Decomposition Integral without Alternatives, its Equivalence to Lebesgue Integral, and Computational Algorithms

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Abstract:
In this paper we present a new class of decomposition integrals called the collection integrals. From this class of integrals we take a closer look on two special types of collection integrals, namely the chain integral and the min-max integral. Superdecomposition version of collection integral is also defined and the superdecomposition duals for the chain and the min-max integrals are presented. Also, the condition on the collection that ensures the coincidence of the collection integral with the Lebesgue integral is presented. Lastly, some computational algorithms are discussed.

Keywords: *decomposition integrals, nonlinear integrals, computational algorithms*

1. Introduction

Theory of linear integration has found many applications throughout mathematics. In the last century some concepts of nonlinear integrals appeared and are under investigation to this day. These nonlinear integrals found many applications also outside the world of mathematics, e.g., in psychology, productivity maximization, and others.

A wide class of nonlinear integrals that contains other nonlinear integrals used today, namely decomposition integrals, were presented not so long ago by Even and Lehrer [4]. In this paper we define a subclass of these integrals and present some results concerning them.

This contribution is organized as follows. Basic building blocks of this paper are introduced in the section 2. In the section 3 we define a special class of decomposition integrals called the collection integrals and in section 4 two such integrals are closely investigated. In the fifth section we are interested in two concepts. Firstly in characterizing a positive bases for the spaces $\mathcal{F}_+$ and $\mathcal{F}$ and, secondly, in characterizing all collection integrals that yield to the Lebesgue integral if restricted to the space of measures. In the section 6, superdecomposition version of the collection integral is presented. The last section of this paper is devoted to the discussion of some computational algorithms.

Recall that, in general, decomposition systems consist of more than one collection. These collections represent some choice alternatives. In this paper, we will consider only singleton decomposition systems, i.e., those consisting of a single collection. Hence the integral with single decomposition alternative will be discussed.

2. Preliminaries

In this paper, without loss of generality, we will consider a fixed finite space $X = \{1, 2, \dots, n\} \subset \mathbb{N}$. A chain on X is any sequence $\{A_i\}_{i=1}^k$ such that $\emptyset \neq A_1 \subsetneq \dots \subsetneq A_k \subseteq X$. A full chain on X is any chain $\{A_i\}_{i=1}^k$ such that $k = n$.

Also, only positive functions on X will be considered, i.e., functions with domain X and co-domain $[0, \infty[$. The class of such functions will be denoted by $\mathcal{F}$. The set of not strictly increasing functions will be denoted by $\mathcal{F}_\uparrow$.

A capacity is any set function $\mu\colon 2^X \to [0, \infty[$ that is grounded, i.e., $\mu(\emptyset) = 0$, and monotone with respect to set inclusion, i.e., $A \subseteq B$ implies $\mu(A) \leq \mu(B)$. The class of all capacities will be denoted by $\mathcal{M}$. A measure is any additive capacity, i.e., if $A, B \subseteq X$ are two disjoint sets then $\mu(A \cup B) = \mu(A) + \mu(B)$ holds. A symbol $\mathcal{M}_+$ denotes the set of all measures on X.

A collection, mostly denoted by $\mathcal{D}$, is any non-empty subset of $2^X \setminus \{\emptyset\}$. A decomposition system $\mathcal{H}$ is any non-empty subset of $2^{2^X \setminus \{\emptyset\}}$, i.e., a decomposition system consists of at least one collection.

Definition 2.1. *A decomposition integral [4, 7] with respect to a decomposition system $\mathcal{H}$ is a mapping $I_\mathcal{H}\colon \mathcal{F} \times \mathcal{M} \to [0, \infty[$ such that $I_\mathcal{H}(f, \mu)$ is equal to*

$$\bigvee_{\mathcal{D} \in \mathcal{H}} \bigvee \left\{ \sum_{A \in \mathcal{D}} a_A \mu(A) \colon a_A \geq 0, \sum_{A \in \mathcal{D}} a_A 1_A \leq f \right\}.$$

Based on the choice of $\mathcal{H}$ we get a different types of decomposition integrals. In the following example some of the well known decomposition integrals are presented.

Example 2.2. *If $\mathcal{H}_1$ consists of all singleton collections, we speak about the Shilkret integral [8], i.e.,*

$$\mathrm{Sh}(f, \mu) = \bigvee \left\{ \mu(A) \min f(A) \colon A \in 2^X \setminus \{\emptyset\} \right\}.$$

Note that we use the following abbreviate notation $\min f(A) = \wedge \{f(x) \colon x \in A\}$. *If $\mathcal{H}_2$ consists only of partitions of X then we speak about the Pan integral [9], i.e.,*

$$\mathrm{Pan}(f, \mu) = \bigvee \left\{ \sum_{A \in \rho} \mu(A) \min f(A) \colon \rho \in \mathrm{Prt}(X) \right\},$$

where $\mathrm{Prt}(X)$ denotes the set of all partitions on X. In case that $\mathcal{H}_3$ is the class of all chains on X then the corresponding integral is the Choquet integral [1], i.e.,

$$\mathrm{Ch}(f, \mu) = \int_0^\infty \mu(f \geq t)\, \mathrm{d}t.$$

Lastly, conceding that $\mathcal{H}_4 = 2^{2^X \setminus \{\emptyset\}}$ we get the concave integral $\text{cov}(f,\mu)$ *introduced by Lehrer [5]. Note that the choice $\mathcal{H}_5 = \left\{2^X \setminus \{\emptyset\}\right\}$ also yields to the concave integral.*

The decomposition integrals represent the integration from below as, for example, the lower Riemann sum. The integration from above is represented by so-called superdecomposition integrals.

