{\varepsilon_n} O(\gamma_n) = o(\gamma_n). \tag{40}$$

Moreover, due to (31), (32), (35), we have

$$G'(\hat{u})\big(\Delta u_n^0 + \tilde{u}_n\big) \le 0 \quad \text{a.e. on} \quad M_0, \tag{41}$$

$$H_u(\hat{w}, \hat{p})\big(\Delta u_n^0 + \tilde{u}_n\big) = 0. \tag{42}$$

Set

$$\bar{u}_n = -u_n' + \tilde{u}_n, \quad \delta u_n = \Delta u_n + \bar{u}_n = \Delta u_n^0 + \tilde{u}_n.$$

Then

$$G'(\hat{u})\delta u_n \le 0 \quad \text{a.e. on} \quad M_0, \quad H_u(\hat{w}, \hat{p})\delta u_n = 0. \tag{43}$$

Also note that

$$\|u_n'\|_1 \le \sqrt{\operatorname{meas} m(\varepsilon_n)}\|u_n'\|_2 = o(\|u_n'\|_2) = o(\sqrt{\gamma_n'}) = o(\sqrt{\gamma_n}).$$

Therefore,

$$\|\bar{u}_n\|_1 = o(\sqrt{\gamma_n}). \tag{44}$$

Step 9°

The equation $\Delta\dot{x}_n = \Delta_n f$ implies

$$\Delta\dot{x}_n = f_x(\hat{w})\Delta x_n + f_u(\hat{w})\Delta u_n + O(|\Delta w_n|^2). \tag{45}$$

There exists $\delta x_n \in W^{1,1}$ such that

$$\delta\dot{x}_n = f_x(\hat{w})\delta x_n + f_u(\hat{w})\delta u_n, \quad \delta x_n(0) = \Delta x_n(0). \tag{46}$$

Then, it follows from equations (45) and (46) that

$$\delta x_n = \Delta x_n + \bar{x}_n,$$

where $\bar{x}_n$ satisfies

$$\dot{\bar{x}}_n = f_x(\hat{w})\bar{x}_n + f_u(\hat{w})\bar{u}_n - O(|\Delta w_n|^2), \quad \bar{x}_n(0) = 0.$$

This implies the following estimate

$$\|\bar{x}_n\|_\infty \le O(\|\bar{u}_n\|_1) + O(\|\Delta w_n\|_2^2) = o(\sqrt{\gamma_n}). \tag{47}$$

Set

$$\bar{w}_n = (\bar{x}_n, \tilde{u}_n), \quad \delta w_n = (\delta x_n, \delta u_n) := \Delta w_n^0 + \bar{w}_n.$$

Then, according to (43) and (46), we see that

$$\delta w_n \in K. \tag{48}$$

Step 10°

Let us compare $\omega(\delta w_n)$ with $\omega(\Delta w_n^0)$. We have

$$\langle H_{ww}(\hat{w},\hat{p})\delta w_n,\delta w_n\rangle = \langle H_{ww}(\hat{w},\hat{p})(\Delta w_n^0+\bar{w}_n),\Delta w_n^0+\bar{w}_n\rangle$$

$$= \langle H_{ww}(\hat{w},\hat{p})\Delta w_n^0,\Delta w_n^0\rangle + 2\langle H_{ww}(\hat{w},\hat{p})\Delta w_n^0,\bar{w}_n\rangle + \langle H_{ww}(\hat{w},\hat{p})\bar{w}_n,\bar{w}_n\rangle.$$

Similarly,

$$\langle F''(\hat{q})\delta q_n,\delta q_n\rangle = \langle F''(\hat{q})(\Delta q_n+\bar{q}_n),\Delta q_n+\bar{q}_n\rangle$$
$$= \langle F''(\hat{q})\Delta q_n,\Delta q_n\rangle + 2\langle F''(\hat{q})\Delta q_n,\bar{q}_n\rangle + \langle F''(\hat{q})\bar{q}_n,\bar{q}_n\rangle,$$

where

$$\delta q_n = (\delta x_n(0),\delta x_n(1)),\quad \Delta q_n = (\Delta x_n(0),\Delta x_n(1)),\quad \bar{q}_n = (\bar{x}_n(0),\bar{x}_n(1)).$$

Therefore,

$$\omega(\delta w_n) = \omega(\Delta w_n^0) + r_\omega(n),$$

where

$$r_\omega(n) = 2\langle F''(\hat{q})\Delta q_n,\bar{q}_n\rangle + \langle F''(\hat{q})\bar{q}_n,\bar{q}_n\rangle$$
$$+\int_0^1 \big(2\langle H_{ww}(\hat{w},\hat{p})\Delta w_n^0,\bar{w}_n\rangle + \langle H_{ww}(\hat{w},\hat{p})\bar{w}_n,\bar{w}_n\rangle\big)\,\mathrm{d}t.$$