Definition 2.3. *A superdecomposition integral [6] with respect to a decomposition system $\mathcal{H}$ is a mapping $I^*_{\mathcal{H}}\colon \mathcal{F}\times\mathcal{M}\to[0,\infty]$ such that $I^*_{\mathcal{H}}(f,\mu)$ is equal to*

$$\bigwedge_{\mathcal{D}\in\mathcal{H}} \bigwedge \left\{ \sum_{A\in\mathcal{D}} a_A\mu(A)\colon a_A \geq 0, \sum_{A\in\mathcal{D}} a_A 1_A \geq f \right\}.$$

Note that the decomposition integral can attain only finite values. In the case of superdecomposition integrals also unbounded values, i.e., ∞, can be attained. Take, for example, $X=\{1,2\}$, $\mathcal{H}=\{\{\{1\}\}\}$, and $f(x)=1$.

Example 2.4. *For decomposition integrals mentioned in previous example there is a corresponding superdecomposition integral. Observe that in the case of the decomposition system $\mathcal{H}_3$ the same integral is obtained.*

In general, the inequality $I_{\mathcal{H}}(f,\mu)\leq I^*_{\mathcal{H}}(f,\mu)$ does not hold and thus the superdecomposition integral can attain values lower than the corresponding decomposition integral.

In this paper we will be interested also in the equivalence of a special class of decomposition integrals with Lebesgue integral. The Lebesgue integral of a function f with respect to a measure μ will be denoted by $\text{Leb}(f,\mu)$.

3. Collection Integral

In this section we will define a collection integral that represents special class of decomposition integrals.

Definition 3.1. *A collection integral with respect to a collection $\mathcal{D}$ is a mapping $\mathcal{I}_{\mathcal{D}}\colon \mathcal{F}\times\mathcal{M}\to[0,\infty[$ such that $\mathcal{I}_{\mathcal{D}}(f,\mu) = I_{\mathcal{H}}(f,\mu)$ where $\mathcal{H}=\{\mathcal{D}\}$. Analogously, super-collection integral is a mapping $\mathcal{I}^*_{\mathcal{D}}\colon \mathcal{F}\times\mathcal{M}\to[0,\infty]$ such that $\mathcal{I}^*_{\mathcal{D}}(f,\mu) = I^*_{\mathcal{H}}(f,\mu)$.*

As already mentioned, the value of a superdecomposition integral might be lower than the value of the corresponding decomposition integral. If we restrict ourselves to only measures and collection integrals this is no longer the case.

Theorem 3.2. *Let $f\in\mathcal{F}$, $\mu\in\mathcal{M}_+$ and let $\mathcal{D}$ be any collection. Then*

$$\mathcal{I}_{\mathcal{D}}(f,\mu)\leq \text{Leb}(f,\mu)\leq \mathcal{I}^*_{\mathcal{D}}(f,\mu).$$

Proof. From the definition of the collection integral we obtain

$$\begin{aligned}\mathcal{I}_{\mathcal{D}}(f,\mu) &= \bigvee\left\{\sum_{A\in\mathcal{D}} a_A\mu(A)\colon \sum_{A\in\mathcal{D}} a_A 1_A \leq f\right\}\\ &= \bigvee\left\{\sum_{x\in X}\mu(\{x\})\sum_{A\in\mathcal{D}} a_A 1_A(x)\colon \sum_{A\in\mathcal{D}} a_A 1_A\leq f\right\}\\ &\leq \sum_{x\in X}\mu(\{x\})f(x) = \text{Leb}(f,\mu).\end{aligned}$$

The inequality $\text{Leb}(f,\mu)\leq\mathcal{I}^*_{\mathcal{D}}(f,\mu)$ can be proved analogously and thus the theorem follows. ☐

We are interested in the problem of finding such collections $\mathcal{D}$ which lead to the equality $\mathcal{I}_{\mathcal{D}} = \text{Leb}$. From the proof of the previous theorem we trivially get the following corollary.

Corollary 3.3. $\mathcal{I}_{\mathcal{D}} = \text{Leb}$ *if and only if for every function $f\in\mathcal{F}$ there exist $a_A\geq 0$, $A\in\mathcal{D}$, such that*

$$\sum_{A\in\mathcal{D}} a_A 1_A = f.$$

4. Examples of Collection Integrals

In this section we will take a closer look to two special types of collection integrals called a chain integral and a min-max integral.

4.1. Chain Integral

The chain integral is a collection integral with respect to a single chain.

Definition 4.1. *Let B be a chain on X. A mapping $\text{ch}_B = \mathcal{I}_B$ is called a chain integral with respect to a chain B.*

The following definition will be useful in proving a recursive equation for the chain integral.

Definition 4.2. *Let $B=\{A_i\}_{i=1}^k$ be a chain. A sequence $\{a_i\}_{i=1}^k$ will be called f-B-feasible if and only if $a_i\geq 0$, $1\leq i\leq k$, and*

$$\sum_{i=1}^k a_i 1_{A_i}\leq f.$$

A f-B-feasible sequence $\{a_i\}_{i=1}^k$ will be called min-f-B-feasible if and only if $a_k = \min f(A_k)$.

Lemma 4.3. *Let $B=\{A_i\}_{i=1}^k$ be a chain. For every f-B-feasible sequence $\{a_i\}_{i=1}^k$ there exists min-f-B-feasible sequence $\{b_i\}_{i=1}^k$ such that*

$$\sum_{i=1}^k a_i\mu(A_i)\leq\sum_{i=1}^k b_i\mu(A_i).$$

Proof. The proof of this lemma will be divided into two cases. Case 1: $\sum_{i=1}^k a_i\leq\min f(A_k)$. Then we can define $\{b_i\}_{i=1}^k$ by

$$b_i = \begin{cases}\min f(A_k), & \text{if } i=k,\\ 0, & \text{otherwise},\end{cases}$$

for $i = 1, 2, \ldots, k$. Truly,

$$\sum_{i=1}^{k} a_i\mu(A_i) \leq \sum_{i=1}^{k} a_i\mu(A_k) = \mu(A_k)\sum_{i=1}^{k} a_i$$
$$\leq \mu(A_k)\min f(A_k) = \sum_{i=1}^{k} b_i\mu(A_i).$$

The fact that $\{b_i\}_{i=1}^k$ is min-f-B-feasible follows from the fact that $\{a_i\}_{i=1}^k$ is f-B-feasible.

Case 2: $\sum_{i=1}^k a_i > \min f(A_k)$. Then there exists $i^* \in \{1, 2, \ldots, k\}$ such that $\sum_{i=i^*+1}^k a_i < \min f(A_k)$ and $\sum_{i=i^*}^k a_i \geq \min f(A_k)$. Then $\{b_i\}_{i=1}^k$ is given by

$$b_i = \begin{cases} \min f(A_k), & \text{if } i = k, \\ \sum_{i=i^*}^k a_i - \min f(A_k), & \text{if } i = i^*, \\ a_i, & \text{if } i = 1, \ldots, i^* - 1, \\ 0, & \text{otherwise,} \end{cases}$$

for $i = 1, 2, \ldots, k$. Indeed,

$$\sum_{i=1}^{k} a_i\mu(A_i)$$
$$= \sum_{i=1}^{i^*-1} a_i\mu(A_i) + a_{i^*}\mu(A_{i^*}) + \sum_{i=i^*+1}^{k} a_i\mu(A_i)$$
$$\leq \sum_{i=1}^{i^*-1} a_i\mu(A_i) + \left(\sum_{i=i^*}^{k} a_i - \min f(A_k)\right)\mu(A_{i^*})$$
$$+ \min f(A_k)\mu(A_k) = \sum_{i=1}^{k} b_i\mu(A_i).$$

Again, the fact that $\{b_i\}_{i=1}^k$ is min-f-B-feasible follows directly from the fact that $\{a_i\}_{i=1}^k$ is f-B-feasible and thus the lemma is proved. □

For the following lemma, let us denote

$$\Xi = \left\{\{a_i\}_{i=1}^k : \{a_i\}_{i=1}^k \text{ is } f\text{-}B\text{-feasible}\right\},$$

and

$$\Theta = \left\{\{a_i\}_{i=1}^k : \{a_i\}_{i=1}^k \text{ is min-}f\text{-}B\text{-feasible}\right\}.$$

Lemma 4.4. *Let $f \in \mathcal{F}$, $\mu \in \mathcal{M}$, and $B = \{A_i\}_{i=1}^k$ be any chain on X. Let us denote*

$$\xi = \bigvee\left\{\sum_{i=1}^{k} a_i\mu(A_i) : \{a_i\}_{i=1}^k \in \Xi\right\}$$

and

$$\theta = \bigvee\left\{\sum_{i=1}^{k} a_i\mu(A_i) : \{a_i\}_{i=1}^k \in \Theta\right\}.$$

Then $\xi = \theta$.

Proof. Note that $\Theta \subseteq \Xi$ and thus $\theta \leq \xi$. On the other hand, based on the previous lemma, for every element $\{a_i\}_{i=1}^k \in \Xi$ there exists an element $\{b_i\}_{i=1}^k \in \Theta$ such that

$$\sum_{i=1}^{k} a_i\mu(A_i) \leq \sum_{i=1}^{k} b_i\mu(A_i)$$

and thus $\xi \leq \theta$ which implies that $\xi = \theta$ as states the lemma. □

Now we can pose and prove a recursive formula for the chain integral.

Theorem 4.5. *Let $B = \{A_i\}_{i=1}^k$ be a chain on X. Let $\tau = \min f(A_k)$ and $\tilde{B} = \{A_i\}_{i=1}^{k-1}$. Then*

$$\mathrm{ch}_B(f, \mu) = \tau\mu(A_k) + \mathrm{ch}_{\tilde{B}}(\tilde{f}, \tilde{\mu}),$$

where $\tilde{f} = f\restriction_{A_{k-1}} - \tau$ and $\tilde{\mu} = \mu\restriction_{2^{A_{k-1}}}$.

Proof. From previous two lemmas we can easily see that

$$\mathrm{ch}_B(f, \mu) = \bigvee\left\{\sum_{i=1}^{k} a_i\mu(A_i) : \{a_i\}_{i=1}^k \in \Xi\right\}$$
$$= \bigvee\left\{\sum_{i=1}^{k} b_i\mu(A_i) : \{b_i\}_{i=1}^k \in \Theta\right\}$$
$$= \mu(A_k)\tau$$
$$+ \bigvee\left\{\sum_{i=1}^{k-1} b_i\mu(A_i) : \{b_i\}_{i=1}^k \text{ is min-}f\text{-}\tilde{B}\text{-feasible}\right\}$$

which proves the theorem. □

Inducing the previous theorem we obtain the following formula.