We show that

$$|r_\omega(n)| = o(\gamma_n). \tag{49}$$

First, we have

$$\langle H_{ww}(\hat{w},\hat{p})\Delta w_n^0,\bar{w}_n\rangle =$$
$$= \langle H_{xx}(\hat{w},\hat{p})\Delta x_n,\bar{x}_n\rangle + \langle H_{xu}(\hat{w},\hat{p})\Delta u_n^0,\bar{x}_n\rangle +$$
$$\langle H_{ux}(\hat{w},\hat{p})\Delta x_n,\tilde{u}_n\rangle + \langle H_{uu}(\hat{w},\hat{p})\Delta u_n^0,\ \tilde{u}_n\rangle.$$

According to (47) and the first estimate in (40) we get

$$\|\Delta x_n\|_\infty\|\bar{x}_n\|_\infty + \|\Delta u_n\|_1\|\bar{x}_n\|_\infty + \|\Delta x_n\|_\infty\|\tilde{u}_n\|_1 = o(\gamma_n).$$

Let us estimate $\||\Delta u_n^0|\cdot|\tilde{u}_n|\|_1$. Using the first estimate in (34), estimate (37) and condition (38), we get

$$\int_0^1 |\Delta u_n^0|\cdot|\tilde{u}_n|\,\mathrm{d}t = \int_0^1 |\Delta u_n^0|\cdot|\tilde{u}_{1n}+\tilde{u}_{2n}|\,\mathrm{d}t \le \|\Delta u_n^0\|_\infty\|\tilde{u}_{1n}\|_1 + \|\Delta u_n^0\|_\infty\|\tilde{u}_{2n}\|_1$$

$$\le \|\Delta u_n^0\|_\infty O(\gamma_n) + \|\Delta u_n^0\|_\infty \frac{1}{\varepsilon_n} O(\gamma_n) = o(\gamma_n). \tag{50}$$

Therefore,

$$\|\langle H_{ww}(\hat{w},\hat{p})\Delta w_n^0,\bar{w}_n\rangle\|_1 = o(\gamma_n).$$

Secondly, we have

$$\langle H_{ww}(\hat{w},\hat{p})\bar{w}_n,\bar{w}_n\rangle = \langle H_{xx}(\hat{w},\hat{p})\bar{x}_n,\bar{x}_n\rangle + \langle 2H_{xu}(\hat{w},\hat{p})\tilde{u}_n,\bar{x}_n\rangle + \langle H_{uu}(\hat{w},\hat{p})\tilde{u}_n,\tilde{u}_n\rangle.$$

Again using (47) and (40) we get

$$\|\bar{x}_n\|_\infty^2 + \|\bar{x}_n\|_\infty\|\tilde{u}_n\|_1 + \|\tilde{u}_n\|_2^2 = o(\gamma_n),$$

and therefore,

$$\|\langle H_{ww}(\hat{w},\hat{p})\bar{w}_n,\bar{w}_n\rangle\|_1 = o(\gamma_n).$$

Consequently,

$$\left|\int_0^1 \big(2\langle H_{ww}(\hat{w},\hat{p})\Delta w_n^0,\bar{w}_n\rangle + \langle H_{ww}(\hat{w},\hat{p})\bar{w}_n,\bar{w}_n\rangle\big)\,\mathrm{d}t\right| = o(\gamma_n).$$

In addition,

$$|\langle 2F''(\hat{q})\Delta q_n,\bar{q}_n\rangle + \langle F''(\hat{q})\bar{q}_n,\bar{q}_n\rangle| \le c\big(\|\Delta x_n\|_\infty\|\bar{x}_n\|_\infty + \|\bar{x}_n\|_\infty^2\big) = o(\gamma_n)$$

with some $c > 0$. This yields the estimate (49). Consequently,

$$\omega(\delta w_n) = \omega(\Delta w_n^0) + o(\gamma_n). \tag{51}$$

Step 11°

Now let us compare $\gamma(\delta w_n)$ with $\gamma_n = \gamma(\Delta w_n)$. We have

$$|\delta x_n|^2 = |\Delta x_n + \bar{x}_n|^2 = |\Delta x_n|^2 + 2\langle\Delta x_n,\bar{x}_n\rangle + |\bar{x}_n|^2.$$

Therefore,

$$\|\delta x_n\|_\infty^2 = \|\Delta x_n\|_\infty^2 + r_x(n),$$

where

$$|r_x(n)| \le \|\Delta x_n\|_\infty\|\bar{x}_n\|_\infty + \|\bar{x}_n\|_\infty^2 = o(\gamma_n). \tag{52}$$

Similarly,

$$|\delta u_n|^2 = |\Delta u_n^0 + \tilde{u}_n|^2 = |\Delta u_n^0|^2 + 2\langle\Delta u_n^0,\tilde{u}_n\rangle + |\tilde{u}_n|^2.$$