Corollary 4.6. *Let $f \in \mathcal{F}$, $\mu \in \mathcal{M}$, and let $B = \{A_i\}_{i=1}^k$ be any chain on X. Then*

$$\mathrm{ch}_B(f, \mu) = \mu(A_k)\min f(A_k)$$
$$+ \sum_{i=1}^{k-1} \mu(A_i)\Big(\min f(A_i) - \min f(A_{i+1})\Big),$$

or

$$\mathrm{ch}_B(f, \mu) = \sum_{i=1}^{k} \mu(A_i)\Big(\min f(A_i) - \min f(A_{i+1})\Big)$$

with convention that $\min f(A_{k+1}) = 0$.

Also from the previous formulae we can find a lower bound on the chain integral as follows.

Corollary 4.7. *Let $f \in \mathcal{F}$, $\mu \in \mathcal{M}$, and let $B = \{A_i\}_{i=1}^k$ be any chain. Then*

$$\mathrm{ch}_B(f, \mu) \geq \mu(A_k)\min f(A_k).$$

From the theory of Choquet integration it is known that

$$\mathrm{Ch}(f, \mu) = \sum_{i=1}^{m} (f_i - f_{i-1})\mu(A_i)$$

where $\{f_i\}_{i=1}^m$ is the increasing enumeration of $\mathrm{Im}(f) \cup \{0\}$ and

$$A_i = \{x \in X : f(x) > f_{i-1}\}$$

for $i = 1, 2, \ldots, m$. Then it can be seen that

$$\mathrm{Ch}(f, \mu) = \sum_{i=1}^{m} \mu(A_i)\Big(\min f(A_i) - \min f(A_{i+1})\Big)$$

with convention that $\min f(A_{m+1}) = 0$.

In other words, for every function $f \in \mathcal{F}$ there exists a chain B such that $\mathrm{ch}_B(f, \mu) = \mathrm{Ch}(f, \mu)$. This chain is called Ch-maximizing chain.

Definition 4.8. *A chain $B = \{A_i\}_{i=1}^k$ is called Ch-maximizing for function f if and only if*

$$\{\min f(A_i) \colon i = 1, 2, \dots, k\} \setminus \{0\} = \mathrm{Im}(f) \setminus \{0\}.$$

Then the following theorem follows from the theory of Choquet integration.

Theorem 4.9. *A chain B is Ch-maximizing for a function f if and only if $\mathrm{ch}_B(f, \mu) = \mathrm{Ch}(f, \mu)$.*

Example 4.10. *Following the original example with workers of Lehrer in [5], let us assume that $X = \{1, 2, 3, 4\}$ represents the set of workers and let $f \colon X \to [0, \infty[$, $f(i) = 5 - i$, denote a non-negative function where $f(i)$ represents the maximum number of working hours for worker $i \in X$. Let us choose a chain $B = \{A_i\}_{i=1}^3$ where $A_1 = \{3\}$, $A_2 = \{2, 3\}$, $A_3 = \{1, 2, 3, 4\}$. This chain can represent the following situation: at the same moment only the all workers, only workers labeled by 2 and 3, and only worker labeled 3 can work at any moment. Let μ represent the number of articles made per hour: $\mu(A_1) = 3$, $\mu(A_2) = 4$ and $\mu(A_3) = 6$. Then $\mathrm{ch}_B(f, \mu)$ represents the maximum number of articles made in this situation. From previous formulae it follows that $\mathrm{ch}_B(f, \mu) = 13$.*

4.2. Min-max Integral

Note that the definition of the Choquet integral can be rewritten to the form

$$\mathrm{Ch}(f, \mu) = \bigvee_{B = \{A_i\}_{i=1}^n} \mathrm{ch}_B(f, \mu)$$

where the supremum operator runs over all full chains B on X. The motivation behind the min-max integral is to replace the first supremum operator by infimum operator.

Definition 4.11. *A min-max integral of a non-negative function $f \in \mathcal{F}$ with respect to a capacity $\mu \in \mathcal{M}$ is defined by*

$$I^{\wedge\vee}(f, \mu) = \bigwedge_{B = \{A_i\}_{i=1}^n} \mathrm{ch}_B(f, \mu)$$

where the infimum operator runs over all full chains B on X.

From the previous discussion on the chain integral we get a lower bound on the min-max integral.

Lemma 4.12. *Let $f \in \mathcal{F}$ and $\mu \in \mathcal{M}$. Then the inequality $I^{\wedge\vee}(f, \mu) \geq \mu(X) \min f(X)$ holds.*

Proof. Let $B = \{A_i\}_{i=1}^n$ be any full chain which implies that $A_n = X$. Then by Corollary 4.7 we obtain that

$$\mathrm{ch}_B(f, \mu) \geq \mu(X) \min f(X)$$

for any full chain B on X which implies that

$$I^{\wedge\vee}(f, \mu) \geq \mu(X) \min f(X)$$

and thus the result follows. □

Now we need to prove that this value is not only the lower bound but also the value of the min-max integral.

Theorem 4.13. *$I^{\wedge\vee}(f, \mu) = \mu(X) \min f(X)$.*

Proof. Following the previous lemma it is enough to find a full chain $B = \{A_i\}_{i=1}^n$ such that $\mathrm{ch}_B(f, \mu) = \mu(X) \min f(X)$. Let $x^* \in X$ be such that $f(x^*) = \min f(X)$. Then let B be any chain such that $A_1 = \{x^*\}$ which implies that $x^* \in A_i$ for all $i \in X$. Then trivially $\mathrm{ch}_B(f, \mu) = \mu(X) \min f(X)$ and thus the theorem follows. □

To this moment we could not really see that the min-max integral belongs to the class of collection integrals. Knowing the formula to compute the value of the min-max integral we can easily see that this integral is indeed the collection integral.

Theorem 4.14. *The min-max integral belongs to the class of the collection integrals, $I^{\wedge\vee} = \mathcal{I}_{\{\{X\}\}}$.*

Example 4.15. *Let f and μ be as in Example 4.10. The value of the min-max integral $I^{\wedge\vee}(f, \mu)$ represents the maximum number of articles made if only all workers can work together. In this setting, $I^{\wedge\vee}(f, \mu) = 6$.*

Remark 4.16. *Observe that the min-max integral is the smallest decomposition integral related to decomposition systems $\mathcal{H}$ dealing with X as an element of some collection from $\mathcal{H}$.*

5. Equivalence of Collection and Lebesgue Integrals

In this section we start by characterization of a positive basis for the space of non-negative functions $\mathcal{F}$ starting with finding a basis for the space of increasing non-negative functions $\mathcal{F}_\uparrow$. This discussion will yield to an easy characterisation of such collections $\mathcal{D}$ that yield to the Lebesgue integral if we restrict ourselves to the class of measures.

Note that both spaces, $\mathcal{F}$ and $\mathcal{F}_\uparrow$, are of dimension n and thus the positive basis will consist of at least n elements.

Definition 5.1. *A positive basis of a function space $\mathcal{S}$ is any sequence $\{E_i\}_{i=1}^m \subseteq \mathcal{S}$ such that for every element $f \in \mathcal{S}$ there are non-negative real numbers $\{\alpha_i\}_{i=1}^m$ such that*

$$\sum_{i=1}^m \alpha_i 1_{E_i} = f.$$

For the set of increasing functions $\mathcal{F}_\uparrow$ we have the following characterization of a positive basis.

Definition 5.2. *A set $\mathcal{B} = \{E_x\}_{x \in X} \subseteq 2^X \setminus \{\emptyset\}$ is called a $\uparrow$-compatible basis if and only if*

1) *for every $x \in X$ we have $\min E_x = x$; and*
2) *if there exists $z \in A_x \cap A_y$ where $z > \max\{x, y\}$ then $z \in A_x \cap A_y$.*

Remark 5.3. *Note that the second condition of Definition 5.2 can be stated as follows: if there exists $z \in A_x \cap A_y$ where $z > x > y$ then $x \in A_y$, or, equivalently, if $x \notin A_y$ where $x > y$ then $z \notin A_y$ for all $z > x$.*

Theorem 5.4. *A sequence $\mathcal{B} = \{E_x\}_{x\in X} \subseteq \mathcal{F}_\uparrow$ is a positive basis of $\mathcal{F}_\uparrow$ if and only if $\mathcal{B}$ is $\uparrow$-compatible basis.*

Proof. Let $\mathcal{B} = \{E_x\}_{x\in X}$ be a positive basis and let $f \in \mathcal{F}_\uparrow$. We will find a positive real numbers $\{\alpha_x\}_{x\in X}$ such that

$$f = \sum_{x\in X} \alpha_x 1_{E_x}.$$

Let us recursively define

$$\alpha_x = f(x) - \sum_{y=1}^{x-1} \alpha_y 1_{E_y}(x).$$

Firstly, we need to prove that $\alpha_x \geq 0$ for all $x = 1, 2, \dots, n$. We will use the proof by induction. For $x = 1$ we obtain that $\alpha_1 = f(1) \geq 0$ and thus α_1 is non-negative. Now, let us assume that $\alpha_1, \alpha_2, \dots, \alpha_x \geq 0$. We want to prove that $\alpha_{x+1} \geq 0$. It follows that

$$\begin{aligned} 0 &\leq f(x+1) - f(x) \\ &= \alpha_{x+1} + \sum_{y=1}^{x} \alpha_y 1_{E_y}(x+1) - \alpha_x - \sum_{y=1}^{x-1} \alpha_y 1_{E_y}(x) \\ &= \alpha_{x+1} - \sum_{y=1}^{x} \alpha_y \Big(1_{E_y}(x) - 1_{E_y}(x+1)\Big) \end{aligned}$$

which implies that

$$\alpha_{x+1} \geq \sum_{y=1}^{x} \alpha_y \Big(1_{E_y}(x) - 1_{E_y}(x+1)\Big).$$

Now it is sufficient to prove that $1_{E_y}(x) \geq 1_{E_y}(x+1)$ for all $y \leq x$. If $y = x$ then the claim holds trivially. Now for $y < x$ let us assume that the claim does not hold, i.e., $1_{E_y}(x) = 0$ and $1_{E_y}(x+1) = 1$, or, equivalently, $x \notin E_y$ and $(x+1) \in E_y$. We have $x \notin A_y$ where $x > y$ and thus based on the Remark 5.3 it follows that $z \notin A_y$ for all $z > x$. Choose $z = x+1$ which contradicts that $(x+1) \in A_y$ and thus $1_{E_y}(x) \geq 1_{E_y}(x+1)$ for all $y \leq x$. This proves that $\alpha_{x+1} \geq 0$ implying that α_x are non-negative for all $x \in X$. Now it is easy to see that

$$\sum_{x\in X} \alpha_x 1_{E_x}(y) = \underbrace{\sum_{x=1}^{y-1} \alpha_x 1_{E_x}(y) + \alpha_y}_{f(y)} + \sum_{x=y+1}^{n} \alpha_x \underbrace{1_{E_x}(y)}_{0} = f(y).$$