Therefore,

$$\|\delta u_n\|_2^2 = \|\Delta u_n^0\|_2^2 + r_u(n),$$

where

$$r_u(n) = 2\int_0^1 \langle\Delta u_n^0,\bar{u}_n\rangle\,\mathrm{d}t + \|\bar{u}_n\|_2^2,$$

and then

$$|r_u(n)| \le \||\Delta u_n^0|\cdot|\tilde{u}_n|\|_1 + \|\tilde{u}_n\|_2^2 = o(\gamma_n). \tag{53}$$

Set $r(n) = r_x(n) + r_u(n)$. Then, in view of (52) and (53),

$$|r(n)| = o(\gamma_n). \tag{54}$$

Consequently,

$$\gamma(\delta w_n) = \gamma_n + o(\gamma_n). \tag{55}$$

Step 12°

Finally, consider the term $\int_0^1 H_u(\hat{w},\hat{p})\Delta u_n^0\,\mathrm{d}t$ in the inequality (30). Let us use (27). Since

$$\begin{aligned}&\langle G''(\hat{u})\delta u_n,\delta u_n\rangle = \langle G''(\hat{u})(\Delta u_n^0+\tilde{u}_n),\Delta u_n^0+\tilde{u}_n\rangle\\ &=\langle G''(\hat{u})\Delta u_n^0,\Delta u_n^0\rangle + 2\langle G''(\hat{u})\Delta u_n^0,\tilde{u}_n\rangle + \langle G''(\hat{u})\tilde{u}_n,\tilde{u}_n\rangle\\ &=\langle G''(\hat{u})\Delta u_n^0,\Delta u_n^0\rangle + r_G(n),\end{aligned}$$

where

$$r_G(n) = 2\langle G''(\hat{u})\Delta u_n^0,\tilde{u}_n\rangle + \langle G''(\hat{u})\tilde{u}_n,\tilde{u}_n\rangle \quad \text{and} \quad \|r_G(n)\|_1 = o(\gamma_n),$$

we obtain from (27) that

$$G'(\hat{u})\Delta u_n^0 + \frac{1}{2}\langle G''(\hat{u})\delta u_n,\delta u_n\rangle \le o(|\Delta u_n^0|^2) + r_G(n) \quad \text{a.e. on} \quad M_0.$$

Due to Assumption 2.1, there is a sequence $\tilde{u}_{Gn}$ such that

$$G'(\hat{u})(\Delta u_n^0 + \tilde{u}_{Gn}) + \frac{1}{2}\langle G''(\hat{u})\delta u_n,\delta u_n\rangle \le 0,$$

$$|\tilde{u}_{Gn}| \le o(|\Delta u_n^0|^2) + c|r_G(n)|,$$

with some $c>0$. Set $\delta v_n = \Delta u_n^0 + \tilde{u}_{Gn}$. Then

$$G'(\hat{u})\delta v_n + \frac{1}{2}\langle G''(\hat{u})\delta u_n,\delta u_n\rangle \le 0, \quad \|\tilde{u}_{Gn}\|_1 = o(\gamma_n), \quad \|\tilde{u}_{Gn}\|_\infty = o(1).$$

Consequently,

$$\int_0^1 H_u(\hat{w},\hat{p})\delta v_n\,\mathrm{d}t = \int_0^1 H_u(\hat{w},\hat{p})\Delta u_n^0\,\mathrm{d}t + o(\gamma_n). \tag{56}$$

Obviously, $\delta v_n \in T_U^{\flat(2)}(\hat{u},\delta u_n)$.

Step 13°

Conditions (30), (51), (55), and (56) imply

$$\omega(\delta w_n) + \int_0^1 H_u(\hat{w},\hat{p})\delta v_n\,\mathrm{d}t \le o(\gamma(\delta w_n)). \tag{57}$$

Since $\delta w_n \in K$ and $\delta v_n \in T_U^{\flat(2)}(\hat{u},\delta u_n)$, condition (57) contradicts Assumption 2.4 in the form (16). The theorem is proven. □

Remark 3.1 *Here we would like to outline some prospects for further research. Recently, together with V. Veliov, we studied sufficient conditions for a strong metric subregularity (SMsR) of the optimality mapping associated with Pontryagin's local maximum principle for a Mayer-type optimal control problem without control constraints. An important role in these conditions was played by the second-order sufficient condition for a weak local minimum. A possible next step in our study is to include the constraint $G(u) \leq 0$ in the problem. We hope that the result obtained in this work will be useful for this purpose.*

There is another goal that we pursued in this work. In our joint works with H. Frankowska, we managed to obtain the necessary second-order conditions for optimal control problems with the constraint $u \in U$, where U is an arbitrary set in $\mathbb{R}^m$. Our results are formulated in terms of first and second order tangents to the set U. It is interesting to obtain similar sufficient conditions for problems with the general control constraint $u \in U$. We hope that the proof of the main result of this paper will allow for such a generalization.

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