Now we need to prove that the reversed claim holds, i.e., if any of the conditions of Theorem 5.4 is omitted then there exist a function that is not decomposeable by $\mathcal{B}$. Let us thus assume that the condition 1 of the positive basis of $\mathcal{F}_\uparrow$ does not hold, i.e., there exists an element $x \in X$ such that $\min E_x \neq x$. Let x^* be the smallest such element and define

$$f(x) = \begin{cases} 1, & \text{if } x \geq x^*, \\ 0, & \text{otherwise.} \end{cases}$$

If $x^* \in E_y$ then $\alpha_y = 0$. On the other hand, if $x^* \notin E_y$ then $1_{E_y}(x^*) = 0$. This implies that

$$\sum_{y\in X} \alpha_y 1_{E_y}(x^*) = 0 \neq 1 = f(x^*)$$

and thus f is not decomposeable by $\mathcal{B}$. If the second condition of Theorem 5.4 is omitted then the function f can be constructed analogously. □

Example 5.5. *Let $X = \{1,2,3,4\}$. The sequences $\{E_x\}_{x\in X}$ given by*
- *$E_1 = \{1\}$, $E_2 = \{2\}$, $E_3 = \{3\}$;*
- *$E_1 = \{1,2,3,4\}$, $E_2 = \{2,3,4\}$, $E_3 = \{3,4\}$;*
- *$E_1 = \{1,2\}$, $E_2 = \{2,3\}$, $E_3 = \{3,4\}$;*

and $E_4 = \{4\}$ form positive bases of the space consisting of increasing non-negative functions $\mathcal{F}_\uparrow$. On the other hand, sequences
- *$E_1 = \{1,2,3,4\}$, $E_2 = \{1,3,4\}$, $E_3 = \{3,4\}$;*
- *$E_1 = \{1,4\}$, $E_2 = \{2,4\}$, $E_3 = \{3,4\}$;*

and $E_4 = \{4\}$ do not form such bases.

Remark 5.6. *Note that the set $\{4\}$ is always part of $\uparrow$-compatible basis.*

Following the results in theory of positive linear dependence [2] we get that every positive basis of $\mathcal{F}_\uparrow$ in spite of Theorem 5.4 is minimal and thus the following result follows.

Theorem 5.7. *Let $\mathcal{D}$ be any collection on X. Then there exist coefficients $\alpha_A \geq 0$, $A \in \mathcal{D}$, such that*

$$\sum_{A\in\mathcal{D}} \alpha_A 1_A = f$$

for all $f \in \mathcal{F}_\uparrow$ if and only if there exist a $\mathcal{B} \subseteq \mathcal{D}$ such that $\mathcal{B}$ is $\uparrow$-compatible basis.

Note that for every non-negative function $f \in \mathcal{F}$ there exists a permutation $\sigma\colon X \to X$ such that $f \circ \sigma$ belongs to $\mathcal{F}_\uparrow$. Thus we can define sets

$$\mathcal{F}_\sigma = \{f \in \mathcal{F}\colon f \circ \sigma \in \mathcal{F}_\uparrow\}.$$

For these sets it is easy to characterize bases.

Theorem 5.8. *Let σ be any permutation of X. A collection $\mathcal{B}$ is a basis of $\mathcal{F}_\sigma$ if and only if*

$$\sigma(\mathcal{B}) = \{\sigma(A)\colon A \in \mathcal{B}\}$$

is $\uparrow$-compatible basis.

Proof. Let $f_\sigma \in \mathcal{F}_\sigma$ and let $\mathcal{B}$ be a collection such that $\sigma(\mathcal{B})$ is a basis in spite of Theorem 5.4. Note that $f \circ \sigma \in \mathcal{F}_\uparrow$ and there are coefficients $a_A \geq 0$, $A \in \sigma(\mathcal{B})$, such that

$$\sum_{A\in\sigma(\mathcal{B})} a_A 1_A = f \circ \sigma.$$

Now apply σ^{-1} on the right and obtain that

$$\sum_{A\in\sigma(\mathcal{B})} a_A 1_A \circ \sigma^{-1} = f.$$

Note that

$$\sum_{A\in\sigma(\mathcal{B})} a_A 1_A \circ \sigma^{-1} = \sum_{\sigma^{-1}(A)\in\mathcal{B}} a_A 1_{\sigma^{-1}(A)}$$
$$= \sum_{A\in\mathcal{B}} a_{\sigma^{-1}(A)} 1_A$$

and thus

$$\sum_{A\in\mathcal{B}} a_{\sigma^{-1}(A)} 1_A = f_\sigma$$

which implies that for every function $f_\sigma \in \mathcal{F}_\sigma$ there are coefficients $b_A = a_{\sigma^{-1}(A)} \geq 0$, $A \in \mathcal{B}$, such that

$$\sum_{A\in\mathcal{B}} b_A 1_A = f_\sigma.$$

Also note that $|\mathcal{B}| = |\sigma(\mathcal{B})| = n$ and thus $\mathcal{B}$ is the minimal basis of $\mathcal{F}_\sigma$ which completes the proof. □

Definition 5.9. *Let σ be any permutation of X. A set $\mathcal{B}$ is called σ-compatible basis if and only if $\sigma(\mathcal{B})$ is $\uparrow$-compatible basis.*

Again, based on the theory of positive linear dependence, we obtain the following result.

Theorem 5.10. *Let $\mathcal{D}$ be any collection on X and let σ be any permutation on X. Then there exist coefficients $a_A \geq 0$, $A \in \mathcal{D}$, such that*

$$\sum_{A\in\mathcal{D}} a_A 1_A = f$$

for all $f \in \mathcal{F}_\sigma$ if and only if there exist $\mathcal{B} \subseteq \mathcal{D}$ such that $\mathcal{B}$ is σ-compatible basis.

Remark 5.11. *Note that the set $\sigma^{-1}(\{4\})$ is always part of σ-compatible basis.*

Now it is trivial to see that

$$\mathcal{F} = \bigcup_\sigma \mathcal{F}_\sigma$$

where the union operator runs through all permutations σ on X. Finally, we can formulate the theorem that characterizes all collections $\mathcal{D}$ that decompose all functions from $\mathcal{F}$.

Theorem 5.12. *Let $\mathcal{D}$ be any collection on X. Then there exist coefficients $a_A \geq 0$, $A \in \mathcal{D}$, such that*

$$\sum_{A\in\mathcal{D}} a_A 1_A = f$$

for all $f \in \mathcal{F}$ if and only if there exists a subset $\mathcal{B}_\sigma \subseteq \mathcal{D}$ such that $\mathcal{B}_\sigma$ is a σ-compatible basis for every permutation σ on X.

Definition 5.13. *A collection $\mathcal{D}$ is called Leb-compatible if and only if there exist $\mathcal{B}_\sigma \subseteq \mathcal{D}$ such that $\mathcal{B}_\sigma$ is σ-compatible basis for every permutation σ on X.*

This definition of Leb-compatible collections might seem hard to imagine. The following theorem gives an easy property that characterizes such collections.

Theorem 5.14. *A collection $\mathcal{D}$ is Leb-compatible if and only if $\{\{x\} : x \in X\} \subseteq \mathcal{D}$.*

Proof. Let us denote $\mathcal{P} = \{\{x\} : x \in X\}$. Firstly, let us assume that $\mathcal{D}$ is Leb-compatible. Then we know that $\sigma^{-1}(\{n\}) \in \mathcal{D}$ for every permutation σ on X which implies that $\mathcal{P} \subseteq \mathcal{D}$. On the other hand, let us assume that $\mathcal{P} \subseteq \mathcal{D}$. Then we want to prove that every function is decomposable by $\mathcal{D}$, i.e., there exist non-negative numbers $a_A \geq 0$, $A \in \mathcal{D}$, such that

$$\sum_{A\in\mathcal{D}} a_A 1_A = f$$

for every $f \in \mathcal{F}$. The choice

$$a_A = \begin{cases} f(x), & \text{if } A = \{x\}, \\ 0, & \text{otherwise}, \end{cases}$$

yields the desired decomposition. □

To this moment we characterised all collections $\mathcal{D}$ that are Leb-compatible, i.e., every function can be decomposed to some non-negative linear combination of elements in $\mathcal{D}$. Now we can formulate the main theorem of this section and characterise all collections $\mathcal{D}$ such that $\mathcal{I}_\mathcal{D}$, restricted to the class of measures, yields to the Lebesgue integral.

Theorem 5.15. *Let $\mathcal{D}$ be any collection on X and let $\mathcal{I}_\mathcal{D}$ be a collection integral with respect to the collection $\mathcal{D}$. Then*

$$\mathcal{I}_\mathcal{D}\restriction_{\mathcal{F}\times\mathcal{M}_+} = \text{Leb}$$

if and only if $\{\{x\} : x \in X\} \subseteq \mathcal{D}$.

Proof. Follows directly from Corollary 3.3 and Theorem 5.14. □

6. Super-collection Integral

In this section we provide the definition of the super-collection integral and discuss superdecomposition duals of the chain and the min-max integral.

Definition 6.1. *A super-collection integral with respect to a collection $\mathcal{D}$ is a mapping $\mathcal{I}^*_\mathcal{D} \colon \mathcal{F} \times \mathcal{M} \to [0,\infty]$ such that $\mathcal{I}^*_\mathcal{D} = I^*_{\{\mathcal{D}\}}$.*

The superdecomposition duals of integrals discussed in Section 4 are presented in the following examples.

Example 6.2. *A super-chain integral of a function $f \in \mathcal{F}$ with respect to a capacity $\mu \in \mathcal{M}$ is defined by $\mathrm{ch}^*_B(f,\mu) = \mathcal{I}^*_B(f,\mu)$.*

Example 6.3. *A max-min integral of a function $f \in \mathcal{F}$ with respect to a capacity $\mu \in \mathcal{M}$ is defined by $I^{\vee\wedge}(f,\mu) = \mu(X) \max f(X)$.*

Remark 6.4. *Analogously to Remark 4.16, the value of max-min integral is the upper bound to the values of those decomposition integrals $I_\mathcal{H}$ that contain X in at least one collection.*

What conditions must $\mathcal{D}$ satisfy to ensure that $\mathcal{I}^*_{\mathcal{D}}$ is equivalent to the Lebesgue integral? From the Theorem 3.2 it follows that the conditions are the same as in the case of $\mathcal{I}_{\mathcal{D}}$.

Theorem 6.5. *Let $\mathcal{D}$ be any collection on X and let $\mathcal{I}^*_{\mathcal{D}}$ be a super-collection integral with respect to the collection $\mathcal{D}$. Then*

$$\mathcal{I}^*_{\mathcal{D}}\restriction_{\mathcal{F}\times\mathcal{M}_+} = \text{Leb}$$

if and only if $\{\{x\} : x \in X\} \subseteq \mathcal{D}$.

7. Computational Algorithms

The last section is devoted to the discussion of known computational algorithms for special types of decomposition integrals.

7.1. Concave Integration as Linear Optimization Problem

Note that the problem of the concave integration can be rewritten to the following optimization problem:

$$\begin{array}{ll} \text{maximize} & \displaystyle\sum_{i=1}^{2^n-1} a_i \mu(P_i) \\ \text{subject to} & A\mathbf{a} \leq \mathbf{f} \text{ and } \mathbf{a} \geq \mathbf{0} \end{array}$$

where $\{P_i\}_{i=1}^{2^n-1}$ is any enumeration of $2^X \setminus \{\emptyset\}$, A is $n \times (2^n - 1)$ matrix with $A_{i,j} = 1_{P_j}(x_i)$, $\mathbf{f}$ is n-dimensional vector whose ith element is $f(x_i)$ and $\mathbf{a}$ is unknown $(2^n - 1)$-dimensional vector.

The following result concerning this optimization problem was proved [3].

Theorem 7.1. *The problem of the concave integration posed as a linear optimization problem is harder than NP.*

7.2. Choquet and Chain Integrals

The Choquet integral can be computed using the ordered values of $\text{Im}(f)$. Ordering of n elements can be done in $O(n \log n)$ steps which yield $O(n \log n)$ algorithm.

Similar approach can be taken for the chain integral. This again yields to $O(n \log n)$ algorithm.

7.3. Min-Max Integral

The computation of the min-max integral is straightforward, i.e.,

$$I^{\wedge\vee}(f, \mu) = \mu(X) \min f(X).$$

The only unknown value is the value of $\min f(X)$. This can be done using only $O(n)$ steps. Thus the algorithm computing the value of the min-max integral will take at most $O(n)$ steps.

7.4. Brute Force Algorithms

With other types of decomposition integrals, e.g., the Shilkret and the Pan integrals, the situation is not so easy. Brute force algorithms, i.e., algorithms that check all possible combinations, must be used.

Theorem 7.2. *Computation of the Shilkret and the Pan integrals belong to at most NP class.*

Proof. To prove this claim it is enough to find polynomial verifiers for both integrals. The solution of Shilkret integral is identified with a set from $2^X \setminus \{\emptyset\}$. Given such set A, the minimum of $f(A)$ can be computed in polynomial time and also the product $\mu(A) \min f(A)$. This gives polynomial verifier for the Shilkret integral. For the Pan integral, the solution is identified with a partition $\{A_i\}_{i\in J}$ of X. Such partition has at most n elements and thus $\min f(A_i)$ for $i \in J$ can be computed using polynomial time algorithm. Also, the sum

$$\sum_{i\in J} \mu(A_i) \min f(A_i)$$

can be computed in polynomial time yielding to a polynomial verifier. Thus the computation of the Shilkret and the Pan integrals belong to at most NP class of computational problems. □

Brute force algorithm for computing the Shilkret integral goes as follows. For every set $A \in 2^X \setminus \{\emptyset\}$ compute $\min f(A)$ and find a minimum of $\mu(A) \min f(A)$. The computation of $\mu(A) \min f(A)$ for any A takes at most $O(n)$ operations. The set $2^X \setminus \{\emptyset\}$ has exactly $(2^n - 1)$ elements which yield to $O(2^n n)$ algorithm.

For the Pan integral we need to check all partitions. The number of partitions of a set win n elements is bounded by Catalan numbers, i.e., to generate all partitions we need $O(3^n)$ operations. For each partition we need to compute at most n minimums which yield to $O(n^2)$ operations per partition and thus the brute force algorithm for the Pan integral takes at least $O(3^n n^2)$ operations.

7.5. Special Classes of Capacities

If we restrict ourselves to a special class of capacities then the computation of decomposition integrals might be simplified. The first such considered class is the class of all measures, i.e., all additive capacities.

Theorem 7.3. *If μ is a measure then*

$$\text{Ch}(f, \mu) = \text{Pan}(f, \mu) = \text{cov}(f, \mu) = \text{Leb}(f, \mu).$$

The same theorem holds if μ is a sub-additive capacity, i.e., $\mu(A \cup B) \leq \mu(A) + \mu(B)$ for all disjoint sets $A, B \in 2^X$. For the super-additive capacities, i.e., set functions μ such that $\mu(A \cup B) \geq \mu(A) + \mu(B)$ for all disjoint sets $A, B \in 2^X$, the situation is more complicated.

A capacity is super-modular if and only if

$$\mu(A \cup B) + \mu(A \cap B) \geq \mu(A) + \mu(B)$$

holds for all $A, B \in 2^X$.

Theorem 7.4. *If μ is a super-modular capacity then* $\text{cov}(f, \mu) = \text{Ch}(f, \mu)$.

8. Conclusion

In this contribution we constructed a special class of decomposition integrals called the collection integral. From this class we took a closer look on two special integrals, namely to the chain integral and the min-max integral and we closely investigated their properties.

Superdecomposition dual of the collection integral, the super-collection integral, was also defined and brief discussion of superdecomposition duals for the chain and the min-max integrals, namely the super-chain and the max-min integrals, is presented.

The main result is in characterizing all collections such that if we restrict ourselves to the class of all measures we obtain the collection integral that coincides with the Lebesgue integral. An interesting question is what conditions must a decomposition system $\mathcal{H}$ fulfill to ensure that the decomposition integral coincides with the Lebesgue integral.

Open problem. Let $\mathcal{H}$ be a decomposition system. What conditions must $\mathcal{H}$ fulfill to ensure that

$$I_{\mathcal{H}}\restriction_{\mathcal{F}\times\mathcal{M}_+} = \text{Leb?}$$

Lastly, basic computational algorithms for computing the value of some decomposition integrals, i.e., the Choquet, the Shilkret and the Pan integrals, are examined. Also algorithms for computing the chain and the min-max integrals are discussed. Nevertheless, the computational complexity of such algorithms is analyzed.

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